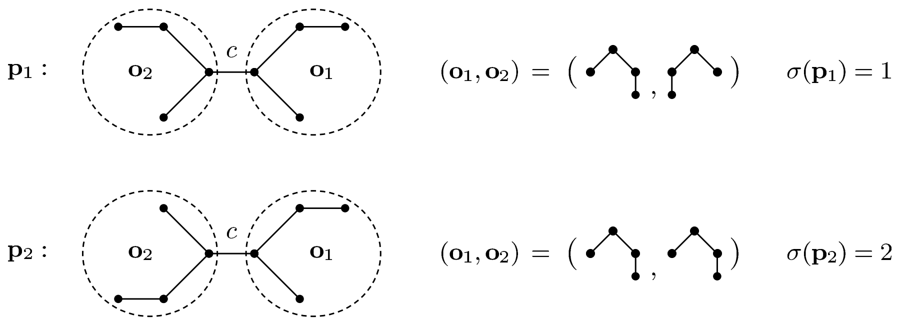

FIGURE 8. $\sigma(\mathbf{p})$: When $c(\mathbf{p})$ is an edge.

Theorem 5. *Given $\mathbf{o}^v \in \mathcal{O}_n^+$, the cardinality of $[\mathbf{o}^v]^+$ is*

$$|[\mathbf{o}^v]^+| = \frac{2\,\epsilon(\mathbf{p})}{\sigma(\mathbf{p})}, \tag{4.3}$$

where $\mathbf{p} = U(\mathrm{d}(\mathbf{o}^v))$, and $\epsilon(\mathbf{p})$ is the number of edges in $\mathbf{p}$.

Proof. Since $\bar{\mathrm{d}}(RL\mathbf{o}^v) = \bar{\mathrm{d}}(\mathbf{o}^v)$ and $U(\mathrm{d}(RL\mathbf{o}^v)) = U(\mathrm{d}(\mathbf{o}^v))$, the size of $[\mathbf{o}^v]^+$ is determined by $\mathbf{p}$. More precisely, the size of $[\mathbf{o}^v]^+$ equals the number of ways of identifying a vertex in $w \in \mathbf{p}$ with $v \in \bar{\mathrm{d}}(\mathbf{o}^v)$. For each vertex w in $\mathbf{p}$, since $\mathbf{p}$ is embedded in the plane, we have $\deg(w)$ distinct ways of identifying w with $v \in \mathrm{d}(\mathbf{o}^v)$. So, if we allow repetition, the number of all possible ways of attaching $\mathbf{p}$ to $\bar{\mathrm{d}}(\mathbf{o}^v)$ is $\sum_{w\in\mathbf{p}} \deg(w) = 2\epsilon(\mathbf{p})$. But by the definition of the symmetry number, each pattern occurs exactly $\sigma(\mathbf{p})$ times. This yields (4.3). $\square$

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PERIODIC DE BRUIJN TRIANGLES: EXACT AND ASYMPTOTIC RESULTS

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ABSTRACT. We study the distribution of the number of permutations with a given periodic up-down sequence w.r.t. the last entry, find exponential generating functions and prove asymptotic formulas for this distribution.

RÉSUMÉ. Nous étudions la distribution du nombre de permutations ayant une suite périodique donnée de montées–descentes par rapport leur derniére entrée. Nous en trouvons les séries génératrices et montrons des formules asymptotiques pour cette distribution.

Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be a permutation of length n. We associate with σ its *up-down sequence* (sometimes called the *shape* of σ, or the *signature* of σ) $\mathcal{P}(\sigma) = (p_1, \dots, p_{n-1})$, which is a binary vector of length $n-1$ such that $p_i = 1$ if $\sigma_i < \sigma_{i+1}$ and $p_i = 0$ otherwise. During the last 120 years, many authors have studied the number $\sharp_n^{\mathcal{P}}$ of all permutations of length n with a given up-down sequence $\mathcal{P}$. Apparently, for the first time this problem was investigated by D. Andrè [An1, An2], who considered the so-called alternating (or up-down) permutations corresponding to the sequence $\mathcal{P} = (1,0,1,0,\dots) = (10)^*$ and proved that the exponential generating function for the number of such permutations is equal to $\tan x + \sec x$. In [An3] he proved that this number grows asymptotically as $2n!(2/\pi)^{n+1}$.

A general approach to this problem was suggested by MacMahon (see [MM]). This approach leads to determinantal formulas for $\sharp_n^{\mathcal{P}}$, rediscovered later by Niven [Ni] from very basic combinatorial considerations. For the relations of this approach to the representation theory of the symmetric group, and for its generalizations, see [Fo, St, BW].

Another, purely combinatorial approach to the same problem was suggested by Carlitz [Ca11]. His general recursive formula for $\sharp_n^{\mathcal{P}}$ is rather difficult to use. However, he managed to obtain explicit expressions for the corresponding exponential generating functions for certain *periodic* cases, that is for up-down sequences of the form $\mathcal{P} = (p)^*$, where p is a binary vector of a fixed length called the *period* of $\mathcal{P}$. In [Ca11, Ca2] he considered the case $\mathcal{P} = (1^k 0)^*$ and expressed the corresponding generating function via the *Olivier functions of the kth order*

$$\varphi_{k,i}(x) = \sum_{j=0}^{\infty} \frac{x^{jk+i}}{(jk+i)!}, \qquad 0 \le i \le k-1.$$

Another case, $\mathcal{P} = (1^2 0^2)^*$, was considered in [CS1, CS2] and solved via Olivier functions of the fourth order. It follows that asymptotically $\sharp_n^{\mathcal{P}}$ in this case grows as $4n!(2/\gamma)^{n+1}$, where $\gamma = 3.7502\dots$ is the smallest positive solution of the equation $\cos t \cosh t + 1 = 0$.

The general periodic problem was solved completely in [CGJN]. As in the two particular cases mentioned above, the answer is expressed via Olivier functions. The techniques used involves matrix Riccati equations, and is rather complicated.

An additional dimension in the problem was introduced by Entringer [En] who studied the distribution of the alternating permutations by the last entry. He observed that the number $\sharp_{i,j}$ of alternating permutations of length i whose last entry equals j satisfy the following equations:

$$\begin{aligned} \natural_{i,j} &= \natural_{i,j-1} + \natural_{i-1,j-1}, & \natural_{i,1} &= 0, & i &= 2k, k > 0, \\ \natural_{i,j} &= \natural_{i,j+1} + \natural_{i-1,j}, & \natural_{i,i} &= 0, & i &= 2k+1, k > 0, \end{aligned} \tag{1}$$

with $\natural_{1,1} = 1$. These equations can be represented graphically as the following triangle displayed on Figure 1.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 1 & & & \\ & & 1 & & 1 & & 0 & & \\ & 0 & & 1 & & 2 & & 2 & \\ 5 & & 5 & & 4 & & 2 & & 0 \end{array}$$

FIGURE 1. The Entringer triangle

Each even row of the triangle starts with 0, and an entry in such a row is equal to the sum of its *left* neighbors in the current and in the previous rows. Similarly, each odd row (except for the first one) ends with 0, and an entry in such a row is equal to the sum of its *right* neighbors in the current and in the previous rows.

The Entringer triangle was studied by many authors. In particular, Arnold [Ar1, Ar2] gave an interpretation of the entries of this triangle in terms of real polynomials with real critical values. Besides, he considered the exponential generating function

$$A(x, y) = \sum_{i \geq 1} \sum_{j=1}^{i} (-1)^{i(i-1)/2} \natural_{i,j} \frac{x^{i-j} y^{j-1}}{(i-j)!(j-1)!}$$

and proved that $A(x, y) = e^y / \cosh(x + y)$. In fact, $A(x, y)$ is the generating function of the *signed Entringer triangle*, which is obtained from the ordinary one by reversing signs in each ith row, where i equals 2 or 3 modulo 4. Observe that the entries $\tilde{\natural}_{i,j}$ of the signed Entringer triangle satisfy relations

$$\tilde{\natural}_{i,j} = \tilde{\natural}_{i,j-1} + \tilde{\natural}_{i-1,j-1} \tag{2}$$

with boundary conditions $\tilde{\natural}_{i,1} = 0$ for $i = 2k$, $\tilde{\natural}_{i,i} = 0$ for $i = 2k+1$, $k > 0$, $\tilde{\natural}_{1,1} = 1$. General triangles satisfying relation (2) with arbitrary boundary conditions were first studied more than 120 years ago by Seidel [Se]. In particular, he proved that the ratio of exponential generating functions for the numbers on the right and on the left sides of such a triangle equals e^x. More recently such triangles where studied, from the combinatorial point of view, in [DV, Du1, Du2]. In particular, it is proved in [DV] that the exponential generating function for a Seidel triangle is equal to $e^y F(x+y)$, where $F(x)$ is the corresponding function for the left side of the triangle.

The case of general up-down sequences was addressed by de Bruijn [dB] (see also [Vi] for another version of the same result). Let $\natural_{i,j}^{\mathcal{P}}$ be the number of permutations of length i whose last entry equals j and whose up-down sequence equals $\mathcal{P} = (p_1, p_2, \dots)$. He proved that these numbers satisfy the following equations:

$$\begin{array}{lll} \sharp_{i,j}^{\mathcal{P}} = \sharp_{i,j-1}^{\mathcal{P}} + \sharp_{i-1,j-1}^{\mathcal{P}}, & \sharp_{i,1}^{\mathcal{P}} = 0, & \text{if } p_{i-1} = 1 \\ \sharp_{i,j}^{\mathcal{P}} = \sharp_{i,j+1}^{\mathcal{P}} + \sharp_{i-1,j}^{\mathcal{P}}, & \sharp_{i,i}^{\mathcal{P}} = 0, & \text{if } p_{i-1} = 0. \end{array}$$

with $\sharp_{1,1}^{\mathcal{P}} = 1$. Evidently, for $\mathcal{P} = (10)^*$ one gets the Entringer relations (1). As before, these equations can be represented graphically as a triangle, and the direction in which one has to advance along the rows of the triangle is governed by the sequence $\mathcal{P}$. We call this triangle the *de Bruijn triangle* corresponding to the up-down sequence $\mathcal{P}$. A de Bruijn triangle is said to be *periodic* if the corresponding up-down sequence is periodic.

The *signed de Bruijn triangle* is obtained from the ordinary de Bruijn triangle by multiplying its ith row by $\varepsilon_i = (-1)^{p_1+p_2+\cdots+p_{i-1}+i-1}$, $i \geq 2$. The corresponding exponential generating function is defined by

$$F^{\mathcal{P}}(x,y) = \sum_{i\geq 1} \sum_{j=1}^{i} \varepsilon_i \sharp_{ij}^{\mathcal{P}} \frac{x^{i-j} y^{j-1}}{(i-j)!(j-1)!}.$$

Let $\mathcal{P}$ be a periodic up-down sequence with period p of length m, and let $i_1 < i_2 < \cdots < i_r$ be the locations of zeros in p. Without loss of generality we assume that $i_r = m$ (otherwise we consider instead of $\mathcal{P}$ the up-down sequence $\bar{\mathcal{P}} = (\bar{p})^*$, where $\bar{p}_i = 1 - p_i$ for $1 \leq i \leq m$; evidently, the de Bruijn triangle for $\bar{\mathcal{P}}$ is obtained from that for $\mathcal{P}$ by the reflection in the main diagonal).

Theorem 1. *The exponential generating function of the signed periodic de Bruijn triangle corresponding to the up-down sequence $\mathcal{P}$ is given by $F^{\mathcal{P}}(x,y) = e^y f^{\mathcal{P}}(x+y)$, where*

$$f^{\mathcal{P}}(t) = \frac{\det \bar{M}^{\mathcal{P}}(t)}{\det M^{\mathcal{P}}(t)}$$

and $M^{\mathcal{P}}(t)$ and $\bar{M}^{\mathcal{P}}(t)$ are $r \times r$ matrices

$$M^{\mathcal{P}}(t) = \begin{pmatrix} \varphi_{m,0} & t^{i_1}\varphi_{m,n-i_1} & t^{i_2}\varphi_{m,n-i_2} & \cdots & t^{i_{r-1}}\varphi_{m,n-i_{r-1}} \\ t^{-i_1}\varphi_{m,i_1} & \varphi_{m,0} & t^{i_1}\varphi_{m,n-i_1} & \cdots & t^{i_{r-2}}\varphi_{m,n-i_{r-2}} \\ t^{-i_2}\varphi_{m,i_2} & t^{-i_1}\varphi_{m,i_1} & \varphi_{m,0} & \cdots & t^{i_{r-3}}\varphi_{m,n-i_{r-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{-i_{r-1}}\varphi_{m,i_{r-1}} & t^{-i_{r-2}}\varphi_{m,i_{r-2}} & t^{-i_{r-3}}\varphi_{m,i_{r-3}} & \cdots & \varphi_{m,0} \end{pmatrix}$$

and

$$\bar{M}^{\mathcal{P}}(t) = \begin{pmatrix} 1 & t^{i_1} & t^{i_2} & \cdots & t^{i_{r-1}} \\ t^{-i_1}\varphi_{m,i_1} & \varphi_{m,0} & t^{i_1}\varphi_{m,n-i_1} & \cdots & t^{i_{r-2}}\varphi_{m,n-i_{r-2}} \\ t^{-i_2}\varphi_{m,i_2} & t^{-i_1}\varphi_{m,i_1} & \varphi_{m,0} & \cdots & t^{i_{r-3}}\varphi_{m,n-i_{r-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{-i_{r-1}}\varphi_{m,i_{r-1}} & t^{-i_{r-2}}\varphi_{m,i_{r-2}} & t^{-i_{r-3}}\varphi_{m,i_{r-3}} & \cdots & \varphi_{m,0} \end{pmatrix}$$

with $\varphi_{m,j} = \varphi_{m,j}(t)$.

The proof of the theorem follows easily from the above mentioned properties of Seidel triangles and an evident identity $e^x = \sum_{i=0}^{m-1} \varphi_{m,i}(x)$. As a corollary we get the result of [CGJN] cited above.

Moreover, the same techniques allows to obtain generating functions for other Seidel triangles with periodic boundary conditions, such as the triangle for Genocchi numbers (see [DV]). It can be also extended to pairs of Seidel triangles with periodic boundary conditions, such as Arnold triangles $L(\beta)$ and $R(\beta)$ for Springer numbers (see [Ar2, Du2]), thus recovering several combinatorial results obtained in [Sp, Ar2].

Let us now consider the asymptotic behavior of the numbers $\sharp^{\mathcal{P}}_{i,j}$. It was observed without a proof in [Ar2, p. 18] that the even rows of the ordinary Entringer triangle approximate, after an appropriate normalization, the function $\sin x$ on the interval $[0,\pi/2]$, while the odd rows approximate $\cos x$. Exact statements with the first two correction terms can be found in [SY].

We generalize this result to arbitrary periodic de Bruijn triangles. Consider the space $\mathcal{L}_2$ of all L_2-functions on the interval $[0,\pi/2]$. Let $\mathfrak{S}_k$ be the linear space of all sequences of real numbers of length k. We define the standard inclusion $i_k\ \mathfrak{S}_k \hookrightarrow \mathcal{L}_2$ as the linear map sending a sequence $\{a_1,\dots,a_k\}$ to the function whose value equals a_j on the interval $[\frac{\pi(j-1)}{2k},\frac{\pi j}{2m})$.

For any l, $0\le l\le m-1$, let $\mathcal{N}^{\mathcal{P}}_{k,l}$ denote the sequence $\{\sharp^{\mathcal{P}}_{kn+l,j}\}\in\mathfrak{S}_{km+l}$, and let $\xi^{\mathcal{P}}_{k,l}=ci_{km+l}(\mathcal{N}^{\mathcal{P}}_{k,l})$, where c is a normalizing coefficient depending on k and l and chosen in such a way that $\int_0^{\pi/2}\xi^{\mathcal{P}}_{k,l}(\theta)d\theta=1$.

Theorem 2. *For any l, $0\le k\le m-1$, the sequence of functions $\xi^{\mathcal{P}}_{k,l}$ converges in the L_2-metric to the normalized first eigenfunction $\Xi^{\mathcal{P}}_l$ of the two-point spectral problem*

$$z^{(m)}=(-1)^r\lambda^m z$$

with m homogeneous boundary conditions

$$\begin{array}{ll} z^{(i)}(0)=0 & \text{if } p_{l+i-1}=1,\\ z^{(i)}(\pi/2)=0 & \text{if } p_{l+i-1}=0,\end{array}$$

where r is the number of zeros in the period p.

The first eigenvalue of the above spectral problem is the minimal positive solution of the equation

$$\det\tilde M^{\mathcal{P}}(\varepsilon\lambda\pi/2)=0,$$

where ε is the mth primitive root of $(-1)^r$ and

$$\tilde M^{\mathcal{P}}(t)=\begin{pmatrix}\varphi_{m,0} & \varphi_{m,i_2-i_1} & \varphi_{m,i_3-i_1} & \cdots & \varphi_{m,i_r-i_1}\\ \varphi_{m,i_1-i_2} & \varphi_{m,0} & \varphi_{m,i_3-i_2} & \cdots & \varphi_{m,i_r-i_2}\\ \varphi_{m,i_1-i_3} & \varphi_{m,i_2-i_3} & \varphi_{m,0} & \cdots & \varphi_{m,i_r-i_3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \varphi_{m,i_1-i_r} & \varphi_{m,i_2-i_r} & \varphi_{m,i_3-i_r} & \cdots & \varphi_{m,0}\end{pmatrix}$$

with $\varphi_{m,j}=\varphi_{m,j}(t)$.

The eigenfunction $\Xi^{\mathcal{P}}_l$ is normalized by the condition $\int_0^{\pi/2}\Xi^{\mathcal{P}}_l(\theta)d\theta=1$, and the indices are understood modulo m.

In particular, for the Entringer numbers $\sharp_{i,j}$ one gets the sine law of [SY]:

$$\begin{cases}\lim_{k\to\infty,\ \frac{j}{2k+1}\to u}\sharp_{2k+1,j}=2(2k)!\left(\frac{2}{\pi}\right)^{2k+1}\cos\frac{\pi u}{2},\\ \lim_{k\to\infty,\ \frac{j}{2k}\to u}\sharp_{2k,j}=2(2k-1)!\left(\frac{2}{\pi}\right)^{2l}\sin\frac{\pi u}{2}.\end{cases}$$

The starting point of this research was the result by the first and the third author that the numbers $\sharp^{\mathcal{P}}_{i,j}$ for $\mathcal{P}=(1^20^2)^*$ arise naturally in counting real rational functions of a certain type, see [SV]. The authors are grateful to Max–Planck–Institut für Mathematik, Bonn for its hospitality in September 2000 and to the Royal Institute of Technology, Stockholm for the financial support of the visit of A. V. to Stockholm in July-August 2001. Sincere thanks goes to Prof. Harold Shapiro from The Royal Institute of Technology for his help with functional analysis in the proof of Theorem 2.

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