

q AND q, t -ANALOGS OF NON-COMMUTATIVE SYMMETRIC FUNCTIONS

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ABSTRACT. We introduce two families of non-commutative symmetric functions that have analogous properties to the Hall-Littlewood and Macdonald symmetric functions.

RÉSUMÉ. Nous présentons deux familles des fonctions symétriques non-commutatives qui ont les propriétés analogues aux fonctions symétriques de Hall-Littlewood et de Macdonald.

1. INTRODUCTION

It was noticed in [16] that many q and q, t analogs that are commonly studied in the space of symmetric functions arise from an unusual q -twisting of the symmetric function found by setting $q = 0$. In particular we see that the Hall-Littlewood and Macdonald [10] symmetric functions arise from this construction by taking a q -analog of operators that add a row on the Schur basis or a column on the Hall-Littlewood symmetric functions.

The q -twisting that was given in that article may easily be expressed in terms of the notation of Hopf algebras (since the symmetric functions form a commutative and co-commutative Hopf algebra structure). Within this context it can be seen that this may be generalized to any graded Hopf algebra. It is not clear when or if this q -analog will be interesting in another setting, but it creates a context for looking for ‘natural’ examples of q -analogs within other spaces.

The non-commutative symmetric functions [5] form such a graded Hopf algebra and are an obvious place to begin searching for such examples. The natural analog of the Schur functions within this space are the ribbon Schur functions.

In a manner analogous to that in the symmetric functions, it is possible to define operators that add a row to the ribbon Schur functions. The q -analog of these operators are a natural analog of the operators that add a row to the Hall-Littlewood symmetric functions. Remarkably, we see that this action gives rise to a family of non-commutative symmetric functions that when expanded in terms of ribbon Schur functions have coefficients that are a power of q .

These NC-symmetric functions have properties that suggest they are an analog of the Hall-Littlewood symmetric functions, but they are not equivalent to the non-commutative Hall-Littlewood symmetric functions of Hivert [7]. They do however share some of the same properties of Hivert’s NC-Hall-Littlewoods including a factorization property by setting q equal to a root of unity (Proposition 13). It should be remarked that the differences between the factorization properties of these and the Hivert functions suggest that the Hivert functions are associated to the length statistic on compositions in the same way that the Hall-Littlewood functions presented here are associated to the size statistic on compositions.

This family of NC-symmetric functions $\{\mathbf{H}_\alpha^q\}_\alpha$ have the following distinguishing properties.

1. They are triangularly related to the ribbon Schur basis. Namely, we have

$$\mathbf{H}_\alpha^q = \mathbf{s}_\alpha + \sum_{\beta > \alpha} c_{\alpha\beta}^q \mathbf{s}_\beta.$$

2. The coefficient of a single ribbon function in \mathbf{H}_α^q is either 0 or a power of q . The coefficient of the ribbon function indexed by a single part in \mathbf{H}_α^q is $q^{n(\alpha)}$ where $n(\alpha) = \sum_i i$ where the sum is over all i which are descents of α .

3. When $q = 1$, \mathbf{H}_α^q becomes \mathbf{h}_α , the non-commutative analogs of the homogeneous symmetric functions. When $q = 0$, \mathbf{H}_α^q specializes to the ribbon Schur function \mathbf{s}_α . When q is a root of unity then \mathbf{H}_α^q specializes to a product of non-commutative symmetric functions.

Considering the morphism χ that sends the non-commutative symmetric function \mathbf{h}_α to the commutative version h_α , the image of the functions \mathbf{H}_α^q are the commutative Hall-Littlewood functions whenever the composition α represents a partition (i.e. when α is a hook). That is,

4. (Proposition 9) $\chi(\mathbf{H}_{(1^a, b)}^q) = H_{(b, 1^a)}^q$ where $H_\lambda^q = \sum_\mu K_{\mu\lambda}(q) s_\mu$.

We introduce an inner product on the space of non-commutative symmetric functions by setting the ribbon Schur functions as ‘semi-self’ dual in the following manner

$$\langle \mathbf{s}_\alpha, \mathbf{s}_\beta \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha\beta^c}.$$

This inner product does not seem to appear elsewhere in the literature, but does share similar properties with the inner product of the symmetric functions and can be a useful tool for calculation within this space. The surprising property that we observe is that the non-commutative analogs of the elementary, homogeneous and Hall-Littlewood bases also share this ‘semi-self’ duality property. That is, we have in addition to the properties mentioned above,

5. (Proposition 8)

$$\langle \mathbf{H}_\alpha^q, \mathbf{H}_\beta^q \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha\beta^c}.$$

Most of these properties are analogous to ones that exist for the non-commutative Hall-Littlewood analogues of Hivert [7], however this last property is not shared by Hivert’s noncommutative Hall-Littlewood functions.

Next, we consider a q, t -analog of the non-commutative symmetric functions where we look for properties that are analogous to the Macdonald symmetric functions. Of course, many q, t -analogues are possible and we consider one that has properties which generalize those for our version of the non-commutative Hall-Littlewood and seem to be analogous to the Macdonald symmetric functions. The family $\{\mathbf{H}_\alpha^{qt}\}_\alpha$ has the following important properties:

1. There is a triangular relation between the family $\{\mathbf{H}_\alpha^t\}_\alpha$ and $\{\mathbf{H}_\alpha^{qt}\}_\alpha$.

$$\mathbf{H}_\alpha^{qt} = \sum_{\beta \leq \alpha} c_{\alpha\beta}^{qt} \mathbf{H}_\beta^t.$$

2. The coefficient of a single ribbon function in \mathbf{H}_α^{qt} is of the form $q^a t^b$ (with $a, b \geq 0$). The coefficient of a ribbon Schur function indexed by a single part in \mathbf{H}_α^{qt} is $t^{n(\alpha)}$, the coefficient of a ribbon Schur function indexed by a composition of 1s is $q^{n(\alpha')}$.

3. We have the specialization $\mathbf{H}_\alpha^{0t} = \mathbf{H}_\alpha^t$, and \mathbf{H}_α^{1t} is a product of non-commutative symmetric functions (Proposition 16).

4. The \mathbf{H}_α^{qt} satisfy the following two relations

$$\begin{aligned} \mathbf{H}_\alpha^{tq} &= \omega' \mathbf{H}_{\alpha'}^{qt}, \\ q^{n(\alpha')} t^{n(\alpha)} \mathbf{H}_\alpha^{q\frac{1}{t}} &= \omega^c \mathbf{H}_\alpha^{qt}. \end{aligned}$$

5. (Proposition 20) $\chi(\mathbf{H}_{(1^a, b)}^{qt}) = H_{(b, 1^a)}^{qt}$, where $H_\lambda^{qt} = \sum_\mu K_{\mu\lambda}(q, t) s_\mu$.

6. (Proposition 17)

$$\langle \mathbf{H}_\alpha^{qt}, \mathbf{H}_\beta^{qt} \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha\beta^c} \prod_{i=1}^{n-1} (1 - q^i t^{n-i}).$$

The most remarkable property that arises from these functions is the existence of an operator ∇ that has the NC-Macdonald functions as eigenfunctions. That is, if we set

$$\nabla \tilde{\mathbf{H}}_\alpha^{qt} = q^{n(\alpha')} t^{n(\alpha)} \tilde{\mathbf{H}}_\alpha^{qt},$$

where $\tilde{\mathbf{H}}_\alpha^{qt} = t^{n(\alpha)} \mathbf{H}_\alpha^{q\frac{1}{t}}$, then this operator has an elegant action on the ribbon Schur functions and shares many of the same properties that exist in the commutative case [3]. Unlike in the commutative case however, formulas for this operator are immediately solvable. In a beautiful analogy, where the commutative version of the operator ∇ produced a q, t grading of the space of parking functions, the non-commutative ∇ produces a grading of the space of preferential arrangements.

In searching for an interesting non-commutative analog of the Macdonald symmetric functions, we considered many possibilities (including the analog considered in [8]). None of the functions except for the one we discuss here had an equivalent ∇ operator and it was this property that indicated to us that these functions are indeed remarkable.

It is not known yet if this family has a representation theoretical model analogous to the $n!$ -conjecture or the diagonal harmonics that motivate the existence of these functions. We do see however that the non-commutative versions of these functions and operators share many of the same properties with the commutative case. Independent of their own interest, it is at least hopeful that they will give some insight into why the some of the conjectures for the commutative case are true.

2. NOTATION FOR COMPOSITIONS, PARTITIONS, HOPF ALGEBRAS, SYMMETRIC, NC-SYMMETRIC, AND QUASI-SYMMETRIC FUNCTIONS

2.1. Compositions. We will say that α is a composition of n and write $\alpha \models n$ if α is a sequence of positive integers such that $\alpha_1 + \alpha_2 + \dots + \alpha_{\ell(\alpha)} = |\alpha| = n$. The length of the sequence is denoted by the symbol $\ell(\alpha)$.

For any two compositions α and β , define the *concatenate* and the *attach* of α and β to be the compositions (respectively)

$$(1) \quad \alpha \cdot \beta = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)}, \beta_1, \beta_2, \dots, \beta_{\ell(\beta)})$$

and

$$(2) \quad \alpha | \beta = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)} + \beta_1, \beta_2, \dots, \beta_{\ell(\beta)}).$$

For a composition α of n define the descent set of α to be the subset of $\{1, 2, 3, \dots, n-1\}$ as $D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \dots + \alpha_{\ell(\alpha)-1}\}$. The size of the descent set of α is one less than it's length and it is easily seen that the compositions of n are in one-to-one correspondence with the subsets of $\{1, 2, \dots, n-1\}$.

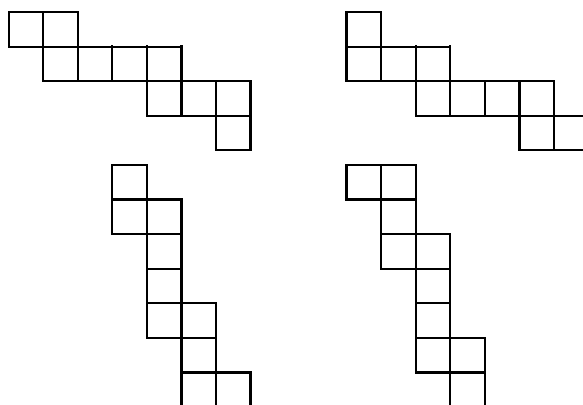


FIGURE 1. Images of the three involutions. If $\alpha = (2, 4, 3, 1)$, then $\bar{\alpha} = (1, 3, 4, 2)$, $\alpha^c = (1, 2, 1, 1, 2, 1, 2)$ and $\alpha' = (2, 1, 2, 1, 1, 2, 1)$.

There is a natural partial order on the compositions of n . Say that a composition α is finer than a composition β (or β is coarser than α) and write $\alpha \leq \beta$ if there exists compositions $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(k)}$ such that $\alpha = \gamma^{(1)} \cdot \gamma^{(2)} \cdot \dots \cdot \gamma^{(k)}$ and $\beta = \gamma^{(1)} | \gamma^{(2)} | \dots | \gamma^{(k)}$. Alternatively, in terms of descent sets we say that $\alpha \leq \beta$ if and only if $D(\beta) \subseteq D(\alpha)$.

There are three standard involutions on the set of compositions. The first involution reverses the order of the sequence. We set $\overleftarrow{\alpha} = (\alpha_{\ell(\alpha)}, \alpha_{\ell(\alpha)-1}, \dots, \alpha_1)$. If the descent set of α is $D(\alpha) = \{i_1, i_2, \dots, i_k\}$ then $D(\overleftarrow{\alpha}) = \{|\alpha| - i_1, |\alpha| - i_2, \dots, |\alpha| - i_k\}$.

The second involution corresponds to taking the complement of the descent set. Define α^c to be the composition with $D(\alpha^c) = \{1, 2, \dots, |\alpha| - 1\} - D(\alpha)$. Notice that if α is a composition of length k then α^c is a composition of length $n + 1 - k$.

Finally, the third involution is the composition of the previously two defined. Let $\alpha' = \overleftarrow{\alpha^c} = \overleftarrow{\alpha}^c$. The composition α' also has length $n + 1 - k$.

For some formulas we will need a total order on the compositions of size n . We set $\phi(\alpha) = \sum_{i \in D(\alpha)} 2^{i-1}$. This map associates each composition with an integer between 0 and $2^{n-1} - 1$. This map induces a total order from the integers on this set which is a refinement of the partial order defined above.

For the composition α , there is a standard statistic we will use frequently given by $n(\alpha) = \sum_{i \in D(\alpha)} i$.

2.2. Partitions. A partition λ of n is a composition of n with the property that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)}$. We will indicate that λ is a partition by the notation that $\lambda \vdash n$. We say that a partition μ is contained in a partition λ and write that $\mu \subseteq \lambda$ if $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for all $1 \leq i \leq \ell(\mu)$. Define then a skew partition to be represented by λ/μ where λ and μ are partitions such that $\mu \subseteq \lambda$.

There is a partial order on the set of partitions. We will say that the partition $\lambda \geq \mu$ if $\lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i$ for all $i \geq 1$.

Partitions will sometimes be represented by Ferrers diagrams, a graphical representation of a partition formed by placing rows of square cells aligned on the left hand edge with λ_i cells in the i^{th} row. We will use the cartesian convention where we place the 1^{st} row of cells on the bottom of the diagram (the matrix convention is to place the 1^{st} row of cells on the top). A skew Ferrers diagram for a skew partition λ/μ is Ferrers diagram for the partition λ where the cells that correspond to the partition μ are not drawn.

Define the conjugate partition to λ to be the partition λ' such that λ'_i is the number of parts of λ that have size greater than or equal to i . This corresponds to the partition formed by flipping λ across the line $x = y$.

Any composition of n may be associated with a ‘ribbon,’ a skew partition with no 2×2 sub-diagrams. This ribbon is usually represented by a skew-Ferrers diagram. The composition $\alpha \vdash n$ is mapped to the skew partition $(\alpha_1 + \alpha_2 + \dots + \alpha_{\ell(\alpha)} - \ell(\alpha) + 1, \alpha_1 + \alpha_2 + \dots + \alpha_{\ell(\alpha)-1} - \ell(\alpha) + 2, \dots, \alpha_1) / (\alpha_1 + \alpha_2 + \dots + \alpha_{\ell(\alpha)-1} - \ell(\alpha) + 1, \alpha_1 + \alpha_2 + \dots + \alpha_{\ell(\alpha)-2} - \ell(\alpha) + 2, \dots, \alpha_1 + \alpha_2 - 2, \alpha_1 - 1)$. For the example in Figure 1, $\alpha = (2, 4, 3, 1)$ is associated to the skew partition $(7, 7, 5, 2) / (6, 4, 1)$.

We will label the cells of the x, y -coordinate lattice and say that a point (i, j) is in the diagram of a partition μ if $1 \leq i \leq \mu_j$. The arm of the cell $s = (i, j)$ in a partition μ is denoted by $a_\mu(s) := \mu_j - i$. The leg will be denoted by the value $l_\mu(s) := a_{\mu'}(j, i)$.

2.3. Hopf algebras. For general facts about Hopf algebras, we refer the reader to [1] or [12].

Let R be a commutative ring and H an R module. We say that H is an algebra if there are maps $\mu : H \otimes H \rightarrow H$ (multiplication) and $\eta : R \rightarrow H$ (unit) that satisfy the following two conditions:

- 1) $\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id)$
- 2) $\mu \circ (\eta \otimes id) = id = \mu \circ (id \otimes \eta)$

We say that H has a co-algebra structure if there are maps $\Delta : H \rightarrow H \otimes H$ (comultiplication) and $\varepsilon : H \rightarrow R$ (counit) that satisfy the following two conditions:

- 1) $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$
- 2) $(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$

If H has at the same time an algebra and a co-algebra structure (H, μ, η) and (H, Δ, ε) and Δ is a homomorphism of algebras then we call H together with these corresponding operations, $(H, \mu, \eta, \Delta, \varepsilon)$, a bialgebra. H is called a Hopf algebra if it is a bialgebra with a map $S : H \rightarrow H$ that satisfies the following identity:

- 1) $\mu \circ S \otimes id \circ \Delta = \mu \circ id \otimes S \circ \Delta = \eta \circ \varepsilon$

If we define the map $\tau : H \otimes H \rightarrow H \otimes H$ by $\tau(a \otimes b) = b \otimes a$ then we say that an algebra H is commutative if $\mu \circ \tau = \mu$ and we say that a co-algebra H is co-commutative if $\tau \circ \Delta = \Delta$. It may be shown that for any Hopf algebra $\Delta \circ S = S \otimes S \circ \tau \circ \Delta$.

If the Hopf algebra is either commutative or co-commutative, then it follows that S is an involution.

An important operation that arises in this setting is the convolution of two operators $f, g \in Hom(H, H)$. We set $f * g = \mu \circ f \otimes g \circ \Delta$. Convolution is an associative binary operation and the element $\eta\varepsilon$ serves as the identity. That is, we have for $V \in Hom(H, H)$

$$(3) \quad \eta\varepsilon * V = V * \eta\varepsilon = V.$$

In addition, it follows from the defining property of the antipode that $id * S = S * id = \eta\varepsilon$.

2.4. Symmetric functions. We refer the reader to [10] for basic facts about the symmetric functions.

Consider the space of symmetric functions as the polynomial ring $\Lambda = \mathbb{Q}[p_1, p_2, \dots]$ in the commuting set of variables $\{p_1, p_2, p_3, \dots\}$. The p_i are the simple power symmetric functions and represent the symmetric formal series $p_k = x_1^k + x_2^k + x_3^k + \dots$ (although in this context we need not consider the variables $\{x_1, x_2, x_3, \dots\}$). Define the degree of the power symmetric function p_k within this space to have degree k . Let Λ^n represent the subspace of polynomials of degree n . Since the p_k commute, we see that Λ^n is spanned by the set of monomials $p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_{\ell(\lambda)}}$ where λ is a partition of n .

Let $n_i(\lambda)$ represent the number of parts of size i in the partition λ , then define $z_\lambda = \prod_{i \geq 1} i^{n_i(\lambda)} n_i(\lambda)!$. The simple elementary symmetric functions are defined to be $e_k = \sum_{\lambda \vdash k} (-1)^{k-\ell(\lambda)} p_\lambda / z_\lambda$ and the simple homogeneous symmetric functions are defined to be $h_k = \sum_{\lambda \vdash k} p_\lambda / z_\lambda$. Set $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$ and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$. The Schur symmetric functions are defined to be $s_\lambda = \det |h_{\lambda_i + i - j}| = \det |e_{\lambda'_i + i - j}|$.

There is a natural scalar product defined on this space that is defined on the power symmetric functions by $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$. It may be shown that the Schur functions are self-dual with respect to the scalar product, that is, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.

We are interested in finding analogs for the two families of symmetric functions $H_\mu^q := \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q) s_\lambda$ and $H_\mu^{qt} := \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q, t) s_\lambda$ (the Hall-Littlewood and Macdonald symmetric functions). For a definition of $K_{\lambda\mu}(q)$ and $K_{\lambda\mu}(q, t)$ and their associated properties, we refer the interested reader to [10].

2.5. Non-commutative symmetric functions. For a more detailed reference about the non-commutative symmetric functions, we refer the reader to [5].

Consider the space of non-commutative symmetric functions as the polynomial ring $NCA = \mathbb{Q} \langle \mathbf{h}_1, \mathbf{h}_2, \dots \rangle$ in the non commuting set of variables $\{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \dots\}$. The degree of a monomial $\mathbf{h}_{i_1} \mathbf{h}_{i_2} \cdots \mathbf{h}_{i_\ell}$ will be the sum of the indices $i_1 + i_2 + \cdots + i_\ell$. The span of the monomials of NCA of degree n will be denoted by NCA^n so that $NCA = \bigoplus_{n \geq 0} NCA^n$ is a graded ring.

We will define $\mathbf{e}_k = \sum_{\alpha \vdash k} (-1)^{k-\ell(\alpha)} \mathbf{h}_\alpha$. These are the analogs of the basic homogeneous and elementary symmetric functions. For any composition we define $\mathbf{h}_\alpha = \mathbf{h}_{\alpha_1} \mathbf{h}_{\alpha_2} \cdots \mathbf{h}_{\alpha_{\ell(\alpha)}}$ and $\mathbf{e}_\alpha = \mathbf{e}_{\alpha_1} \mathbf{e}_{\alpha_2} \cdots \mathbf{e}_{\alpha_{\ell(\alpha)}}$. The sets $\{\mathbf{h}_\alpha\}_{\alpha \vdash n}$ and $\{\mathbf{e}_\alpha\}_{\alpha \vdash n}$ are all bases for the space of non-commutative symmetric functions of degree n .

The ribbon Schur functions are defined to be $\mathbf{s}_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \mathbf{h}_\beta$. It is well known that the set $\{\mathbf{s}_\alpha\}_{\alpha \vdash n}$ also defines a basis for NCA^n . It is normal to define a pairing of the NC-symmetric functions with the Quasi-symmetric functions. Instead here we define a scalar product on this space such that the ribbon Schur functions are self-dual, that is,

$$(4) \quad \langle \mathbf{s}_\alpha, \mathbf{s}_\beta \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha\beta^c}.$$

We remark that this product has the property that $\langle f, g \rangle = (-1)^{\deg(f)+1} \langle g, f \rangle$. This follows from the relation $\ell(\alpha) + \ell(\alpha^c) = |\alpha| + 1$. We will prove in full generality in later sections (although it is not difficult to show) that we have the following relations

Proposition 1.

$$(5) \quad \langle \mathbf{s}_\alpha, \mathbf{s}_\beta \rangle = \langle \mathbf{h}_\alpha, \mathbf{h}_\beta \rangle = \langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha\beta^c}.$$

Corollary 2. For any $f \in NCA^n$, we have

$$(6) \quad f = \sum_{\alpha \vdash n} (-1)^{\ell(\alpha)+1} \langle \mathbf{s}_{\alpha^c}, f \rangle \mathbf{s}_\alpha = \sum_{\alpha \vdash n} (-1)^{|\alpha| + \ell(\alpha)} \langle f, \mathbf{s}_{\alpha^c} \rangle \mathbf{s}_\alpha.$$

A similar statement can be made for the \mathbf{h}_α and \mathbf{e}_α bases.

There is a co-commutative Hopf algebra structure on the space of non-commutative symmetric functions. Multiplication μ , the unit η and the counit ε as defined in the usual manner. The comultiplication is defined as the algebra homomorphism that sends $\Delta(\mathbf{h}_k) = \sum_{i=0}^k \mathbf{h}_i \otimes \mathbf{h}_{k-i}$ and $\Delta(\mathbf{e}_k) = \sum_{i=0}^k \mathbf{e}_i \otimes \mathbf{e}_{k-i}$. The antipode is defined by $S(\mathbf{s}_\alpha) = (-1)^{|\alpha|} \mathbf{s}_{\alpha'}$, $S(\mathbf{h}_\alpha) = (-1)^{|\alpha|} \mathbf{e}_{\overline{\alpha}}$ and $S(\mathbf{e}_\alpha) = (-1)^{|\alpha|} \mathbf{h}_{\overline{\alpha}}$. Elementary properties of the antipode and the scalar product show that $\langle S(f), S(g) \rangle = \langle g, f \rangle$.

There are three standard involutions that correspond to those that exist for the compositions. Set $\omega'(\mathbf{s}_\alpha) = \mathbf{s}_{\alpha'}$, $\overleftarrow{\omega}(\mathbf{s}_\alpha) = \mathbf{s}_{\overline{\alpha}}$ and $\omega^c(\mathbf{s}_\alpha) = \mathbf{s}_{\alpha^c}$. It is important to note that we

have the relations $\omega' \overleftarrow{\omega} = \overleftarrow{\omega} \omega' = \omega^c$. These operations may be expressed on other bases as well yielding the following expressions:

$$(7) \quad \begin{aligned} \omega'(\mathbf{h}_\alpha) &= \mathbf{e}_{\overline{\alpha}} & \omega'(\mathbf{e}_\alpha) &= \mathbf{h}_{\overline{\alpha}} \\ \overleftarrow{\omega}(\mathbf{h}_\alpha) &= \mathbf{h}_{\overline{\alpha}} & \overleftarrow{\omega}(\mathbf{e}_\alpha) &= \mathbf{e}_{\overline{\alpha}} \\ \omega^c(\mathbf{h}_\alpha) &= \mathbf{e}_\alpha & \omega^c(\mathbf{e}_\alpha) &= \mathbf{h}_\alpha \end{aligned}$$

Of course we also see that $\langle \overleftarrow{\omega} f, \overleftarrow{\omega} g \rangle = \langle f, g \rangle$ and $\langle \omega' f, \omega' g \rangle = \langle \omega^c f, \omega^c g \rangle = \langle g, f \rangle$.

We will sometimes wish to look at the commutative versions of the non-commutative symmetric functions. To this end, we introduce the surjection $\chi : NCL \rightarrow \Lambda$ which sends \mathbf{h}_α to the symmetric function $h_{\alpha_1} h_{\alpha_2} \dots h_{\alpha_k}$.

2.6. The quasi-symmetric functions. Consider the space of polynomials in the commuting set of variables $x_1, x_2, x_3, \dots, x_n$. The quasi-symmetric functions will be denoted by $Qsym$ which will be the subspace of polynomials spanned by the functions

$$(8) \quad M_\alpha = \sum_f x_{f(1)}^{\alpha_1} x_{f(2)}^{\alpha_2} \cdots x_{f(\ell(\alpha))}^{\alpha_{\ell(\alpha)}},$$

where the sum is over all functions $f : [\ell(\alpha)] \rightarrow [n]$ such that $f(i) < f(i + 1)$.

These functions are the analogs of the monomial symmetric functions within the space of symmetric functions. There is a standard pairing between the quasi-symmetric functions and space of non-commutative symmetric functions. This pairing is defined by setting non-commutative homogeneous symmetric functions as dual to the M_α basis, that is $[M_\alpha, \mathbf{h}_\beta] = \delta_{\alpha\beta}$. This is the pairing that makes $Qsym$ and NCL graded dual Hopf algebras [11].

The ribbon quasi-symmetric functions, F_α are then defined as the elements of $Qsym$ such that $[F_\alpha, \mathbf{s}_\beta] = \delta_{\alpha\beta}$. Clearly, any $A \in Qsym$ may be expanded in these bases by using the formula

$$(9) \quad A = \sum_{\beta \models \deg(A)} [A, \mathbf{s}_\beta] F_\beta = \sum_{\beta \models \deg(A)} [A, \mathbf{h}_\beta] M_\beta.$$

Similarly, for any element $\mathbf{A} \in NCL$, \mathbf{A} may be expanded in terms of the \mathbf{s}_β and \mathbf{h}_β bases if we know the pairing between \mathbf{A} and F_β or M_β .

$$(10) \quad \mathbf{A} = \sum_{\beta \models \deg(\mathbf{A})} [F_\beta, \mathbf{A}] \mathbf{s}_\beta = \sum_{\beta \models \deg(\mathbf{A})} [M_\beta, \mathbf{A}] \mathbf{h}_\beta$$

There is a simple relation between the $Qsym/NCL$ pairing and the scalar product on NCL . This is expressed with the following proposition.

Proposition 3. $A \in Qsym$ and $\mathbf{A} \in NCL$ are such that for all $\mathbf{B} \in NCL$

$$[A, \mathbf{B}] = \langle \mathbf{A}, \mathbf{B} \rangle,$$

if and only if A and \mathbf{A} have the following relationship

$$\mathbf{A} = \sum_{\beta \models n} (-1)^{\ell(\beta)+1} [A, \mathbf{s}_\beta] \mathbf{s}_{\beta^c} = \sum_{\beta \models n} (-1)^{\ell(\beta)+1} [A, \mathbf{h}_\beta] \mathbf{h}_{\beta^c},$$

$$(11) \quad A = \sum_{\beta \models n} \langle \mathbf{A}, \mathbf{s}_\beta \rangle F_\beta = \sum_{\beta \models n} \langle \mathbf{A}, \mathbf{h}_\beta \rangle M_\beta.$$

Proof. By applying Corollary 2,

$$(12) \quad \begin{aligned} \langle \mathbf{A}, \mathbf{B} \rangle &= \left\langle \sum_{\beta \models n} (-1)^{\ell(\beta)+1} [A, \mathbf{s}_\beta] \mathbf{s}_{\beta^c}, \mathbf{B} \right\rangle \\ &= \left[A, \sum_{\beta \models n} (-1)^{\ell(\beta)+1} \langle \mathbf{s}_{\beta^c}, \mathbf{B} \rangle \mathbf{s}_\beta \right] = [A, \mathbf{B}]. \end{aligned}$$

□

This last proposition implies that for a basis \mathbf{A}_α such that $\langle \mathbf{A}_\alpha, \mathbf{A}_\beta \rangle = (-1)^{n+\ell(\alpha)} \delta_{\alpha\beta^c}$, then to compute its dual basis in $Qsym$ it is sufficient to calculate the values of the scalar products $\langle \mathbf{A}_\alpha, \mathbf{s}_\beta \rangle$, since by equation (11), we have that

$$\left[(-1)^{n+\ell(\alpha^c)} \sum_{\beta \models n} \langle \mathbf{A}_{\alpha^c}, \mathbf{s}_\beta \rangle F_\beta, \mathbf{A}_\beta \right] = (-1)^{n+\ell(\alpha^c)} \langle \mathbf{A}_{\alpha^c}, \mathbf{A}_\beta \rangle = \delta_{\alpha\beta}.$$

3. q -ANALOG BASES

3.1. Scrambled Hopf algebra operators. Consider the following transformation on $Hom(A, A)$ that seems to arise in a natural way when considered as an operation on symmetric functions [17]. If A is a Hopf algebra, then for $V \in Hom(A, A)$ we define $\overline{V} = \mu \circ id \otimes (VS) \circ \Delta = id * (VS)$. We may show that in any co-commutative Hopf algebra A , the bar operation on $V \in Hom(A, A)$ is an involution. That is, we have

Proposition 4. *For any $V \in Hom(A, A)$ with $(A, \mu, \eta, \Delta, \varepsilon, S)$ a co-commutative Hopf algebra, we have that $\overline{\overline{V}} = V$.*

Our q -analog arises by starting with a graded co-commutative Hopf algebra with a function R^q such that for any element $f \in A$ that is of homogeneous degree $R^q(f) = q^{deg(f)} f$. The important properties of this function are that $R^q|_{q=0} = \eta\varepsilon$ and $R^q|_{q=1} = id$. Now for any $V \in Hom(A, A)$, we set

$$(13) \quad \tilde{V}^q = \overline{\overline{R^q V}}.$$

It is not at all obvious that the effect on V will be necessarily interesting. We do however have many examples of where this q -analog arises within the theory of symmetric functions [16]. One may only hope that within other co-commutative Hopf algebras that this operation is also interesting. In this exposition we will show that there is an application of this q -analog within the non-commutative symmetric functions.

Proposition 5. *Let A be a Hopf algebra, with the property that $\overline{V} := id * VS$ is an involution for all $V \in Hom(A, A)$. Also assume that there is an operator R^q such that $R^q|_{q=0} = \eta\varepsilon$ and $R^q|_{q=1} = id$. If V does not depend on q , then the operator $\tilde{V}^q = \overline{\overline{R^q V}}$ is a q -analog of V (recovered by setting $q = 0$) and $\overline{\overline{V}} \eta\varepsilon$ (by setting $q = 1$).*

Proof. Since $id * S = \eta\varepsilon$, we have that $\overline{id} = \eta\varepsilon$. It follows then that

$$(14) \quad \tilde{V}^q|_{q=0} = \overline{\overline{\eta\varepsilon}} = \overline{\overline{V}} = V,$$

$$(15) \quad \tilde{V}^q|_{q=1} = \overline{\overline{id}} = \overline{\overline{\eta\varepsilon}}.$$

□

3.2. A q -twisting of NCA operators. Define an operator A_α that sends the ribbon Schur function \mathbf{s}_β to the ribbon Schur function $\mathbf{s}_{\beta\alpha}$ (the concatenate operation) and then extend this operator linearly to the space NCA . Similarly define an operator B_α that sends the ribbon Schur function \mathbf{s}_β to the ribbon Schur function $\mathbf{s}_{\beta\alpha}$ (the attach operation) and extend the function linearly.

Define \widetilde{A}_α^q as the q -analog of the operator A_α using equation (13). From Proposition 5, \widetilde{A}_α^q has the property that when $q = 0$ it is A_α . When $q = 1$ we see that

$$(16) \quad \widetilde{A}_\alpha^q(\mathbf{s}_\beta)|_{q=1} = \overline{A_\alpha \eta \varepsilon}(\mathbf{s}_\beta) = id * (\overline{A_\alpha \eta \varepsilon S})(\mathbf{s}_\beta) = \mathbf{s}_\beta \overline{A_\alpha}(1).$$

It is easily shown that $\overline{A_\alpha}(1) = \mathbf{s}_\alpha$, hence we see that $\widetilde{A}_\alpha^q(\mathbf{s}_\beta)|_{q=1} = \mathbf{s}_\beta \mathbf{s}_\alpha = A_\alpha(\mathbf{s}_\beta) + B_\alpha(\mathbf{s}_\beta)$. In fact we may derive that there is a simple formula for the action of \widetilde{A}_α^q .

Proposition 6. *Let β be a composition of $n > 0$, then*

$$\widetilde{A}_\alpha^q(\mathbf{s}_\beta) = A_\alpha(\mathbf{s}_\beta) + q^n B_\alpha(\mathbf{s}_\beta).$$

Also $\widetilde{A}_\alpha^q(1) = \mathbf{s}_\alpha$.

Proof. Define the numbers $C_{\beta\gamma}^\alpha$ as the coefficients that arise in the coproduct $\Delta(\mathbf{s}_\alpha) = \sum_{\beta, \gamma} C_{\beta\gamma}^\alpha \mathbf{s}_\beta \otimes \mathbf{s}_\gamma$. From the defining property of the antipode we have the relation

$$(17) \quad \sum_{\beta\gamma} (-1)^{|\beta|} C_{\beta\gamma}^\alpha \mathbf{s}_{\beta'} \otimes \mathbf{s}_\gamma = \sum_{\beta\gamma} (-1)^{|\gamma|} C_{\beta\gamma}^\alpha \mathbf{s}_\beta \otimes \mathbf{s}_{\gamma'} = 0$$

as long as α is not the empty composition.

The notation for the q analog of any operator is given in equation (13). To show that the q -analog of A_α satisfies the proposition we must give a definition in terms of Hopf algebra operations and then demonstrate that the action reduces dramatically. For any operator, (13) may be restated as

$$(18) \quad \widetilde{V}^q = \mu(1 \otimes V)(\mu \otimes S)(1 \otimes \Delta)(1 \otimes \mu)(1 \otimes S \otimes R^q)(1 \otimes \Delta)\Delta.$$

At this point it is a direct computation within the Hopf algebra of the non-commutative symmetric functions using relation (17) and the definition of A_α to arrive at the formula stated in the proposition. Since the computation is detailed and not necessary for the remainder of this exposition, we leave it to the reader as an exercise. \square

3.3. An example of q -non-commutative symmetric functions. Define for a pair of compositions $\alpha, \beta \models n$ the statistic $c(\alpha, \beta) = \sum_{i \in D(\alpha) \cap D(\beta)} i$.

Proposition 7. *Let $\mathbf{H}_\alpha^q = \sum_{\beta} q^{c(\alpha, \beta^c)} \mathbf{s}_\beta$ where the sum is over all compositions β of $|\alpha|$ such that $\alpha \leq \beta$ then*

$$\mathbf{H}_\alpha^q = \widetilde{A}_{\alpha_{\ell(\alpha)}}^q \widetilde{A}_{\alpha_{\ell(\alpha)-1}}^q \cdots \widetilde{A}_{\alpha_1}^q 1.$$

Notice that when $q = 0$, then $\mathbf{H}_\alpha^0 = \mathbf{s}_\alpha$. We also have that when $q = 1$, then $\mathbf{H}_\alpha^1 = \mathbf{h}_\alpha$. We will think of this family as a non-commutative analog of the Hall-Littlewood symmetric functions because of the following two properties. The first says that this family shares a sort of self-duality property similar to the \mathbf{s} , \mathbf{h} and \mathbf{e} bases of NCA , the second says that the commutative versions of these symmetric functions agree with the Hall-Littlewood symmetric functions when indexed by a composition that is equivalent to a partition.

Proposition 8.

$$(19) \quad \langle \mathbf{H}_\alpha^q, \mathbf{H}_\beta^q \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha\beta^c}.$$

Proof. Relation (19) will follow from Proposition 17 by setting $t = q$ and $q = 0$. \square

We also have the following remarkable connection with the Hall-Littlewood basis, H_λ^q .

Proposition 9. *If $\alpha = (1^a, b)$, then*

$$\chi(\mathbf{H}_\alpha^q) = H_{(b, 1^a)}^q.$$

Proof. Due to the rule given in ([10], (5.7), p. 228) we have the following important recurrence for the Hall-Littlewood symmetric functions indexed by a hook.

$$(20) \quad h_b H_{(1^a)}^q = H_{(b, 1^a)}^q + (1 - q^a) H_{(b+1, 1^{a-1})}^q$$

Also consider the recurrence that we have developed for these non-commutative symmetric functions.

$$(21) \quad \begin{aligned} \mathbf{H}_{(1^a)}^q \mathbf{h}_b &= (A_{(b)} + q^a B_{(b)} + (1 - q^a) B_{(b)}) (\mathbf{H}_{(1^a)}^q) \\ &= \mathbf{H}_{(1^a, b)}^q + (1 - q^a) B_{(b)} (\mathbf{H}_{(1^a)}^q) \end{aligned}$$

We also have that $\mathbf{H}_{(1^a)}^q = (A_{(1)} + q^{a-1} B_{(1)}) (\mathbf{H}_{(1^{a-1})}^q)$. Since we have that $B_{(r)} B_{(s)} = B_{(r+s)}$ and $B_{(r)} A_{(s)} = A_{(r+s)}$, then we see $B_{(b)} (\mathbf{H}_{(1^a)}^q) = \mathbf{H}_{(1^{a-1}, b+1)}^q$.

This implies

$$(22) \quad \mathbf{H}_{(1^a)}^q \mathbf{h}_b = \mathbf{H}_{(1^a, b)}^q + (1 - q^a) \mathbf{H}_{(1^{a-1}, b+1)}^q.$$

By induction on the length of the hook we see that $\chi(\mathbf{H}_\alpha^q) = H_{(b, 1^a)}^q$ (which is obviously true when either $a = 0$ or $b = 1$). \square

Using Proposition 8 we are able to derive an equation for the basis in $Qsym$ dual to the \mathbf{H}_α^q . We notice that $\delta_{\alpha\beta} = (-1)^{\ell(\alpha)+1} \langle \mathbf{H}_{\alpha^c}^q, \mathbf{H}_\beta^q \rangle$. This implies that if

$$(23) \quad [P_\alpha^q, \mathbf{H}_\beta^q] = \delta_{\alpha\beta},$$

then $P_\alpha^q = \sum_{\beta=|\alpha|} \langle (-1)^{\ell(\alpha)+1} \mathbf{H}_{\alpha^c}^q, \mathbf{s}_\beta \rangle F_\beta$. A simple calculation using Proposition 3 implies that

Corollary 10.

$$(24) \quad P_\alpha^q = \sum_{\beta \leq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} q^{c(\alpha^c, \beta)} F_\beta$$

is the basis of $Qsym$ which is dual to the family \mathbf{H}_α^q with respect to the $Qsym/NCA$ pairing.

We remark that these non-commutative symmetric functions are not equivalent to those defined in [7]. They are however remarkably similar and do agree for $\alpha = (1^a, b)$. We show this in the following proposition.

Say that $D(\alpha) = \{a_1 < a_2 < \dots < a_{\ell(\alpha)-1}\}$ and $D(\beta) = \{b_1 < b_2 < \dots < b_{\ell(\beta)-1}\}$. Let $Bre(\alpha, \beta)$ be the composition of $\ell(\alpha)$ with the descent set equal to $D(Bre(\alpha, \beta)) = \{\#\{a_j : a_j \leq b_i\} : 1 \leq i \leq \ell(\beta) - 1\}$. Let

$$\mathbf{W}_\alpha^q = \sum_{\beta \geq \alpha} q^{n(Bre(\alpha, \beta)^c)} \mathbf{s}_\beta.$$

This is the definition of the non-commutative analogs of the Hall-Littlewood symmetric functions given in Theorem 6.13 of [7]. We may easily see that the family \mathbf{W}_α^q satisfies the following recurrence

Proposition 11.

$$(25) \quad \mathbf{W}_{\alpha \cdot (m)}^q = A_{(m)}(\mathbf{W}_{\alpha}^q) + q^{\ell(\alpha)} B_{(m)}(\mathbf{W}_{\alpha}^q)$$

Proof. $D(\text{Bre}(\alpha \cdot (m), \beta \cdot (m))) = D(\text{Bre}(\alpha, \beta)) \cup \{|\alpha|\}$. At the same time we have $D(\text{Bre}(\alpha \cdot (m), \beta|(m))) = D(\text{Bre}(\alpha, \beta))$. Both $\text{Bre}(\alpha \cdot (m), \beta \cdot (m))$ and $\text{Bre}(\alpha \cdot (m), \beta|(m))$ are compositions of $\ell(\alpha) + 1$, hence the proposition follows. \square

This proposition should be compared to Proposition 6. This also shows the following corollary.

Corollary 12. *For $\alpha = (1^a, b)$ we have*

$$\mathbf{W}_{\alpha}^q = \mathbf{H}_{\alpha}^q.$$

One open question that arises from this definition is: Is there a Hecke algebra action on $Qsym$ such that the functions P_{α}^q are invariants under its action? This is the case of the functions of Hivert that are dual to the non-commutative symmetric functions \mathbf{W}_{α}^q and are given by the formula:

$$(26) \quad G_{\alpha}^q = \sum_{\beta \leq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} q^{n(\overline{\text{Bre}(\beta, \alpha)})} F_{\beta}.$$

The functions \mathbf{H}_{α}^q have a factorization property that is very similar to that held by the functions of Hivert \mathbf{W}_{α}^q and the commutative Hall-Littlewood symmetric functions [7].

Theorem 13. *Let ζ be an r^{th} root of unity. Then*

$$\mathbf{H}_{\alpha}^{\zeta} = \mathbf{H}_{\alpha^{(1)}}^{\zeta} \mathbf{H}_{\alpha^{(2)}}^{\zeta} \cdots \mathbf{H}_{\alpha^{(k)}}^{\zeta}$$

for any decomposition $\alpha = \alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(k)}$, where for $1 \leq i \leq k - 1$, $\alpha^{(i)}$ is a composition of a multiple of r .

This theorem follows from the following derivation of the product rule for these functions and then evaluating q at a root of unity.

Proposition 14.

$$(27) \quad \mathbf{H}_{\alpha}^q \mathbf{H}_{\beta}^q = \sum_{\gamma \geq \beta} f_{\alpha\beta}^{\gamma}(q) (\mathbf{H}_{\alpha \cdot \gamma}^q + (1 - q^{|\alpha|}) \mathbf{H}_{\alpha|\gamma}^q),$$

where

$$(28) \quad f_{\alpha\beta}^{\gamma}(q) = q^{c(\beta, \gamma^c)} (1 - q^{|\alpha|})^{\ell(\beta) - \ell(\gamma)}.$$

Before proceeding with the proof we introduce the following lemma.

Lemma 15.

$$(29) \quad \mathbf{H}_{\alpha}^q \mathbf{s}_{\beta} = \sum_{\gamma \geq \beta} g_{\alpha\beta}^{\gamma}(q) (\mathbf{H}_{\alpha \cdot \gamma}^q + (1 - q^{|\alpha|}) \mathbf{H}_{\alpha|\gamma}^q),$$

where

$$(30) \quad g_{\alpha\beta}^{\gamma}(q) = (-1)^{\ell(\beta) - \ell(\gamma)} q^{c(\alpha, \beta, (\alpha \cdot \gamma)^c)}.$$

Proof. Let $n = |\alpha| + |\beta|$. We will take the scalar product defined in equation (4) of $\mathbf{H}_\alpha^q \mathbf{s}_\beta$ and $\mathbf{H}_{\theta^c}^q$. This will give the coefficient of $(-1)^{\ell(\theta)+n} \mathbf{H}_\theta^q$ in the expression $\mathbf{H}_\alpha^q \mathbf{s}_\beta$. By expanding \mathbf{H}_α^q in terms of \mathbf{s}_γ and using that $\langle \mathbf{s}_\alpha, \mathbf{H}_\beta^q \rangle = (-1)^{|\alpha| + \ell(\alpha)} q^{c(\alpha, \beta)} \chi(\alpha \leq \beta^c)$, we see

$$\begin{aligned}
(-1)^{\ell(\theta)+n} \langle \mathbf{H}_\alpha^q \mathbf{s}_\beta, \mathbf{H}_{\theta^c}^q \rangle &= (-1)^{\ell(\theta)+n} \sum_{\gamma \geq \alpha} q^{c(\alpha, \gamma^c)} \langle \mathbf{s}_{\gamma \cdot \beta} + \mathbf{s}_{\gamma | \beta}, \mathbf{H}_{\theta^c}^q \rangle \\
(31) \qquad &= \sum_{\gamma \geq \alpha} q^{c(\alpha, \gamma^c)} (-1)^{\ell(\theta) + \ell(\gamma \cdot \beta)} q^{c(\gamma \cdot \beta, \theta^c)} \chi(\gamma \cdot \beta \leq \theta) \\
&\quad + \sum_{\gamma \geq \alpha} q^{c(\alpha, \gamma^c)} (-1)^{\ell(\theta) + \ell(\gamma | \beta)} q^{c(\gamma | \beta, \theta^c)} \chi(\gamma | \beta \leq \theta).
\end{aligned}$$

Break the composition θ into the composition consisting of the first $|\alpha|$ ‘cells’ of θ : $\mu = \theta|_{1 \dots |\alpha|}$, and the last $|\beta|$ ‘cells’: $\nu = \theta|_{|\alpha|+1 \dots n}$, so that either $\theta = \mu \cdot \nu$ or $\theta = \mu | \nu$.

If $\nu < \beta$ or $\mu < \alpha$ then clearly both sums will be 0.

If $\mu > \alpha$ then $D(\mu^c) \cap D(\alpha)$ is non empty and contains at least one element a . For each γ in the sums, either $D(\gamma)$ contains a or it doesn’t. The terms with γ such that $a \in D(\gamma)$ are of opposite sign but the same q coefficient as those γ such that $a \notin D(\gamma)$. Therefore the two sums will again be 0.

We need only consider the θ where $\mu = \alpha$ and $\nu \geq \beta$. If $\theta = \alpha \cdot \nu$, then the second sum is clearly 0 and first sum contains only 1 term, $(-1)^{\ell(\nu) + \ell(\beta)} q^{c(\alpha \cdot \beta, \theta^c)}$ (which agrees with the statement of the lemma). If $\theta = \alpha | \nu$, then both sums have exactly one non-zero term, and the scalar product is

$$\begin{aligned}
(32) \qquad &(-1)^{\ell(\nu) + \ell(\beta) - 1} q^{c(\alpha \cdot \beta, (\alpha | \nu)^c)} + (-1)^{\ell(\nu) + \ell(\beta)} q^{c(\alpha | \beta, (\alpha | \nu)^c)} \\
&= (-1)^{\ell(\nu) + \ell(\beta)} q^{c(\alpha | \beta, (\alpha | \nu)^c)} (1 - q^{|\alpha|}).
\end{aligned}$$

□

Proof of Proposition 14. Expanding \mathbf{H}_β^q in terms of \mathbf{s}_γ and using Lemma 15, yields

$$\begin{aligned}
(33) \qquad &(-1)^{\ell(\theta)+n} \langle \mathbf{H}_\alpha^q \mathbf{H}_\beta^q, \mathbf{H}_{\theta^c}^q \rangle \\
&= (-1)^{\ell(\theta)+n} \sum_{\gamma \geq \beta} \sum_{\mu \geq \gamma} q^{c(\beta, \gamma^c)} \langle g_{\alpha\gamma}^\mu(q) (\mathbf{H}_{\alpha \cdot \mu}^q + (1 - q^{|\alpha|}) \mathbf{H}_{\alpha | \mu}^q), \mathbf{H}_{\theta^c}^q \rangle \\
&= \sum_{\gamma \geq \beta} \sum_{\mu \geq \gamma} q^{c(\beta, \gamma^c)} g_{\alpha\gamma}^\mu(q) \delta_{\alpha \cdot \mu, \theta} + q^{c(\beta, \gamma^c)} g_{\alpha\gamma}^\mu(q) (1 - q^{|\alpha|}) \delta_{\alpha | \mu, \theta}.
\end{aligned}$$

Now if $|\alpha|$ is in $D(\theta)$ then the inner product is

$$\begin{aligned}
(34) \qquad &= \sum_{\mu \geq \gamma \geq \beta} q^{c(\beta, \gamma^c)} (-1)^{\ell(\gamma) - \ell(\mu)} q^{c((\alpha \cdot \gamma), (\alpha \cdot \mu)^c)} \\
&= \sum_{\mu \geq \gamma \geq \beta} q^{c(\beta, \gamma^c) + c(\gamma, \mu^c)} (-q^{|\alpha|})^{\ell(\gamma) - \ell(\mu)} \\
&= q^{c(\beta, \mu^c)} (1 - q^{|\alpha|})^{\ell(\beta) - \ell(\mu)}.
\end{aligned}$$

This agrees with the formula given for $f_{\alpha\beta}^\mu(q)$. If $|\alpha|$ is not in $D(\theta)$, the result is $(1 - q^{|\alpha|})$ times this result. □

4. q, t -ANALOGS OF NON-COMMUTATIVE SYMMETRIC FUNCTIONS

Define the following q, t -non commutative symmetric function.

$$(35) \quad \mathbf{H}_\alpha^{qt} = \sum_{\beta \models |\alpha|} t^{c(\alpha, \beta^c)} q^{c(\alpha', \overleftarrow{\beta})} \mathbf{s}_\beta$$

Clearly from this definition, if $q = 0$ and $t = q$, then all terms such that $D(\alpha') \cap D(\overleftarrow{\beta})$ is non empty vanish and we have $\mathbf{H}_\alpha^{0q} = \mathbf{H}_\alpha^q$. Therefore we also have the specializations, $\mathbf{H}_\alpha^{00} = \mathbf{s}_\alpha$ and $\mathbf{H}_\alpha^{01} = \mathbf{h}_\alpha$, and $\mathbf{H}_\alpha^{10} = \mathbf{e}_{\alpha^c}$.

Moreover, \mathbf{H}_α^{qt} satisfies the following relations which are similar to those held by the Macdonald symmetric functions in the commutative case:

$$(36) \quad \mathbf{H}_\alpha^{tq} = \omega' \mathbf{H}_\alpha^{qt}.$$

$$(37) \quad q^{n(\alpha')} t^{n(\alpha)} \mathbf{H}_\alpha^{\frac{1}{q} \frac{1}{t}} = \omega^c \mathbf{H}_\alpha^{qt}.$$

When we set $q = 1$, the \mathbf{H}_α^{1t} become products of some non-standard non-commutative symmetric functions, as seen in the following proposition.

Proposition 16. *Define the non-commutative symmetric functions*

$\mathbf{H}_{(m)}^{q(i)} = \sum_{\beta \models m} q^{(\ell(\beta)-1)i+n(\overleftarrow{\beta})} \mathbf{s}_\beta$. *For a composition α such that $k = \ell(\alpha)$, we have*

$$(38) \quad \mathbf{H}_\alpha^{q1} = \mathbf{H}_{(\alpha_1)}^{q(\sum_{i>1} \alpha_i)} \mathbf{H}_{(\alpha_2)}^{q(\sum_{i>2} \alpha_i)} \dots \mathbf{H}_{(\alpha_k)}^{q(0)}.$$

Proof. Fix α and for $1 \leq i \leq \ell(\alpha)$ let $\gamma^{(i)}$ be a composition of α_i . The coefficient of $\mathbf{s}_{\gamma^{(1)}} \mathbf{s}_{\gamma^{(2)}} \dots \mathbf{s}_{\gamma^{(\ell(\alpha))}}$ in the right hand side of equation (38) is q raised to the power of

$$(39) \quad \sum_i n(\overleftarrow{\gamma^{(i)}}) + \sum_i (\ell(\gamma^{(i)}) - 1) \sum_{j>i} \alpha_j = \sum_i \left(\sum_{d \in D(\overleftarrow{\gamma^{(i)}})} i + \sum_{j>i} \alpha_j \right).$$

This agrees with $c(\alpha', \overleftarrow{\beta})$ where β is attach and concatenate of the $\gamma^{(i)}$ and hence agrees with the q coefficient on the left hand side of equation (38). \square

We also have the following two additional Propositions that lead us to believe that it is an interesting generalization of the family \mathbf{H}_α^q .

Proposition 17. *Let $\alpha \models n$, then*

$$(40) \quad \langle \mathbf{H}_\alpha^{qt}, \mathbf{H}_\beta^{qt} \rangle = (-1)^{|\alpha|+\ell(\alpha)} \delta_{\alpha\beta^c} \prod_{i=1}^{n-1} (1 - q^i t^{n-i}).$$

Proof.

$$(41) \quad \begin{aligned} \langle \mathbf{H}_\alpha^{qt}, \mathbf{H}_\beta^{qt} \rangle &= \sum_\gamma \sum_\theta t^{c(\alpha, \gamma^c) + c(\beta, \theta^c)} q^{c(\alpha', \overleftarrow{\gamma}) + c(\beta', \overleftarrow{\theta})} \langle \mathbf{s}_\gamma, \mathbf{s}_\theta \rangle \\ &= \sum_\gamma (-1)^{n+\ell(\gamma)} t^{c(\alpha, \gamma^c) + c(\beta, \gamma)} q^{c(\alpha', \overleftarrow{\gamma}) + c(\beta', \gamma')}. \end{aligned}$$

If $\alpha \neq \beta^c$, then $(D(\alpha) \cap D(\beta)) \cup (D(\alpha^c) \cap D(\beta^c))$ is non empty. Take the smallest element i of this set (although any will do) and consider the involution ϕ on the set of compositions such that the compositions that contain i in the descent set are sent to the compositions that do not contain the element i (in the most natural manner). For each $\gamma \models n$, the terms corresponding to γ and $\phi(\gamma)$ have the same weight but opposite sign, hence the sum is 0.

If $\alpha = \beta^c$, then the sum reduces to

$$(42) \quad \begin{aligned} &= \sum_{\gamma} (-1)^{n+\ell(\gamma)} t^{c(\alpha, \gamma^c) + c(\alpha^c, \gamma)} q^{c(\alpha', \bar{\gamma}) + c(\bar{\alpha}, \gamma')} \\ &= \sum_{S \subseteq \{1, \dots, n\}} (-1)^{n+|S \cap D(\alpha^c)| + |S^c \cap D(\alpha)| + 1} t^{\sum_{i \in S} i} q^{\sum_{i \in S} n+1-i}, \end{aligned}$$

where the subsets S represent the sets $(D(\alpha) \cap D(\gamma^c)) \cup (D(\alpha^c) \cap D(\gamma))$. This is clearly equal to the product stated in the proposition. \square

Corollary 18. *The family*

$$(43) \quad P_{\alpha}^{qt} = \prod_{i=1}^{n-1} \frac{1}{1 - q^i t^{n-i}} \sum_{\beta} (-1)^{\ell(\beta) - \ell(\alpha)} t^{c(\alpha^c, \beta)} q^{c(\bar{\alpha}, \beta')} F_{\beta},$$

has the property that $[P_{\alpha}^{qt}, \mathbf{H}_{\beta}^{qt}] = \delta_{\alpha\beta}$.

Proof. If we wish that $[P_{\alpha}^{qt}, \mathbf{H}_{\beta}^{qt}] = \delta_{\alpha\beta}$, then using equation (11),

$$P_{\alpha}^{qt} = \sum_{\beta \models |\alpha|} \langle \mathbf{A}, \mathbf{s}_{\beta} \rangle F_{\beta},$$

where $\mathbf{A} = (-1)^{\ell(\alpha)+1} \prod_{i=1}^{n-1} \frac{1}{1 - q^i t^{n-i}} \mathbf{H}_{\alpha^c}^{qt}$, since

$$(44) \quad \delta_{\alpha\beta} = (-1)^{\ell(\alpha)+1} \prod_{i=1}^{n-1} \frac{1}{1 - q^i t^{n-i}} \langle \mathbf{H}_{\alpha^c}^{qt}, \mathbf{H}_{\beta}^{qt} \rangle.$$

A simple calculation yields the equation stated in the corollary. \square

There is a characterization of the non-commutative q, t analogs \mathbf{H}_{α}^{qt} in terms of properties that are similar to those shared by the Macdonald symmetric functions. This characterization is not particularly important for our treatment, but it should not be ignored because of the similarities that it shares with the commutative case.

Define a family of non-commutative symmetric functions \mathbf{P}_{α}^{qt} by the following three conditions.

1. $\mathbf{P}_{\alpha}^{qt} = \mathbf{H}_{\alpha}^t + \sum_{\beta < \alpha} c_{\alpha\beta}(q, t) \mathbf{H}_{\beta}^t$ for some coefficients $c_{\alpha\beta}(q, t)$ that are rational functions in the parameters q and t .
2. $\omega' \mathbf{P}_{\alpha}^{qt} = a_{\alpha}(q, t) \mathbf{P}_{\alpha'}^{tq}$ for some coefficients $a_{\alpha}(q, t)$.
3. $\langle \mathbf{P}_{\alpha}^{qt}, \mathbf{P}_{\beta}^{qt} \rangle = 0$ if $\alpha \neq \beta^c$.

Theorem 19. *The family \mathbf{P}_{α}^{qt} are defined by the three conditions listed above and, moreover, $\mathbf{P}_{\alpha}^{qt} = r_{\alpha} \mathbf{H}_{\alpha}^{qt}$ where $r_{\alpha} = 1 / \prod_{i \in D(\alpha^c)} (1 - q^{n-i} t^i)$. The coefficients $c_{\alpha\beta}(q, t)$ are given by the formula*

$$c_{\alpha\beta}(q, t) = \prod_{i \in D(\alpha^c) \cap D(\beta)} q^{n-i} / (1 - t^i q^{n-i}).$$

The coefficients $a_{\alpha}(q, t)$ mentioned in the second condition are given by the formula $a_{\alpha}(q, t) = \prod_{i \in D(\alpha)} (1 - q^{n-i} t^i) / \prod_{i \in D(\alpha^c)} (1 - q^{n-i} t^i)$.

Proof. The proof proceeds by induction, for there is a method of calculating the coefficients $c_{\alpha\beta}(q, t)$ from ones preceding it in some order. Say that $\mathbf{P}_\alpha^{qt} = \sum_{\beta < \alpha} c_{\alpha\beta}(q, t) \mathbf{H}_\beta^t$ where we assume that $c_{\alpha\alpha}(q, t) = 1$ and that this family satisfies the three conditions given above.

Assume that the coefficients $c_{\gamma\delta}(q, t)$ are known and given by the formula stated in the theorem for all γ such that $|D(\gamma)| > |D(\alpha)|$ or for $\gamma = \alpha$ and $\delta > \beta$. To determine $c_{\alpha\beta}(q, t)$ we take the scalar product of \mathbf{P}_α^{qt} and $\omega' \mathbf{P}_\beta^{tq}$ since $\beta < \alpha$, $|D(\overline{\beta})| > |D(\alpha)|$ and all coefficients in \mathbf{P}_β^{qt} have been calculated already. Since $\omega' \mathbf{P}_\beta^{tq} = \mathbf{P}_{\beta^c}^{qt} = \sum_{\theta \leq \beta^c} c_{\beta^c\theta}(q, t) \mathbf{H}_\theta^t$, hence we have the expression

$$\begin{aligned} \langle \mathbf{P}_\alpha^{qt}, \omega' \mathbf{P}_\beta^{tq} \rangle &= \langle \mathbf{P}_\alpha^{qt}, a_{\overline{\beta}}(t, q) \mathbf{P}_{\beta^c}^{qt} \rangle = 0 \\ &= a_{\overline{\beta}}(t, q) \sum_{\alpha \geq \theta \geq \beta} c_{\alpha\theta}(q, t) c_{\beta^c\theta^c}(q, t) (-1)^{n+\ell(\theta)} \\ &= a_{\overline{\beta}}(t, q) c_{\alpha\beta}(q, t) (-1)^{n+\ell(\beta)} + \\ &\quad a_{\overline{\beta}}(t, q) \sum_{\alpha \geq \theta > \beta} c_{\alpha\theta}(q, t) c_{\beta^c\theta^c}(q, t) (-1)^{n+\ell(\theta)}. \end{aligned}$$

Those values of $c_{\beta^c\theta^c}(q, t)$ may be calculated from what we have already determined since

$$c_{\beta^c\theta^c}(q, t) = \langle \mathbf{H}_\theta^t, \mathbf{P}_{\beta^c}^{qt} \rangle (-1)^{n+\ell(\theta)} = \langle \mathbf{H}_\theta^t, \omega' \mathbf{P}_\beta^{tq} \rangle / a_{\overline{\beta}}(t, q) (-1)^{n+\ell(\theta)}.$$

Hence we see that

$$c_{\alpha\beta}(q, t) = \frac{(-1)^{n+1+\ell(\beta)}}{a_{\overline{\beta}}(t, q)} \sum_{\alpha \geq \theta > \beta} c_{\alpha\theta}(q, t) \langle \mathbf{H}_\theta^t, \omega' \mathbf{P}_\beta^{tq} \rangle.$$

In addition we may calculate $a_{\overline{\beta}}(t, q)$ since

$$\langle \mathbf{H}_\beta^t, \omega \mathbf{P}_\beta^{tq} \rangle = \langle \mathbf{H}_\beta^t, a_{\overline{\beta}}(t, q) \mathbf{P}_{\beta^c}^{qt} \rangle = a_{\overline{\beta}}(t, q) (-1)^{n+\ell(\beta)}.$$

Although we may use these formulas to calculate the coefficients, the only conclusion that we are going to draw from them is that the coefficients $c_{\alpha\beta}(q, t)$ are determined by assuming that the defining conditions are true, hence the family \mathbf{P}_α^{qt} which satisfies these conditions is unique.

It remains to show then that $\mathbf{P}_\alpha^{qt} = \mathbf{H}_\alpha^{qt} / \prod_{i \in D(\alpha^c)} (1 - q^{n-i} t^i)$ satisfies the conditions listed above. Clearly they satisfy conditions 2 and 3. It remains to show that the \mathbf{H}_α^{qt} have the correct expansion in terms of \mathbf{H}_α^t .

$$\begin{aligned}
 \langle \mathbf{H}_{\alpha^c}^t, \mathbf{H}_{\alpha}^{qt} \rangle &= \sum_{\beta \geq \alpha^c} t^{c(\alpha^c, \beta^c)} \langle \mathbf{s}_{\beta}, \mathbf{H}_{\alpha}^{qt} \rangle \\
 &= \sum_{\beta \geq \alpha^c} (-1)^{n+\ell(\beta)} t^{c(\alpha, \beta) + c(\alpha^c, \beta^c)} q^{c(\alpha', \beta')} \\
 &= \sum_{\beta \geq \alpha^c} (-1)^{n+\ell(\beta)} t^{c(\alpha^c, \beta^c)} q^{c(\alpha', \beta')} \\
 &= \sum_{S \subseteq D(\alpha^c)} (-1)^{n+1+|S|} t^{\sum_{i \in S} i} q^{\sum_{i \in S} n-i} \\
 &= (-1)^{n+\ell(\alpha^c)} \prod_{i \in D(\alpha^c)} (1 - q^{n-i} t^i)
 \end{aligned}$$

We also see for β is not strictly smaller than α then $D(\alpha) \cap D(\beta^c)$ is non-empty and

$$\begin{aligned}
 \langle \mathbf{H}_{\beta^c}^t, \mathbf{H}_{\alpha}^{qt} \rangle &= \sum_{\gamma \geq \beta^c} t^{c(\beta^c, \gamma^c)} \langle \mathbf{s}_{\gamma}, \mathbf{H}_{\alpha}^{qt} \rangle \\
 &= \sum_{\gamma \geq \beta^c} t^{c(\beta^c, \gamma^c)} t^{c(\alpha, \gamma)} q^{c(\alpha', \gamma')} (-1)^{n+\ell(\gamma)}.
 \end{aligned}$$

Since there is some element a in $D(\alpha) \cap D(\beta^c)$, every composition $\gamma \geq \beta^c$ either has $a \in D(\gamma)$ or $a \in D(\gamma^c)$. There is an obvious involution between these two sets of compositions and they have the same weight but opposite sign, hence the sum is 0 in this case. \square

As in the case of the family with one parameter, when the functions are indexed by composition representing a partition (i.e. a hook), then they are a generalization of the Macdonald symmetric function.

Proposition 20. *If $\alpha = (1^a, b)$, then*

$$(45) \quad \chi(\mathbf{H}_{\alpha}^{qt}) = H_{(b, 1^a)}^{qt}.$$

Proof. Idea: same as in q case. Show that

$$(46) \quad \mathbf{H}_{(1^a)}^{qt} \mathbf{H}_{(b)}^{qt} = \frac{1-t^a}{1-q^b t^a} \mathbf{H}_{(1^{a-1}, b+1)}^{qt} + \frac{1-q^b}{1-q^b t^a} \mathbf{H}_{(1^a, b)}^{qt}$$

and by a formula ([10] eq. (6.24) p. 340) we have the same recurrence in the commutative case. That is,

$$(47) \quad H_{(1^a)}^{qt} H_{(b)}^{qt} = \frac{1-t^a}{1-q^b t^a} H_{(b+1, 1^{a-1})}^{qt} + \frac{1-q^b}{1-q^b t^a} H_{(b, 1^a)}^{qt}.$$

By induction this implies that the commutative versions agree on hooks.

Consider the product $\mathbf{H}_{(1^a)}^{qt} \mathbf{H}_{(b)}^{qt}$. This is equal to

$$\begin{aligned}
 (48) \quad \mathbf{H}_{(1^a)}^{qt} \mathbf{H}_{(b)}^{qt} &= \sum_{\gamma \models a} \sum_{\theta \models b} t^{n(\gamma^c)} q^{n(\bar{\theta})} \mathbf{s}_{\gamma} \mathbf{s}_{\theta} \\
 &= \sum_{\gamma \models a} \sum_{\theta \models b} t^{n(\gamma^c)} q^{n(\bar{\theta})} (A_{\theta}(\mathbf{s}_{\gamma}) + B_{\theta}(\mathbf{s}_{\gamma})).
 \end{aligned}$$

We also have since $(1^a, b)' = (1^{b-1}, a + 1)$,

$$(49) \quad \begin{aligned} \mathbf{H}_{(1^a, b)}^{qt} &= \sum_{\beta \models a+b} t^{c((1^a, b), \beta^c)} q^{c((1^{b-1}, a+1), \overleftarrow{\beta})} \mathbf{s}_\beta \\ &= \sum_{\gamma \models a} \sum_{\theta \models b} t^{n(\gamma^c)} q^{n(\overleftarrow{\theta})} (A_\theta(\mathbf{s}_\gamma) + t^a B_\theta(\mathbf{s}_\gamma)). \end{aligned}$$

While at the same time

$$(50) \quad \begin{aligned} \mathbf{H}_{(1^{a-1}, b+1)}^{qt} &= \sum_{\beta \models a+b} t^{c((1^{a-1}, b+1), \beta^c)} q^{c((1^b, a), \overleftarrow{\beta})} \mathbf{s}_\beta \\ &= \sum_{\gamma \models a} \sum_{\theta \models b} t^{n(\gamma^c)} q^{n(\overleftarrow{\theta})} (q^b A_\theta(\mathbf{s}_\gamma) + B_\theta(\mathbf{s}_\gamma)). \end{aligned}$$

From here is easily shown that

$$(51) \quad (1 - q^b t^a) \mathbf{H}_{(1^a)}^{qt} \mathbf{H}_{(b)}^{qt} = (1 - q^b) \mathbf{H}_{(1^a, b)}^{qt} + (1 - t^a) \mathbf{H}_{(1^{a-1}, b+1)}^{qt}.$$

□

These two properties are only an indication that \mathbf{H}_α^{qt} are an important generalization of the Macdonald symmetric functions. The first property does not occur in many families of non-commutative symmetric functions, the second, however, could appear for many different families (since the functions of Hivert also have the same property that they have the 'correct' statistic on hooks).

The most important indication that the family \mathbf{H}_α^{qt} is indeed an important analog to the Macdonald symmetric functions is the appearance of an operator 'nabla' that is analogous to the operator introduced in [2] for the symmetric functions.

First we define the analog $\tilde{\mathbf{H}}_\alpha^{qt} = t^{n(\alpha)} \mathbf{H}_\alpha^{q\frac{1}{t}} = \sum_{\beta \models |\alpha|} t^{c(\alpha, \beta)} q^{c(\alpha', \overleftarrow{\beta})} \mathbf{s}_\beta$. Next, define ∇ to be a linear operator with the property that $\nabla(\tilde{\mathbf{H}}_\alpha^{qt}) = t^{n(\alpha)} q^{n(\alpha')} \tilde{\mathbf{H}}_\alpha^{qt}$. For our scalar product, we have that

$$(52) \quad \langle \nabla(f), \nabla(g) \rangle = q^{\binom{n}{2}} t^{\binom{n}{2}} \langle f, g \rangle.$$

As we will see, this operator has many properties that are analogous to those seen in the commutative case. In the non-commutative case the situation is somewhat simpler and we are able to state precisely the action of the operator on the ribbon basis.

Proposition 21. *If $\alpha \models n$, then*

$$(53) \quad \nabla(\mathbf{s}_\alpha) = (-1)^{n+\ell(\alpha)} q^{n(\alpha')} t^{n(\alpha^c)} \sum_{\beta \leq \alpha^c} \prod_{i \in D(\alpha) \cap D(\beta)} (t^i + q^{n-i}) \mathbf{s}_\beta.$$

To prove this formula we will need several lemmas for the action of these operators on various bases. By choosing good notation for these operators, the proofs become almost transparent. We will order the ribbons by their descent sets using the total order described section 2.1.

For a tensor product of two matrices with $B = [b_{ij}]_{1 \leq i, j \leq n}$ we will use the convention that

$$(54) \quad A \otimes B = [b_{ij} A]_{1 \leq i, j \leq n}.$$

That is, the (r, s) entry in this matrix is

$$b_{(r \operatorname{div} n)+1, (s \operatorname{div} n)+1} a_{(r \operatorname{mod} n)+1, (s \operatorname{mod} n)+1}.$$

Lemma 22. *Let \mathbf{s} be a column vector of \mathbf{s}_α and $\tilde{\mathbf{H}}$ is a column vector of $\tilde{\mathbf{H}}_\alpha^{qt}$ (both using the total order of section 2.1), then*

$$(55) \quad \tilde{\mathbf{H}} = \begin{bmatrix} 1 & q^{n-1} \\ 1 & t \end{bmatrix} \otimes \begin{bmatrix} 1 & q^{n-2} \\ 1 & t^2 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & q \\ 1 & t^{n-1} \end{bmatrix} \mathbf{s}.$$

This lemma follows by realizing that if $\phi(\alpha) = k$, then the entries in the k^{th} row of the tensor product matrix agrees with the formula for the coefficients of \mathbf{s}_β in $\tilde{\mathbf{H}}_\alpha^{qt}$. By taking the inverse of this tensor product matrix we derive the inverse relation.

Corollary 23. *Let \mathbf{s} be a column vector of \mathbf{s}_α and $\tilde{\mathbf{H}}$ is a column vector of $\tilde{\mathbf{H}}_\alpha^{qt}$, then*

$$(56) \quad \mathbf{s} = \left(\prod_{i=1}^{n-1} \frac{1}{t^i - q^{n-i}} \right) \begin{bmatrix} t & -q^{n-1} \\ -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} t^2 & -q^{n-2} \\ -1 & 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} t^{n-1} & -q \\ -1 & 1 \end{bmatrix} \tilde{\mathbf{H}}.$$

Lemma 24. *Let $\tilde{\mathbf{H}}$ be a column vector of $\tilde{\mathbf{H}}_\alpha^{qt}$, then*

$$(57) \quad \nabla \tilde{\mathbf{H}} = \begin{bmatrix} q^{n-1} & 0 \\ 0 & t \end{bmatrix} \otimes \begin{bmatrix} q^{n-2} & 0 \\ 0 & t^2 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} q & 0 \\ 0 & t^{n-1} \end{bmatrix} \tilde{\mathbf{H}}.$$

The proof of this lemma again follows by calculating the entry in the row indexed by $\phi(\alpha)$.

Proof. (of Proposition 21) We calculate the action of ∇ on the column vector \mathbf{s} . This follows by first expressing \mathbf{s} in terms of $\tilde{\mathbf{H}}$ using equation (56), then using the action of ∇ on $\tilde{\mathbf{H}}$, then reexpressing the answer in terms of \mathbf{s} using (55). We calculate that

$$(58) \quad \frac{1}{t^i - q^{n-i}} \begin{bmatrix} t^i & -q^{n-i} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q^{n-i} & 0 \\ 0 & t^i \end{bmatrix} \begin{bmatrix} 1 & q^{n-i} \\ 1 & t^i \end{bmatrix} = \begin{bmatrix} 0 & -q^{n-i}t^i \\ 1 & (t^i + q^{n-i}) \end{bmatrix}.$$

Therefore we see that the action of ∇ on \mathbf{s} is given by the equation

$$(59) \quad \nabla(\mathbf{s}) = \begin{bmatrix} 0 & -q^{n-1}t \\ 1 & (t + q^{n-1}) \end{bmatrix} \otimes \begin{bmatrix} 0 & -q^{n-2}t^2 \\ 1 & (t^2 + q^{n-2}) \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & -qt^{n-1} \\ 1 & (t^{n-1} + q) \end{bmatrix} \mathbf{s}.$$

Now translate this tensor product directly to the action on the \mathbf{s}_α basis to arrive at the formula stated in equation (53). \square

Interesting connections arise with combinatorics and representation theory that are analogous to the commutative case. Recall that for the standard symmetric functions we have that $\langle \nabla(e_n), h_{1^n} \rangle$ is a q, t analog of the number $(n+1)^{n-1}$, which is the number of parking functions (a function $f : [n] \rightarrow [n]$ is a parking function if the sequence $(f(1), f(2), \dots, f(n))$ when sorted in increasing order (a_1, a_2, \dots, a_n) satisfies $a_i \leq i$). We also know that $\nabla(e_n)|_{t=1}$ is a graded Frobenius series for the parking function module [6]. Moreover, the top component $\nabla(e_n)|_{e_n}$ is known to be a q, t -analog of the number of increasing parking functions which is given by the Catalan numbers [4].

In the non-commutative case, these statements occur with exact analogy. We will see that analogs of the parking functions are the preferential arrangements ([13] p. 146). An exponential generating function for the number of preferential arrangements is given by $(2 - e^x)^{-1}$ and the number of increasing preferential arrangements is 2^{n-1} .

Proposition 25. *The quantity $\langle \chi(\nabla(\mathbf{e}_n)), h_{1^n} \rangle$ is a q, t analog for the number of preferential arrangements (the maps $f : [n] \rightarrow [k]$ where $1 \leq k \leq n$ which are onto for some k). Moreover, the quantity $\langle \chi(\nabla(\mathbf{e}_n)), e_n \rangle = \prod_{i=1}^{n-1} (q^{n-i} + t^i)$ is a q, t analog of 2^{n-1} .*

This proposition is a consequence of the statement that appears in full generality just below. For the moment we will provide the following example:

Example 26. At $n = 4$, there are $125 = (4 + 1)^{4-1}$ parking functions, and $14 = C_4$ weakly increasing parking functions represented by the following list. The first number is the number of parking functions such that $(f(1), f(2), f(3), f(4))$ when sorted is the adjacent sequence. The sum of these numbers is 125.

$$1 \times 1111 \quad 4 \times 1112 \quad 4 \times 1113 \quad 12 \times 1223 \quad 12 \times 1134 \quad 12 \times 1123 \quad 12 \times 1124$$

$$4 \times 1222 \quad 4 \times 1114 \quad 6 \times 1133 \quad 6 \times 1122 \quad 12 \times 1224 \quad 12 \times 1233 \quad 24 \times 1234$$

$75 = (2 - e^x)^{-1}|_{x^4} 4!$ of the parking functions do not ‘skip’ an integer, these are the preferential arrangements. Exactly $8 = 2^3$ of the preferential arrangements are weakly increasing, those given by the following list:

$$1111 \quad 1112 \quad 1122 \quad 1123 \quad 1222 \quad 1223 \quad 1233 \quad 1234$$

We remark that every preferential arrangement is also a parking function. This is a natural subset of the parking functions which we will denote by $Pref_n$. Just as in the case of the parking functions, there is a natural S_n action on this set formed by permuting the values of the function (that is $(\sigma f)(i) = f(\sigma_i)$) and hence $Pref_n$ forms an S_n module by defining a vector space with $Pref_n$ as the basis.

Let f be a preferential arrangement and define the content of a preferential arrangement to be the composition $\alpha^{(f)}$ such that the i^{th} component is $|f^{-1}(i)|$. We remark that two preferential arrangements are in the same S_n -orbit if $\alpha^{(f)} = \alpha^{(g)}$. The Frobenius series of the S_n module generated by the preferential arrangements of content α is given by the homogeneous symmetric function h_α .

It follows that the preferential arrangement module may be graded by a statistic on the content of the preferential arrangements. If we choose our grading to be $q^{n(\alpha')}$ (this agrees with the ‘area’ statistic on Dyck paths), then clearly the Frobenius series for the module of preferential arrangements is given by

$$(60) \quad \mathcal{F}_{Pref_n}(q) = \sum_{\alpha \models n} q^{n(\alpha')} h_\alpha.$$

In the commutative case it is known that

$$(61) \quad \nabla(e_n)|_{t=1} = \sum_{\mu \subseteq \delta_n} q^{\binom{n}{2} - |\mu|} e_{\lambda(\mu)},$$

where $\delta_n = (n - 1, n - 2, \dots, 1, 0)$ and $\lambda(\mu)$ is the sequence $(m_1(\mu), \dots, m_{n-1}(\mu), n - \sum_{i=1}^{n-1} m_i(\mu))$ and $m_i(\mu)$ is the number of parts of size i in the partition μ . This is related to the Frobenius series for the module of parking functions by an application of the involution ω . In exact analogy with the commutative case we have the following proposition.

Proposition 27. Set $t = 1$ in the equation for the action of ∇ on e_n , then

$$(62) \quad \nabla(e_n)|_{t=1} = \sum_{\alpha \models n} q^{n(\alpha')} e_\alpha.$$

Proof. With $t = 1$ the action of ∇ on $\mathbf{s}_{(1^n)}$ is given from equation (53)

$$\begin{aligned}
(63) \quad \nabla(\mathbf{e}_n) &= \sum_{\beta} \prod_{i \in D(\overline{\beta})} (1 + q^i) \mathbf{s}_{\beta} \\
&= \sum_{\beta} \sum_{\gamma \geq \overline{\beta}} q^{n(\gamma)} \mathbf{s}_{\beta} \\
&= \sum_{\gamma} \sum_{\overline{\beta} \leq \gamma} q^{n(\gamma)} \mathbf{s}_{\beta} \\
&= \sum_{\gamma} q^{n(\gamma)} \mathbf{e}_{\gamma'} = \sum_{\gamma} q^{n(\gamma')} \mathbf{e}_{\gamma}.
\end{aligned}$$

□

This may be used to show that the quantity $\nabla(e_n)|_{t=1} - \chi(\nabla(e_n))|_{t=1}$ is e -positive (the coefficients are polynomials in q with non-negative integer coefficients when the expression is expressed in the elementary basis).

We may use property (52) and (53) to calculate the inverse of this function as well.

Proposition 28.

$$(64) \quad \nabla^{-1}(\mathbf{s}_{\alpha}) = (-1)^{\ell(\alpha)+1} \sum_{\alpha^c \leq \beta} q^{-n(\beta')} t^{-n(\beta^c)} \prod_{i \in D(\beta^c) \cap D(\alpha^c)} (t^i + q^{n-i}) \mathbf{s}_{\beta}.$$

Proof.

$$\begin{aligned}
(65) \quad \nabla^{-1}(\mathbf{s}_{\alpha}) &= \sum_{\beta \models n} (-1)^{n+\ell(\beta)} \langle \nabla^{-1}(\mathbf{s}_{\alpha}), \mathbf{s}_{\beta^c} \rangle \mathbf{s}_{\beta} \\
&= \sum_{\beta \models n} (-1)^{n+\ell(\beta)} q^{-\binom{n}{2}} t^{-\binom{n}{2}} \langle \mathbf{s}_{\alpha}, \nabla(\mathbf{s}_{\beta^c}) \rangle \mathbf{s}_{\beta} \\
&= \sum_{\beta \models n} \sum_{\gamma \leq \beta} (-1)^{\ell(\beta)+\ell(\beta^c)} q^{n(\overline{\beta})-\binom{n}{2}} \\
&\quad t^{n(\beta)-\binom{n}{2}} \prod_{i \in D(\beta^c) \cap D(\gamma)} (t^i + q^{n-i}) \langle \mathbf{s}_{\alpha}, \mathbf{s}_{\gamma} \rangle \mathbf{s}_{\beta} \\
&= \sum_{\alpha^c \leq \beta} (-1)^{\ell(\alpha)+1} q^{-n(\beta')} t^{-n(\beta^c)} \prod_{i \in D(\beta^c) \cap D(\alpha^c)} (t^i + q^{n-i}) \mathbf{s}_{\beta}.
\end{aligned}$$

□

5. APPENDIX: TRANSITION MATRICES BETWEEN \mathbf{H}_{α}^q AND \mathbf{s}_{β}

$$\begin{aligned}
&\left[\begin{array}{c|ccc} 2 & 1 & 0 & \\ \hline 11 & q & 1 & \end{array} \right] \\
&\left[\begin{array}{c|cccc} 3 & 1 & 0 & 0 & 0 \\ \hline 12 & q & 1 & 0 & 0 \\ 21 & q^2 & 0 & 1 & 0 \\ \hline 111 & q^3 & q^2 & q & 1 \end{array} \right]
\end{aligned}$$

$$\left[\begin{array}{c|cccccccc} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 13 & q & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 22 & q^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 112 & q^3 & q^2 & q & 1 & 0 & 0 & 0 & 0 \\ 31 & q^3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 121 & q^4 & q^3 & 0 & 0 & q & 1 & 0 & 0 \\ 211 & q^5 & 0 & q^3 & 0 & q^2 & 0 & 1 & 0 \\ 1111 & q^6 & q^5 & q^4 & q^3 & q^3 & q^2 & q & 1 \end{array} \right]$$

6. APPENDIX: TRANSITION MATRICES BETWEEN $\tilde{\mathbf{H}}_\alpha^{qt}$ AND \mathbf{s}_β

$$\left[\begin{array}{c|cc} 2 & 1 & q \\ 11 & 1 & t \end{array} \right]$$

$$\left[\begin{array}{c|cccc} 3 & 1 & q^2 & q & q^3 \\ 12 & 1 & t & q & tq \\ 21 & 1 & q^2 & t^2 & t^2q^2 \\ 111 & 1 & t & t^2 & t^3 \end{array} \right]$$

$$\left[\begin{array}{c|cccccccc} 4 & 1 & q^3 & q^2 & q^5 & q & q^4 & q^3 & q^6 \\ 13 & 1 & t & q^2 & tq^2 & q & tq & q^3 & tq^3 \\ 22 & 1 & q^3 & t^2 & t^2q^3 & q & q^4 & t^2q & t^2q^4 \\ 112 & 1 & t & t^2 & t^3 & q & tq & t^2q & t^3q \\ 31 & 1 & q^3 & q^2 & q^5 & t^3 & t^3q^3 & t^3q^2 & t^3q^5 \\ 121 & 1 & t & q^2 & tq^2 & t^3 & t^4 & t^3q^2 & t^4q^2 \\ 211 & 1 & q^3 & t^2 & t^2q^3 & t^3 & t^3q^3 & t^5 & t^5q^3 \\ 1111 & 1 & t & t^2 & t^3 & t^3 & t^4 & t^5 & t^6 \end{array} \right]$$

7. APPENDIX: TRANSITION MATRICES BETWEEN $\nabla(\mathbf{s}_\alpha)$ AND \mathbf{s}_β

$$\left[\begin{array}{c|cc} 2 & 0 & -qt \\ 11 & 1 & t+q \end{array} \right]$$

$$\left[\begin{array}{c|cccc} 3 & 0 & 0 & 0 & q^3t^3 \\ 12 & 0 & 0 & -qt^2 & -(t+q^2)qt^2 \\ 21 & 0 & -q^2t & 0 & -(t^2+q)q^2t \\ 111 & 1 & t+q^2 & t^2+q & (t+q^2)(t^2+q) \end{array} \right]$$

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