THE COINVARIANT ALGEBRA OF THE SYMMETRIC GROUP AS A DIRECT SUM OF INDUCED MODULES

HIDEAKI MORITA AND TATSUHIRO NAKAJIMA

Abstract. Let R_n be the coinvariant algebra of the symmetric group S_n. The algebra has a natural gradation. For a fixed ℓ $(1 \leq \ell \leq n)$, let $R_n(k;\ell)$ $(0 \leq k \leq \ell - 1)$ be the direct sum of all the homogeneous components of R_n whose degrees are congruent to k modulo ℓ. In this article, we will show that for each ℓ there exists a subgroup H_ℓ of S_n and a representation $\Psi(k;\ell)$ of H_ℓ such that each $R_n(k;\ell)$ is induced by $\Psi(k;\ell)$.

Résumé. Soit R_n l'alègbre des coinvariants du groupe symétrique S_n. Cette algèbre a une graduation naturelle. Pour un entier ℓ $(1 \leq \ell \leq n)$ fixe, soit $R_n(k;\ell)$ $(0 \leq k \leq \ell - 1)$ la somme directe de toutes les composantes homogènes de R_n dont les degrés sont congrus à k modulo ℓ. Dans cet article, nous montrerons que pour chaque ℓ il existe un sous-groupe H_ℓ de S_n et une représentation $\Psi(k;\ell)$ de H_ℓ tel que $R_n(k;\ell)$ est induite par $\Psi(k;\ell)$.

1. Introduction

A *partition* of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of nonnegative integers with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We also denote the partition λ by $(1^{m_1} 2^{m_2} \cdots n^{m_n})$, where m_i is the multiplicity of i in λ for $1 \leq i \leq n$. If λ is a partition of n, we simply write $\lambda \vdash n$. The *Young diagram* of a partition λ is a set of points

$$Y_\lambda = \{(i,j) \in \mathbf{Z}^2 | 1 \leq j \leq \lambda_i\},$$

in which we regard the coordinates increase from left to right, and from top to bottom. Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. A *standard tableau* T of shape λ is a bijection $T : Y_\lambda \to [n]$ with the condition that the assigned numbers strictly increase along both the rows and the columns in Y_λ. We illustrate the Young diagram Y_λ and a standard tableau T for $\lambda = (3,2,2) \vdash 7$ in the following:

$$Y_\lambda = \begin{matrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{matrix} \quad , \qquad T = \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & 7 & \end{matrix} \quad .$$

We denote by $\mathrm{STab}(\lambda)$ the set of all the standard tableaux of shape λ.

For a standard tableau T of shape $\lambda \vdash n$, define the *descent set* $\mathrm{Des}(T)$ by

$$\mathrm{Des}(T) := \{i \in [n-1] \mid i+1 \text{ is located in a lower row than } i \text{ in } T\}.$$

We call the sum of the elements of $\mathrm{D}(T)$ the *major index* of T, and denote it by $\mathrm{maj}(T)$. In the preceding example, $\mathrm{Des}(T) = \{1,4,5\}$ and $\mathrm{maj}(T) = 1+4+5 = 10$.

Let S_n be the symmetric group of degree n, and

$$P_n = \mathbb{C}[x_1, x_2, \ldots, x_n]$$

denote the polynomial ring with n variables over $\mathbb{C}$. As customary, S_n acts on P_n from the left as permutations of variables by setting

$$(wf)(x_1, x_2, \ldots, x_n) = f(x_{w(1)}, x_{w(2)}, \ldots, x_{w(n)}) ,$$

where $w \in S_n$ and $f(x_1, x_2, \ldots, x_n) \in P_n$. Let $I_n = \oplus_{d \geq 0} I^d$ denote the graded S_n-stable ideal of P_n generated by the elementary symmetric functions. Hence the quotient algebra $R_n = P_n / I_n$ is also a graded S_n-module. We write its homogeneous decomposition as

$$R_n = \bigoplus_{d \geq 0} R_n^d,$$

and call R_n the *coinvariant algebra* of S_n It is well known that the coinvariant algebra R_n affords the left regular representation of S_n.

Let us consider, for each integer $k = 0, \ldots, n-1$, the direct sum $R_n(k;n)$ of homogeneous components of R whose degrees are congruent to k modulo n, i.e.,

$$R_n(k;n) = \bigoplus_{d \equiv k \mod n} R_n^d .$$

Since each homogeneous component R_n^d is S_n-invariant, these subspaces also afford representations of S_n, and the dimensions of these representations do not depend on k, i.e.,

$$\dim R_n(k;n) = (n-1)!$$

for all $k = 0, \ldots, n-1$.

In [KW], W. Kraskiewicz and J. Weymann consider these S_n-modules, and prove that each $R_n(k;n)$ is induced from a corresponding irreducible representation of a cyclic subgroup of S_n (see also [G, Proposition 8.2] [R, Theorem 8.9]). Precisely, let γ be the cyclic permutation $(12 \cdots n)$, and C_n the subgroup of S_n generated by γ. The cyclic subgroup C_n of degree n has n inequivalent irreducible representations

$$\psi^{(k)} \ : \ C_n \longrightarrow \mathbb{C}^{\times} \ , \quad \gamma \longmapsto \zeta_n^k \ ,$$

where ζ_n is the primitive root of unity, and the following equivalence of S_n-modules holds for each $k = 0, \ldots n-1$:

$$R_n(k;n) \cong_{S_n} \operatorname{Ind}_{C_n}^{S_n} \left(\psi^{(k)} \right) .$$

(*Remark* : In fact, the number n by which we take modulo is the *Coxeter number* of S_n, i.e., the order of the Coxeter elements of the Coxeter group of type A_{n-1}. They also obtain similar results for Coxeter groups of type B_n and D_n. Stembridge obtains more general results [S]. He treats the Complex reflection groups G and shows that the coinvariant algebra of G has the similar properties for the irreducible representation of the cyclic subgroup of G generated by a *Springer's regular element* [Sp]. We can easily see that the Coxeter elements are regular.)

They also prove that the multiplicity of a irreducible representation of S_n in R_n^d $(d \geq 0)$ is described by the major index of standard tableaux. It is well known that the irreducible representations of S_n are in one to one correspondence with the partitions of n. For $\lambda \vdash n$ let V^λ denote the corresponding irreducible representation of S_n. They showed that the multiplicity $[\, R_n^d : V^\lambda \,]$ of V^λ on R_n^d equals the number of standard tableaux whose major index are d :

$$[\, R_n^d : V^\lambda \,] = \sharp \{\, T \in \mathrm{STab}(\lambda) \mid \mathrm{maj}(T) = d \}.$$

(See also [G, Theorem 8.6] [R, Theorem 8.8]. A different approach to the result of Kraskiewicz-Weymann, using the *multi major index*, is discussed by A. Jöllenbeck and M. Schocker [JS].) Combining these results, the multiplicities of the irreducible representation V^λ on the induced representations $\mathrm{Ind}_{C_n}^{S_n}\left(\psi^{(k)}\right) \cong_{S_n} R_n(k;n)$ are easily follows :

$$[R_n(k;n) : V^\lambda] = \sharp\{\, T \in \mathrm{STab}(\lambda) \mid \mathrm{maj}(T) \equiv k \bmod n\}.$$

It should be mentioned here that a more refined result is obtained by R. Adin, F. Brenti and Y. Roichman [ADR] recently. For each subset $S \subseteq [n-1]$, they construct an S_n-module R_S satisfying

$$R_n^d = \bigoplus_S R_n^S \ ,$$

where the direct sum is taken over the subsets $S \subseteq [n-1]$ such that $\sum_{i \in S} i = d$, and describe the multiplicities of irreducible constituents on R_n^S as follows :

$$[\, R_n^S : V^\lambda\,] = \sharp\{T \in \mathrm{STab}(\lambda) \mid \mathrm{Des}(T) = S\}\ .$$

They also consider an analogue of the theorem of Kraskiewicz and Weymann for the Weyl groups of type B, and obtain a finer result on the irreducible decompositions of the coinvariant algebras of type B than one already obtained by Stembridge in [S].

The aim of the present article is to achieve a generalization of the results of [KW] in the following sense. Fix an integer $\ell \in [n]$), and consider subspaces of R_n obtained by gathering homogeneous components whose degrees are congruent modulo ℓ. Precisely, for each $k = 0, \ldots, \ell - 1$ we will consider

$$R_n(k;\ell) = \bigoplus_{d \equiv k \bmod \ell} R_n^d.$$

We can see that the dimension of the space $R_n(k;\ell)$ is independent of k, i.e.,

$$\dim R_n(k;\ell) = \frac{n!}{\ell}$$

for all $0 \le k \le \ell - 1$. In this article we will seek out a systematic realization of each submodule $R_n(k;\ell)$ as a S_n-module induced from a subgroup of S_n that is determined by ℓ. First we settle a subgroup H_ℓ of S_n for each $\ell \in [n]$, then construct a representation $\Psi(k;\ell)$ of H_ℓ for each $k = 0, \ldots, \ell - 1$. Finally, we will show that

$$R_n(k;\ell) \cong_{S_n} \mathrm{Ind}_{H_\ell}^{S_n}\left(\Psi(k;\ell)\right)$$

for each ℓ and k. We will give here a more precise information. For an fixed ℓ, say $n = d\ell + r$ $(0 \le r \le \ell - 1)$. Then we can choose a subgroup H_ℓ of S_n isomorphic to a direct product of a cyclic groups of degree ℓ and the symmetric group of degree r:

$$H_\ell \cong C_\ell \times S_r\ .$$

We construct a representation $\Psi(k;\ell)$ of H_ℓ, which is not necessarily irreducible, in a simple manner. Comparing their graded characters as polynomials in q modulo $q^\ell - 1$, we can verify that, for each k, the representation $R_n(k;\ell)$ of S_n is induced by the representation $\Psi(k;\ell)$

of H_ℓ. We can easily obtain the multiplicity $[R(k;\ell):\psi^\lambda]$ of the irreducible representation V^λ ($\lambda \vdash n$) in $R_n(k;\ell)$ as

$$[\,R_n(k;\ell):\psi^\lambda\,] = \sharp\{\,T \in \mathrm{STab}(\lambda) \mid \mathrm{maj}(T) \equiv k \bmod \ell\}$$

by the theorem of Kraskiewicz and Weymann.

2. Coinvariant algebra and its graded character

Let $R_n = \oplus_{d\geq 0} R_n^d$ be the coinvariant algebra of S_n and its homogeneous decomposition. Let $\ell \in [n]$ be a fixed integer. For each $k = 0, 1, \ldots, \ell-1$, define

$$R_n(k;\ell) := \bigoplus_{d\equiv k \bmod \ell} R_n^d \ ,$$

i.e.,

$$R_n = \bigoplus_{k=0}^{\ell-1} R_n(k;\ell) \ .$$

Let q be an indeterminate over $\mathbb{C}$. Define the graded character of R_n by

$$X_n(q) = \sum_{d\geq 0} q^d \chi^{n,d},$$

where $\chi^{n,d}$ is the character of the representation R_n^d of S_n. We denote by $X_{n,\rho}(q)$ and $\chi_\rho^{n,d}$ the value of $X_n(q)$ and $\chi^{n,d}$ at elements of cycle-type $\rho \vdash n$, respectively. Precisely, $X_{n,\rho}(q)$ is a polynomial in q whose coefficient in q^d is $\chi_\rho^{n,d}$.

The graded character of R_n evaluated at a partition $\rho = (1^{m_1}2^{m_2}\cdots n^{m_n}) \vdash n$ is given by

$$X_{n,\rho}(q) = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^{m_1}(1-q^2)^{m_2}\cdots(1-q^n)^{m_n}}$$

([Gr, Appendix], see also [G, Proposition 8.1]). By the formula for the graded character we obtain the following results, which play a key role in the proof of main theorem.

Proposition 1. *Fix a integer $\ell \in [n]$. Let p be a divisor of ℓ, $n = ep + s$ $(0 \leq s \leq p-1)$, and θ a primitive p-th root of unity. If $\lambda \vdash n$ satisfies*

$$X_{n,\rho}(\theta) \neq 0,$$

then $\rho = (1^{m_1}\cdots s^{m_s}p^e)$, where $m_1 + \cdots + sm_s = s$.

Proposition 2. *Let $\ell \in [n]$ be a fixed integer. Then the dimension of $R_n(k;\ell)$ is independent of the choice of $k = 0, 1, \ldots, \ell-1$, i.e., we have*

$$\dim R_n(k;\ell) = \frac{n!}{\ell}$$

for all $k = 0, 1, \ldots, \ell-1$.

Proposition 3. *Let n be a positive integer, and choose an integer ℓ $(1 \leq \ell \leq n)$. If $n = d\ell + r$ $(0 \leq r < \ell)$, then we have*

$$X_n(q) \equiv \mathrm{Ind}_{S_{d\ell}\times S_r}^{S_n}\left(X_{d\ell}(q)\, X_r(q)\right) \mod q^\ell - 1 \ .$$

Note that the polynomial $X_{n,\rho}(q)$ is also known as a *Green polynomial* $Q_\rho^{(1^n)}(q)$ of type A [Gr][Mac,III.7]. Translating Proposition 1 and Proposition 3 into the language of the Green polynomials, we obtain a formula for the Green polynomials at a root of unity (Cf. [LLT]).

Corollary 4. *Let $n > \ell$ be positive integers, p a divisor of ℓ, and θ a primitive p-th root of unity. If we write $n = d\ell + r = ep + s$ $(0 \le r \le \ell - 1,\ 0 \le s \le p-1)$, then*

(a) $Q_\rho^{(1^n)}(\theta) = 0$ *unless* $\rho = (1^{m_1} \cdots s^{m_s} p^e)$ *and* $m_1 + 2m_2 + \cdots + sm_s = s$.

(b) *If* $\rho = (1^{m_1} \cdots s^{m_s} p^e)$,

$$Q_\rho^{(1^n)}(q) \equiv Q_{\rho^1}^{(1^{d\ell})}(q)\, Q_{\rho^2}^{(1^r)}(q) \bmod q^\ell - 1\,,$$

where $\rho^1 = (p^{e-f}) \vdash d\ell$ *and* $\rho^2 = (1^{m_1} \cdots s^{m_s} p^f) \vdash r$.

3. Main result

Let n be a positive integer, and suppose that $n = d\ell + r$, where $0 \le r \le \ell - 1$.

First we consider the case of $r = 0$, that is $n = d\ell$. Let C_ℓ be the cyclic group of degree ℓ, and we embed C_ℓ into S_n by

$$C_\ell \cong \langle \gamma_1 \gamma_2 \cdots \gamma_d \rangle \subset S_n\ ,$$

where $\gamma_1 = (1, 2, \ldots, \ell)$, $\gamma_2 = (\ell+1, \ell+1, \ldots, 2\ell)$, $\ldots$, $\gamma_d = ((d-1)\ell+1, \ldots, d\ell)$. The cyclic group C_ℓ has ℓ inequivalent irreducible representations $\psi^{(0)}, \ldots, \psi^{(\ell-1)}$, i.e.,

$$\psi^{(k)}\ :\ C_\ell \longrightarrow \mathbb{C}^\times\ ,\quad \gamma_1\gamma_2\cdots\gamma_d \longmapsto \zeta_\ell^k\ ,$$

where ζ_ℓ denotes a primitive ℓ-th root of unity. Let

$$\tau^{(k)} := \frac{1}{\ell} \sum_{i=0}^{\ell-1} \zeta_\ell^{-ik} (\gamma_1 \cdots \gamma_d)^i \quad (k = 1, 2, \ldots, \ell)\,.$$

We can easily check that each $\tau^{(k)}$ is an idempotent by a direct calculation.

Let $\mathbb{C}[S_n]$ be the group algebra of S_n, and $\tau^{(k)}$ an idempotent of $\mathbb{C}[S_n]$ defined above. Consider the representation of S_n afforded by the left ideal $\mathbb{C}[S_n]\tau^{(k)}$, which is equivalent to the induced representation $\mathrm{Ind}_{C_\ell}^{S_n}\left(\psi^{(k)}\right)$. Its character $\chi[\mathbb{C}[S_n]\tau^{(k)}]$ is given by $\Gamma_n \tau^{(k)}$, where Γ_n is an operator defined by

$$\Gamma_n\ :\ \mathbb{C}[S_n] \longrightarrow \mathbb{C}[S_n]\ ,\quad \rho \longmapsto \sum_{w \in S_n} w^{-1} \rho\, w$$

(see e.g., [G, Proposition 5.2] [R, Lemma 8.4]). Here we regard an element $\rho = \sum_{w \in S_n} \rho_w\, w \in \mathbb{C}[S_n]$ as a function on S_n that maps $w \in S_n$ to the coefficient ρ_w. Equivalently,

$$\mathrm{Ind}_{C_\ell}^{S_n}\left(\chi[\psi^{(k)}]\right) = \Gamma_n \tau^{(k)}\ ,$$

where $\chi\left[\psi^{(k)}\right]$ stands for the C_ℓ-character of $\psi^{(k)}$.

By Proposition 2, the dimension of the space

$$R_n(k;\ell) = \bigoplus_{d \equiv k \bmod \ell} R_n^d$$

is constant with respect to $k = 0, \ldots, \ell - 1$. This fact seems to imply that every $R_n(k;\ell)$ $(k = 0, \ldots, \ell)$ are induced from the same dimensional representations of some subgroup of S_n. We verify in the following that a irreducible representation of C_ℓ yields each $R_n(k;\ell)$.

Proposition 5. *Let n be a positive integer and ℓ a divisor of n. Write $d = n/\ell$. Let $\gamma_i = ((i-1)\ell + 1, (i-1)\ell + 2, \ldots, i\ell) \in S_n$ $(i = 1, \ldots, d)$ be a cyclic permutation, C_ℓ the cyclic subgroup of S_n generated by $\gamma_1 \cdots \gamma_d$, and $\psi^{(k)}$ $(k = 0, \ldots, \ell - 1)$ its irreducible representation. Then, we have an isomorphism of S_n-modules*

$$R_n(k;\ell) \cong_{S_n} \operatorname{Ind}_{C_\ell}^{S_n}\left(\psi^{(k)}\right) \quad (k = 0, 1, \ldots, \ell - 1).$$

Next we consider the case of $n = d\ell + r$ and $r \neq 0$. For each $\ell = 1, 2, \ldots, n$, we define a subgroup H_ℓ of S_n by

$$\begin{aligned} H_\ell &= \langle \gamma_1\gamma_2 \cdots \gamma_d \rangle \times S_r \\ &\cong C_\ell \times S_r \ , \end{aligned}$$

where γ_i is the cyclic permutation $((i-1)\ell+1, (i-1)\ell+2, \ldots, i\ell)$, and the symmetric group S_r of degree r is identified as a subgroup $\{w \in S_n \mid w(i) = i \text{ for all } i = 1, 2, \ldots, n-r\}$ of S_n.

For each $k = 0, 1, \ldots, \ell - 1$, we construct a representation $\Psi(k;\ell)$ of H_ℓ as follows :

$$\Psi(k;\ell) := \bigoplus_{\lambda \vdash r} \bigoplus_{T \in \mathrm{STab}(\lambda)} \psi^{(\overline{k - \mathrm{maj}(T)})} \otimes V^\lambda \ ,$$

where $\overline{k - \mathrm{maj}(T)} = k - \mathrm{maj}(T) \mod \ell$, and $\psi^{(i)}$ $(i = 0, \ldots, \ell)$ and V^λ $(\lambda \vdash r)$ are the irreducible representations of C_ℓ and S_r, respectively. Then it can be seen that the degree of $\Psi(k;\ell)$ does not depend on k and hence so does $\deg \operatorname{Ind}_{H_\ell}^{S_n}(\Psi(k;\ell))$. Actually, since $\dim V^\lambda = \sharp\, \mathrm{STab}(\lambda)$ and $\sum_{\lambda \vdash r} \sharp \mathrm{STab}(\lambda)^2 = r!$, we have

$$\begin{aligned} \deg \Psi(k;\ell) &= \sum_{\lambda \vdash r} \sum_{T \in \mathrm{STab}(\lambda)} \deg\left(\psi^{(\overline{k - \mathrm{maj}(T)})} \otimes V^\lambda\right) \\ &= \sum_{\lambda \vdash r} \sum_{T \in \mathrm{STab}(\lambda)} \sharp \mathrm{STab}(\lambda) = r! \ , \end{aligned}$$

and $\deg \operatorname{Ind}_{H_\ell}^{S_n}(\Psi(k;\ell)) = r!n!/r!\ell = n!/\ell$, which coincides with the dimension of $R_n(k;\ell)$. Moreover, we show that these two representations are equivalent.

Theorem 6 (Main result)**.** *Let n be a positive integer. Fix an integer $\ell \in [n]$ and write $n = d\ell + r$ $(0 \leq r \leq \ell)$. Let $H_\ell \cong C_\ell \times S_r$ be the subgroup of S_n and $\Psi(k;\ell)$ $(k = 0, 1, \ldots, \ell-1)$ its representations defined by*

$$\Psi(k;\ell) := \bigoplus_{\lambda \vdash r} \bigoplus_{T \in \mathrm{STab}(\lambda)} \psi^{(\overline{k - \mathrm{maj}(T)})} \otimes V^\lambda \ ,$$

where $\psi^{(i)}$ and V^λ stand for the irreducible representations of C_ℓ and S_r, respectively. Then, for $k = 0, 1, \ldots, \ell - 1$, there is an isomorphism

$$R_n(k;\ell) \cong_{S_n} \operatorname{Ind}_{H_\ell}^{S_n}(\Psi(k;\ell)) \ .$$

as an S_n-module.

When $r = 0$ or 1, H_ℓ is a cyclic group and $\Psi(k;\ell)$ is irreducible. In this case, the generator of H_ℓ coincides with a regular element of S_n defined by Springer [Sp].

The following Corollary follows trivially from Theorem 6 and the Theorem of Kraskiewicz-Weymann.

Corollary 7. *The multiplicity of the irreducible representation V^λ in $R_n(k;\ell)$ is equal to the number of standard Young tableaux of shape λ with major index congruent to k modulo ℓ, that is,*

$$[R_n(k;\ell) : V^\lambda] = \sharp\{T \in \mathrm{STab}(\lambda) \ : \ \mathrm{maj}(T) \equiv k \bmod \ell\} \ .$$

Example 8. In the case of $n = 5$ and $\ell = 3$, the subgroup H_3 is $\langle(123)\rangle \times \langle(45)\rangle$, which is isomorphic to $C_3 \times S_2$. Then we have

$$R^{(5)}(k;3) \cong_{S_5} \mathrm{Ind}_{H_3}^{S_5} \left(\psi^{(k)} \otimes V^{(2)}\right)$$

for each $k = 0, 1, 2$.

If we consider the case $n = 11$ and $\ell = 4$ (thus $r = 3$), then the subgroup H_4 is $\langle(1234)(5678)\rangle \times \langle(9,10),(10,11)\rangle$ isomorphic to $C_4 \times S_3$. Hence, for each $R^{(11)}(k;4)$ $(k = 0,1,2,3)$ is isomorphic to the representation induced by

$$\Psi(0;4) = (\psi^{(0)} \otimes V^{(3)}) \oplus (\psi^{(3)} \otimes V^{(2,1)}) \oplus (\psi^{(2)} \otimes V^{(2,1)}) \oplus (\psi^{(1)} \otimes V^{(1,1,1)}) \ ,$$
$$\Psi(1;4) = (\psi^{(1)} \otimes V^{(3)}) \oplus (\psi^{(0)} \otimes V^{(2,1)}) \oplus (\psi^{(3)} \otimes V^{(2,1)}) \oplus (\psi^{(2)} \otimes V^{(1,1,1)}) \ ,$$
$$\Psi(2;4) = (\psi^{(2)} \otimes V^{(3)}) \oplus (\psi^{(1)} \otimes V^{(2,1)}) \oplus (\psi^{(0)} \otimes V^{(2,1)}) \oplus (\psi^{(3)} \otimes V^{(1,1,1)}) \ ,$$
$$\Psi(3;4) = (\psi^{(3)} \otimes V^{(3)}) \oplus (\psi^{(2)} \otimes V^{(2,1)}) \oplus (\psi^{(1)} \otimes V^{(2,1)}) \oplus (\psi^{(0)} \otimes V^{(1,1,1)}) \ .$$

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H. Morita: School of Science, Tokai University, Hiratsuka 259-1292, Japan
E-mail address: `moritah@msg.biglobe.ne.jp`

T. Nakajima: Faculty of Economics, Meikai University, Urayasu 279-8550, Japan
E-mail address: `tatsu@ka2.so-net.ne.jp`