(BI-)COHEN-MACAULAY SIMPLICIAL COMPLEXES AND THEIR ASSOCIATED COHERENT SHEAVES

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Abstract. Via the BGG correspondence a simplicial complex Δ on $[n]$ is transformed into a complex of coherent sheaves on $\mathbf{P}^{n-1}$. We show that this complex reduces to a coherent sheaf $\mathcal{F}$ exactly when the Alexander dual Δ^* is Cohen-Macaulay.

We then determine when both Δ and Δ^* are Cohen-Macaulay. This corresponds to $\mathcal{F}$ being a locally Cohen-Macaulay sheaf.

Lastly we conjecture for which range of invariants of such Δ it must be a cone.

Abstrait. Sous la correspondance BGG un complexe simplicial Δ, ayant les vertices $1, \ldots, n$, est transformé en un complexe des faisceau cohérents sur $\mathbf{P}^{n-1}$. On démontre que ce complexe se réduit à un seul faisceau $\mathcal{F}$ precisement quand le dual d'Alexander Δ^* est de Cohen-Macaulay.

La condition que tous les deux Δ et Δ^* soient de Cohen-Macualay corresponde dans la même façon à la condition que $\mathcal{F}$ soit un faisceau locallement de Cohen-Macaulay.

Finalement, on pose une conjecture concernant la question: pour quelles valeurs des invariants de Δ est Δ nécessairement un cône.

Introduction

To a simplicial complex Δ on the set $[n] = \{1, \ldots, n\}$ is associated a monomial ideal I_Δ in the exterior algebra E on a vector space of dimension n. Lately there has been a renewed interest in the Bernstein-Gel'fand-Gel'fand (BGG) correspondence which associates to a graded module M over the exterior algebra E a complex of coherent sheaves on the projective space $\mathbf{P}^{n-1}$ (see [6], [1]). In this paper we study simplicial complexes in light of this correspondence. Thus to each simplicial complex Δ we get associated a complex of coherent sheaves on $\mathbf{P}^{n-1}$. Our basic result is that this complex reduces to a single coherent sheaf $\mathcal{F}$ if and only if the Alexander dual Δ^* is a Cohen-Macaulay simplicial complex. So we in yet a new way establish the naturality of the concept of a simplicial complex being Cohen-Macaulay in addition to the well established interpretations via the topological realization, and via commutative algebra and Stanley-Reisner rings.

It also opens up the possibility to study simplicial complexes from the point of view of algebraic geometry. A simple fact is that Δ is a cone if and only if the support of $\mathcal{F}$ is contained in a hyperplane. Now the nicest coherent sheaves on projective space may be said to be vector bundles, or more generally those sheaves which when projected down as far as possible, to a projective space of dimension equal to the dimension of the support of the sheaf, become vector bundles. This is the class of locally Cohen-Macaulay sheaves (of pure dimension). We show that the coherent sheaf $\mathcal{F}$ is a locally Cohen-Macaulay sheaf iff both Δ and Δ^* are Cohen-Macaulay simplicial complexes. We call such Δ bi-Cohen-Macaulay and try to describe this class as well as possible.

In Section 1 we recall basic facts about the BGG-correspondence. In Section 2 we apply this to simplicial complexes and show the basic theorem, that we get a coherent sheaf $\mathcal{F}$ via the BGG-correspondence iff Δ^* is Cohen-Macaulay. We are also able to give a kind of geometric interpretation of the h-vector of Δ^* in terms of the sheaf $\mathcal{F}$.

In Section 3 we consider bi-Cohen-Macaulay simplicial complexes Δ. Then the associated sheaf $\mathcal{F}$ on $\mathbf{P}^{n-1}$ when projected down to $\mathbf{P}^{s-1}$, where $s-1$ is the dimension of the support of $\mathcal{F}$, becomes one of the sheaves of differentials $\Omega^c_{\mathbf{P}^{s-1}}$. This gives quite restrictive conditions on the face vector of such Δ. It is parametrized by three parameters, namely n, c and s.

When $c = 0$, Δ is just the empty simplex. When $c = 1$ a result of Fröberg [9] enables a combinatorial description of such Δ. If Δ has dimension $d-1$ equal to 1 it is a tree and in general Δ is what is called a $d-1$-tree. When $c \geq 2$ a combinatorial description seems less tractable. There is a classical example of Reisner [10] of a triangulation of the real projective plane which is bi-Cohen-Macaulay if char $k \neq 2$ but neither Δ nor Δ^* are Cohen-Macaulay if char $k = 2$. In particular Δ is not shellable.

Now suppose $\mathcal{F}$ projects down to $\Omega^c_{\mathbf{P}^{s-1}}$. A natural question to ask is whether $\mathcal{F}$ is degenerate or not (the support contained in a hyperplane or not). This corresponds to Δ being a cone or not. In the last section, Section 5, we conjecture that there exists a bi-Cohen-Macaulay Δ which is not a cone if and only if $n \leq (c+1)(s-c)$. We prove this conjecture when $c = 1$ and give examples to show the plausibility of this conjecture for any c. For the boundary values $n = (c+1)(s-c)$ we construct particularly nice examples of bi-Cohen-Macaulay Δ with invariants n, c, and s which are not cones.

In the end we would like to make one additional point for the naturality of this approach. Monomial ideals of a polynomial ring correspond to multicomplexes while monomial ideals of the exterior algebra correspond to simplicial complexes.

Finally we express our appreciation of Macaulay2. In our investigations in this paper we have repeatedly had the benefit of computing resolutions over the exterior algebra using Macaulay2.

1. The BGG correspondence

We start by recalling some facts about the BGG correspondence originating from [3]. Our main reference is [6].

Tate resolutions. Let V be a finite dimensional vector space of dimension n over a field k. Let $E(V) = \oplus \wedge^i V$ be the exterior algebra and for short denote it by E. Given a graded (left) E-module $M = \oplus M_i$ we can take a minimal projective resolution of M

$$P : \quad \cdots \to P^{-2} \to P^{-1} \to M$$

where

$$P^{-p} = \oplus_{a \in \mathbf{Z}} E(a) \otimes_k \tilde{V}^{-p}_{-a}.$$

Now the canonical module ω_E, which is $\mathrm{Hom}_k(E, k)$, is the injective envelope of k. Hence we can take a minimal injective resolution

$$I \,:\, M \to I^0 \to I^1 \to \cdots$$

where

$$I^p = \oplus_{a \in \mathbf{Z}} \omega_E(a) \otimes_k V^p_{-a}.$$

(For $-p < 0$ we put $V^{-p}_{-a} = \tilde{V}^{-p}_{-a-n}$.) By fixing an isomorphism $k \to \wedge^n V^*$ where V^* is the dual space of V, we get an isomorphism of E and $\omega_E(-n)$ as left E-modules, where we have given V degree 1 and V^* degree -1.

We can then join together P and I into an unbounded acyclic complex $T(M)$, called the Tate resolution of M

$$\cdots \to \oplus_a \omega_E(a) \otimes_k V^p_{-a} \xrightarrow{d_p} \oplus_a \omega_E(a) \otimes_k V^{p+1}_{-a} \to \cdots$$

such that M is $\ker d^0$ and also $\mathrm{im}\, d^{-1}$. (One should use ω_E instead of E in this complex since ω_E is the natural thing to use in the framework of Koszul duality and hence in the BGG correspondence, see [6].)

BGG correspondence. The terms T^i have natural algebraic geometric interpretations via the BGG correspondence. Let V have a basis $\{e_a\}$ and let $W = V^*$ be the dual space of V with dual basis $\{x_a\}$. Let $S = S(W)$ be the symmetric algebra on W. To M we then associate a complex of free S-modules

$$L(M) \,:\, \cdots \to S(i) \otimes_k M_i \xrightarrow{\delta^i} S(i+1) \otimes_k M_{i+1} \to \cdots$$

where

$$\delta^i(s \otimes m) = \sum_a s x_a \otimes e_a m.$$

If we sheafify $L(M)$ we get a complex of coherent sheaves on the projective space $\mathbf{P}(W)$

$$\tilde{L}(M) \,:\, \cdots \to \mathcal{O}_{\mathbf{P}(W)}(i) \otimes_k M_i \to \mathcal{O}_{\mathbf{P}(W)}(i+1) \otimes_k M_{i+1} \to \cdots .$$

This, in short, is the BGG correspondence between finitely generated graded (left) E-modules and complexes of coherent sheaves on $\mathbf{P}(W)$.

Remark 1.1. The BGG correspondence induces an equivalence of triangulated categories between the stable module category of finitely generated graded modules over E and the bounded derived category of coherent sheaves on $\mathbf{P}(W)$

$$E - \underline{\text{mod}} \overset{\tilde{L}}{\simeq} D^b(\text{coh}/\mathbf{P}(W)).$$

Due to this remark we may also start with a coherent sheaf $\mathcal{F}$, and there will be a module M over E such that $\tilde{L}(M)$ only has non-zero cohomology in degree 0, equal to $\mathcal{F}$. Forming the Tate resolution $T(M)$ we also denote it as $T(\mathcal{F})$ and say it is the Tate resolution of $\mathcal{F}$.

Duals. Consider $\wedge^n W$ as a module situated in degree $-n$ and let $M^\vee$ be $\text{Hom}_k(M, \wedge^n W)$. Since $\tilde{L}(\wedge^n W)$ naturally identifies with the canonical sheaf $\omega_{\mathbf{P}(W)}$ on $\mathbf{P}(W)$ shifted n places to the left, we see that

$$\tilde{L}(M^\vee) = \text{Hom}_k(\tilde{L}(M), \omega_{\mathbf{P}(W)})[n].$$

Hence if $\tilde{L}(M)$ has only one nonvanishing cohomology group $\mathcal{F}$ in cohomological degree p, then

$$\mathcal{E}xt^i(\mathcal{F}, \omega_{\mathbf{P}(W)}) = H^{i-p-n}\tilde{L}(M^\vee). \tag{1}$$

Since ω_E naturally identifies with $\text{Hom}_k(\omega_E, \wedge^n W)$ we also get that the Tate resolution of $M^\vee$ is the dual $\text{Hom}_k(T(M), \wedge^n W)$ of the Tate resolution of M.

Projections. Given a subspace $U \subseteq W$ we get a projection $\pi : \mathbf{P}(W) \dashrightarrow \mathbf{P}(U)$. If the support of the coherent sheaf $\mathcal{F}$ does not intersect the center of projection $\mathbf{P}(W/U) \subseteq \mathbf{P}(W)$ we get a coherent sheaf $\pi_*(\mathcal{F})$ on $\mathbf{P}(U)$. How is the Tate resolution of $\pi_*\mathcal{F}$ related to that of $\mathcal{F}$? Via the epimorphism $E \to E(U^*)$ the latter becomes an E-module. It then turns out that the Tate resolution

$$T(\pi_*\mathcal{F}) = \text{Hom}_E(E(U^*), T(\mathcal{F})).$$

Note that

$$\text{Hom}_E(E(U^*), \omega_E) = \omega_{E(U^*)}.$$

Hence

$$T(\pi_*\mathcal{F}) \, : \, \cdots \to \oplus_i \omega_{E(U^*)}(p-i) \otimes_k H^i\mathcal{F}(p-i) \to \cdots .$$

In particular we see that the cohomology groups $H^i\pi_*\mathcal{F}(p-i)$ and $H^i\mathcal{F}(p-i)$ are equal.

Linear subspaces. If $U \to W$ is a surjection, we get an inclusion of linear subspaces $i : \mathbf{P}(W) \hookrightarrow \mathbf{P}(U)$. Then by [7, 1.4 (21)] the Tate resolution of $i_*\mathcal{F}$ is

$$\text{Hom}_E(E(U^*), T(\mathcal{F})).$$

2. Simplicial complexes giving coherent sheaves.

The BGG-correspondence applied to simplicial complexes. Let Δ be a simplicial complex on the set $[n] = \{1, \ldots, n\}$. Then we get a monomial ideal I_Δ in E which is generated by the monomials $e_{i_1} \cdots e_{i_r}$ such that $\{i_1, \ldots, i_r\}$ is not in Δ. Dualizing the inclusion $I_\Delta \subseteq E(V)$ we get an exact sequence

$$0 \to C_\Delta \to E(W) \to (I_\Delta)^* \to 0. \tag{2}$$

Note that $E(W)$ is a coalgebra and that C_Δ is the subcoalgebra generated by all $x_{i_1} \cdots x_{i_r}$ such that $\{i_1, \ldots i_r\}$ is in Δ.

Now think of $\omega_E = E(W)$ as a left $E(V)$-module; then C_Δ is a submodule of ω_E. Then we can use the BGG correspondence. A natural question to ask is: When does $\tilde{L}(C_\Delta)$ have only one non-vanishing cohomology group, a coherent sheaf $\mathcal{F}$? It turns out that this happens exactly when the Alexander dual simplicial complex Δ^* is Cohen-Macaulay. Let us recall this and some other notions.

A simplicial complex Δ is *Cohen-Macaulay* if its Stanley-Reisner ring $k[\Delta]$ is a Cohen-Macaulay ring. For more on this see Stanley's book [11].

The *Alexander dual* Δ^* of Δ consists of subsets F of $[n]$ such that $[n] - F$ is not a face of Δ. Via the isomorphism $\omega_E \cong E(n)$, the submodule C_{Δ^*} corresponds to the ideal I_Δ in E. So we get from (2) an exact sequence

$$0 \to C_\Delta \to \omega_E \to (C_{\Delta^*})^\vee \to 0. \tag{3}$$

Dualizing this we get

$$0 \to C_{\Delta^*} \to \omega_E \to (C_\Delta)^\vee \to 0. \tag{4}$$

Main theorem. A coherent sheaf $\mathcal{F}$ on a projective space is *locally Cohen-Macaulay* of pure dimension n if for all the localizations $\mathcal{F}_P$ we have depth $\mathcal{F}_P = \dim \mathcal{F}_P = n$. This is equivalent to all intermediate cohomology groups $H^i\mathcal{F}(p)$ vanishing for $0 < i < n$ when p is large positive or negative. It is also equivalent to $\mathcal{F}$ projecting down to a vector bundle on $\mathbf{P}^n$.

Let c be the largest integer such that all $c-1$-simplexes of $[n]$ are contained in Δ.

Theorem 2.1. a) *The complex $\tilde{L}(C_\Delta)$ has at most one non-vanishing cohomology group, a coherent sheaf $\mathcal{F}$, if and only if Δ^* is Cohen-Macaulay. In this case $\mathcal{F}$ is $H^{-c}\tilde{L}(C_\Delta)$.*

b) *$\mathcal{F}$ is locally Cohen-Macaulay of pure dimension if and only if both Δ and Δ^* are Cohen-Macaulay.*

c) *The support of $\mathcal{F}$ is contained in a hyperplane if and only if Δ (or equivalently Δ^*) is a cone.*

We include the proof of parts a) and b):

Proof. By [5] Δ^* is Cohen-Macaulay if and only if the associated ideal of Δ in the symmetric algebra has a linear resolution. By [1, Cor.2.2.2] this happens exactly when I_Δ has a linear resolution over the exterior algebra. Now note that since I_Δ in E is generated by exterior monomials, in any case a resolution will have terms

$$I_\Delta \leftarrow \oplus_{a\geq c+1} E(-a)\otimes_k \tilde{V}_a^1 \leftarrow \oplus_{a\geq c+2} E(-a)\otimes_k \tilde{V}_a^2 \leftarrow \cdots$$

with all $\tilde{V}^i_{c+i}$ non-zero. But then the injective resolution of the vector space dual $(I_\Delta)^*$ will have "pure" terms $\omega_E(a)\otimes_k \tilde{V}^{a+a_0}_{-a}$ for $a \gg 0$, meaning $\tilde{L}(C_\Delta)$ is a coherent sheaf, if and only if I_Δ has a linear resolution from the very start and this then happens exactly when Δ^* is Cohen-Macaulay.

The fact that $\mathcal{F}$ is locally Cohen-Macaulay means that the terms in the Tate resolution are $\omega_E(a)\otimes_k \tilde{V}^{a+a_0}_{-a}$ for $a \gg 0$ and similarly for $a \ll 0$.

Now by the dual sequences (3) and (4), the dual of the Tate resolution of C_Δ is the Tate resolution of C_{Δ^*}. Thus we get that the condition just stated for the Tate resolution of $\mathcal{F}$ must mean that both Δ and Δ^* are Cohen-Macaulay. □

Definition 2.2. If Δ^* is Cohen-Macaulay we denote the corresponding coherent sheaf by $\mathcal{S}(\Delta)$.

When Δ and Δ^* are both Cohen-Macaulay we say that Δ is *bi-Cohen-Macaulay.*

Proposition 2.3. *When Δ^* is CM the complex*

$$\tilde{L}((C_{\Delta^*})^\vee)[-c-1] : \mathcal{O}_{\mathbf{P}(W)}(-c-1)^{f^*_{d^*}} \leftarrow \cdots \leftarrow \mathcal{O}_{\mathbf{P}(W)}(-n)$$

is a resolution of $\mathcal{S}(\Delta)$.

Numerical invariants. For a simplicial complex Δ on n vertices, let f_i be the number of i-dimensional simplices. The f-polynomial of Δ is

$$f_\Delta(t) = 1 + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d$$

where $d-1$ is the dimension of Δ.

If we form the cone $C\Delta$ of Δ over a new vertex, then the f-polynomial of $C\Delta$ is

$$f_{C\Delta}(t) = (1+t) f_\Delta(t).$$

The f-polynomial of the Alexander dual Δ^* is related to f by

$$f_i^* + f_{n-i-2} = \binom{n}{i+1}.$$

Note that the invariants c^* and d^* of Δ^* are related to those of Δ by

$$c^* + d + 1 = n, \quad c + d^* + 1 = n.$$

Proposition 2.4. *Suppose Δ^* is Cohen-Macaulay. The Hilbert series of $\mathcal{S}(\Delta)$ is given by*

$$\sum_k h^0(\mathcal{S}(\Delta)(k)) t^k = (-1)^{c+1} + (-1)^c f_\Delta(-t)/(1-t)^n.$$

If f_Δ is $(1+t)^{n-s}f$ where $f(1)$ is non-zero, then the support of $\mathcal{S}(\Delta)$ has dimension $s-1$.

There is also another equivalent set of numerical invariants of Δ. They are related to the f_i's by the following polynomial equation

$$t^d + f_0 t^{d-1} + \cdots + f_{d-1} = (1+t)^d + h_1(1+t)^{d-1} + \cdots + h_d. \tag{5}$$

When Δ is Cohen-Macaulay all the $h_i \geq 0$, [11, II.3].

There is no geometric interpretation of the h_i's in terms of the topological realization of Δ. However the following gives a kind of geometric interpretation of the h_i^*'s for a CM simplicial complex Δ^* in terms of the sheaf $\mathcal{S}(\Delta)$.

Proposition 2.5. *If Δ^* is CM then in the Grothendieck group of sheaves on $\mathbf{P}(W)$*

$$[\mathcal{S}(\Delta)(c+1)] = h^*_{d^*}[\mathcal{O}_{\mathbf{P}^{n-1}}] + h^*_{d^*-1}[\mathcal{O}_{\mathbf{P}^{n-2}}] + \cdots + h^*_0[\mathcal{O}_{\mathbf{P}^c}] \tag{6}$$

Remark 2.6. We thus see that with larger and larger c we are situated in a smaller and smaller part of the Grothendieck group.

3. Bi-Cohen-Macaulay simplicial complexes

Numerical invariants. The basic types of bi-Cohen-Macaulay simplicial complexes turn out to be the skeletons of simplices of various dimensions. So let

$$f_{s,c}(t) = 1 + st + \binom{s}{2}t^2 + \cdots + \binom{s}{c}t^c \tag{7}$$

be the f-polynomial of the $(c-1)$-dimensional skeleton of the $(s-1)$-simplex.

Proposition 3.1. *If Δ is bi-CM then*

$$f_\Delta(t) = (1+t)^{n-s} f_{s,c}(t)$$

for some s. We then say that Δ is of type (n, c, s).

Remark 3.2. If Δ is Cohen-Macaulay, the terms of the h-vector are all non-negative. If Δ is bi-CM of type (n, c, s), the terms $h_{c+1} = h_{c+2} = \cdots = 0$. So the bi-CM simplicial complexes are in a way numerically extremal in the class of Cohen-Macaulay complexes.

Algebraic geometric description of bi-CM simplicial complexes. Let $\Omega^c_{\mathbf{P}^{n-1}}$ be the sheaf of c-differentials on $\mathbf{P}^{n-1}$.

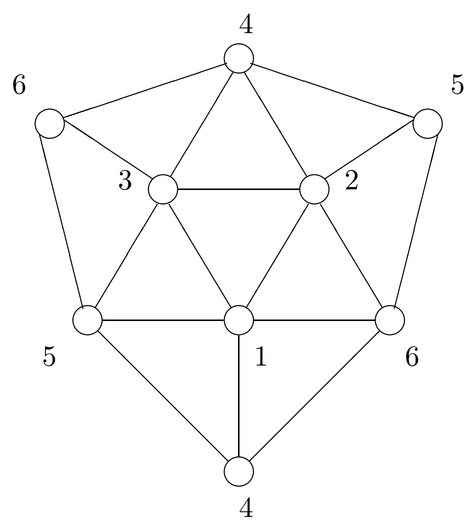
Proposition 3.3. *a) Let Δ be the $(c-1)$-skeleton of a simplex on n vertices. Then Δ is bi-CM with $\mathcal{S}(\Delta) = \Omega^c_{\mathbf{P}^{n-1}}$.*

b) When Δ is bi-CM of type (n, c, s) then $\pi_*\mathcal{S}(\Delta) = \Omega^c_{\mathbf{P}^{s-1}}$ where π is a projection $\mathbf{P}^{n-1} \dashrightarrow \mathbf{P}^{s-1}$ whose center is disjoint from the support of $\mathcal{S}(\Delta)$.

Topological description of bi-CM simplicial complexes. The bi-CM simplicial complexes Δ correspond by [5] to Stanley-Reisner rings $k[\Delta]$ which are CM and have a linear resolution over the polynomial ring. Since the generators of the ideal of $k[\Delta]$ will have degree $c+1$, we say the resolution if $c+1$-linear.

In [9] R. Fröberg studies Stanley-Reisner rings $k[\Delta]$ with 2-linear resolution. When Δ is CM (so Δ is bi-CM with $c=1$) he shows that Δ is what is called a $d-1$-tree. (Stictly speaking he uses this term only for the 1-skeleton of Δ.) They arise as inductively as follows. Start with a $d-1$-simplex, then attach $d-1$ simplices, one at a time, by identifying one (and only one) $d-2$-face of Δ with one (and only one) $d-2$-face of the simplex to be attached. This thus describes bi-CM Δ with $c=1$. When $c \geq 2$ things appear to be less tractable as the following example shows.

Example 3.4. The following example was first noted in [10]. Consider the simplicial complex of dimension 2 with invariants (n, c, s) equal to $(6, 2, 4)$:



This simplicial complex is a triangulation of the real projective plane. It is isomorphic to its Alexander dual. Over any field of characteristic different from two, it is bi-Cohen-Macaulay. However, it has homology in dimension one over $\mathbb{Z}/2\mathbb{Z}$, so it is not Cohen-Macaulay over that field. In particular, it is not shellable.

4. When are CM-simplicial complexes cones?

The following proposition gives rise to the problems and results addressed in this section. In particular we are interested in determining for which range of invariants (n, c, s) a bi-CM simplicial complex necessarily is a cone.

Proposition 4.1. *Let f be a polynomial. Then there exists $e(f)$ such that for $e > e(f)$ if Δ is a CM simplicial complex with $f_\Delta = (1+t)^e f$, then Δ is a cone.*

We now pose the following.

Problem 4.2. *For each polynomial f with $f(-1)$ non-zero, determine the least number, call it $e(f)$, such that when Δ is Cohen-Macaulay with $f_\Delta = (1+t)^e f$ and not a cone, then $e \leq e(f)$.*

In the case where f is $f_{s,c}$, see (7), we propose the following conjecture for the value of the upper bound of $e = n - s$ when Δ is not a cone.

Conjecture 1. *Suppose Δ is bi-CM of type (n, c, s) and not a cone. Then $n - s \leq c(s - c - 1)$.*

Conjecture 2. *Suppose $\mathcal{F}$ is a non-degenerate coherent sheaf on $\mathbf{P}^{n-1}$ which projects down to $\Omega^c_{\mathbf{P}^{s-1}}$ on $\mathbf{P}^{s-1}$. Then $n - s \leq c(s - c - 1)$.*

Conjecture 1 is true for $c = 1$.

Proposition 4.3. *If Δ is bi-CM of type $(n, 1, s)$ and not a cone, then $n - s \leq s - 2$.*

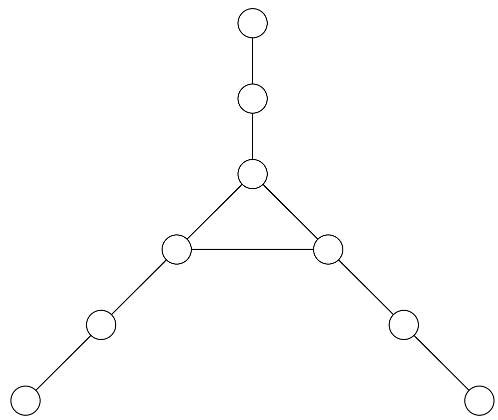
Clearly Conjecture 2 implies Conjecture 1 by letting $\mathcal{F}$ be $\mathcal{S}(\Delta)$. The following shows the existence of non-degenerate coherent sheaves $\mathcal{F}$ attaining the bound in Conjecture 2 and which cannot be lifted further.

Proposition 4.4. *The sheaf $\mathcal{O}(-c - 1, 0)$ on the Segre embedding of $\mathbf{P}^c \times \mathbf{P}^{s-c-1}$ in $\mathbf{P}^{(c+1)(s-c)-1}$ projects down to $\Omega^c_{\mathbf{P}^{s-1}}$.*

Since the Segre embedding is smooth and projectively normal, this line bundle cannot be lifted further.

The sheaf $\mathcal{O}(-c - 1, 0)$ above cannot come from a simplicial complex since it is not supported on a union of linear spaces. The following gives an example of a simplicial complex attaining the bound in Conjecture 1. There are many others. For instance, one can take I_Δ to be the polarization of the $(s - c)$-th power of the irrelevant ideal in a polynomial ring with $c + 1$ variables.

Definition 4.5. Given integers c and m, define the complex $\Delta = \Delta(c, m)$ as follows. Consider a graph G with $(c + 1)m$ vertices, arranged as a complete graph on $c + 1$ vertices, with tails of m vertices (including the inner vertex, contained in the complete graph) going out from each of these vertices. Δ is now the simplicial complex on the vertices of G whose facets are given by connected subgraphs with cm vertices. E.g. for $(c, m) = (2, 3)$ G is the graph



Proposition 4.6. *Given integers c, m, the complex Δ constructed above is bi-Cohen-Macaulay, and is of type $(n, c, s) = ((c+1)m, c, m+c)$. Δ is not a cone.*

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