ORTHOGONAL POLYNOMIALS ARISING FROM THE WREATH PRODUCTS

HIROKI AKAZAWA AND HIROSHI MIZUKAWA

ABSTRACT. Zonal spherical functions of the Gelfand pair ($W(B_n)$, S_n) are expressed in terms of the Krawtchouk polynomials which are a special family of Gauss' hypergeometric functions. Its generalizations are considered in this abstract. Some class of orthogonal polynomials are discussed in this abstract which are expressed in terms of $(n+1, m+1)$-hypergeometric functions. The orthogonality comes from that of zonal spherical functions of certain Gelfand pairs of wreath product.

RÉSUMÉ. Des fonctions sphériques zonales de la paire $(W(B_n), S_n)$ de Gelfand sont exprimées en termes de polynômex de Krawtchouk qui sont une famille spéciale des fonctions hypergéométriques des Gauss. Ses généralisations sont considérées dans cet abstrait. Une certaine classe des polynômex orthogonaux sont discutées dans cet abstrait qui sont exprimés en termes de fonctions $(n+1, m+1)$-hypergéométriques. L'orthogonalité vient de celle des fonctions sphériques zonales de certaines paires de Gelfand de produits en couronne.

1. INTRODUCTION

Askey-Wilson polynomials and q-Racah polynomials are fundamental orthogonal polynomials which are described by the basic hypergeometric functions. Roughly speaking there are two points of view of orthogonal polynomials. One is through the Riemannian symmetric spaces which are homogeneous spaces of Lie groups. The other is through the finite groups. In this abstract we discuss some discrete orthogonal polynomials arising from Gelfand pairs [15, 16] of wreath products.

A pair of groups (G, H) is called a Gelfand pair if the induced representation $1_H^G \cong C(G/H)$, where $C(G/H)$ is a complex valued functions defined over G/H, is multiplicity free as a G-module. In this situation there exists a unique H-invariant element in each irreducible component of 1_H^G which is called the zonal spherical function. There are interesting relations between zonal spherical functions on finite groups [15] and hypergeometric functions [5]. A hypergeometric function of

Classification number :33C45,05E35,05E05.

one variable is by definition

$$ {}_\ell F_m(a_1, a_2, \cdots, a_\ell; b_1, b_2, \cdots, b_m; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_\ell)_k}{(b_1)_k (b_2)_k \cdots (b_m)_k} \frac{x^k}{k!}. $$

In the following we recall some known results: Zonal spherical functions of the Gelfand pair, $W(B_n)$, the hyperoctahedral group, and S_n, the symmetric group, are expressed in terms of the Krawtchouk polynomials. The Krawtchouk polynomials is a special family of Gauss' hypergeometric polynomials [5, 6, 7, 17, 18];

$$ {}_2F_1(-k, -j; -n; 2). $$

Here k and j are nonnegative integers at most n. A generalization is given by the Gelfand pair $(S_q \wr S_n, S_{q-1} \wr S_n)$. In this case the zonal spherical functions are realized as follows [5, 17, 18];

$$ {}_2F_1(-k, -j; -n; \frac{q}{q-1}). $$

By considering another Gelfand pair, S_k and its maximal parabolic subgroup $S_{v-i} \times S_i (1 \leq i \leq [r/2])$, we have the Hahn polynomials [5];

$$ {}_3F_2(-i, -j, -v - \ell + j; -k, -v + k; 1). $$

We remark that in this case the classical hypergeometric functions of one variable still occur, since the Gelfand pairs $(S_q \wr S_n, S_{q-1} \wr S_n)$ and $(S_k, S_{v-i} \times S_i)$ are of "rank one". One might expect a multivariate version of hypergeometric functions arise naturally as zonal spherical functions for certain types of Gelfand pairs. General hypergeometric functions are known as $(n+1, m+1)$-hypergeometric functions [3, 9, 10, 12, 13, 19];

$$ F(\alpha, \beta; \gamma, X) = \sum_{(a_{ij}) \in M_{n,m-n-1}(\mathbb{N}_0)} \frac{\prod_{i=1}^{n} (\alpha_i)_{\sum_{j=1}^{m-n-1} a_{ij}} \prod_{i=1}^{m-n-1} (\beta_i)_{\sum_{j=1}^{n} a_{ji}}}{(\gamma)_{\sum_{ij} a_{ij}}} \frac{\prod x_{ij}^{a_{ij}}}{\prod a_{ij}!}, $$

which are originally due to K. Aomoto and I. M. Gelfand. Here we denote by X the set of variables x_{ij} $(1 \leq i \leq n, 1 \leq j \leq m-n-1)$.

In this abstract we consider another generalization of the Gelfand pair $(S_q \wr S_n, S_{q-1} \wr S_n)$. We will see that its zonal spherical functions are realized by means of a discrete orthogonal polynomials coming from $F(\alpha, \beta; \gamma, X)$.

2. Main Results

We denote the shifted factorial of an indeterminate x by

$$ (x)_m = x(x+1)(x+2) \cdots (x+m-1) $$

for $m \in \mathbb{Z}_{>0}$ and

$$ (x)_0 = 1. $$

Now if $-N$ is a negative integer, then we define the finite series called the $(n+1, m+1)$-*hypergeometric functions*[3, 19];

$$F(\alpha, \beta; -N, X) = \sum_{\substack{\sum_{i,j} a_{ij} \leq N \\ (a_{ij}) \in M_{n,m-n-1}(\mathbb{N}_0)}} \frac{\prod_{i=1}^{n} (\alpha_i)_{\sum_{j=1}^{m-n-1} a_{ij}} \prod_{j=1}^{m-n-1} (\beta_j)_{\sum_{i=1}^{n} a_{ij}}}{(-N)_{\sum_{i,j} a_{ij}}} \frac{\prod x_{ij}^{a_{ij}}}{\prod a_{ij}!}$$

for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n$, $\beta = (\beta_1, \cdots, \beta_{m-n-1}) \in \mathbb{C}^{m-n-1}$. Our purpose of this paper is to obtain the following orthogonality relations.

Theorem 2.1. *For a positive integer r, we assume that $k = (k_0, ..., k_{r-1})$, $k' = (k'_0, ..., k'_{r-1})$ and $\ell = (\ell_0, ..., \ell_{r-1})$ are elements of $\mathbb{N}_0^r$ such that $\sum_{i=0}^{r-1} k_i = \sum_{i=0}^{r-1} k'_i = \sum_{i=0}^{r-1} \ell_i = n$. We put $\tilde{\ell} = (\ell_1, ..., \ell_{r-1})$ for $\ell = (\ell_0, \ell_1, ..., \ell_{r-1})$, $\xi = \exp(2\pi\sqrt{-1}/r)$ and $\tilde{\Xi}_r = (1 - \xi^{ij})_{1 \leq i,j \leq r-1}$. Then we have*

$$\frac{1}{r^n} \sum_{\ell_0 + \cdots + \ell_{r-1} = n} \binom{n}{\ell_0, ..., \ell_{r-1}} F(-\tilde{\ell}, -\tilde{k}; -n; \tilde{\Xi}_r) \overline{F(-\tilde{\ell}, -\tilde{k}'; -n; \tilde{\Xi}_r)}$$

$$= \binom{n}{k_0, ..., k_{r-1}}^{-1} \prod \delta_{kk'}.$$

Theorem 2.2. *For a positive integer $m = [r/2]$, we assume that $k = (k_0, ..., k_m)$, $k' = (k'_0, ..., k'_m)$ and $\ell = (\ell_0, ..., \ell_m)$ are elements of $\mathbb{N}_0^{m+1}$ such that $\sum_{i=0}^{r-1} k_i = \sum_{i=0}^{r-1} k'_i = \sum_{i=0}^{r-1} \ell_i = n$. We put $\tilde{\Theta}_r = (1 - \cos(2\pi ij/r))_{1 \leq i,j \leq m}$. Then we have*

(1) *If r is an odd positive integer,*

$$\frac{1}{r^n} \sum_{\ell_0 + \cdots + \ell_m = n} 2^{n-\ell_0} \binom{n}{\ell_0, ..., \ell_m} F(-\tilde{\ell}, -\tilde{k}; -n; \tilde{\Theta}_r) F(-\tilde{\ell}, -\tilde{k}'; -n; \tilde{\Theta}_r)$$

$$= 2^{-n+k_0} \binom{n}{k_0, ..., k_m}^{-1} \delta_{kk'}.$$

(2) *If r is an even positive integer,*

$$\frac{1}{r^n} \sum_{\ell_0 + \cdots + \ell_m = n} 2^{n-\ell_0-\ell_m} \binom{n}{\ell_0, ..., \ell_m} F(-\tilde{\ell}, -\tilde{k}; -n; \tilde{\Theta}_r) F(-\tilde{\ell}, -\tilde{k}'; -n; \tilde{\Theta}_r)$$

$$= 2^{-n+k_0+k_m} \binom{n}{k_0, ..., k_m}^{-1} \delta_{kk'}.$$

Actually these relations are obtained from orthogonality of zonal spherical functions of the Gelfand pair of finite groups.

3. Theory of Zonal Spherical Functions on Finite Groups

Let G be a finite group and H be its subgroup.

Definition 3.1. If the induced representation 1_H^G is multiplicity free, then the pair (G, H) is called a *Gelfand pair*.

Assume from now that (G, H) is a Gelfand pair and the induced representation is decomposed as G-module;

$$V = 1_H^G = \bigoplus_{i=1}^{s} V_i, \; V_i \not\cong V_j \; (i \neq j).$$

It is a well known fact that s equals $|H\backslash G/H|$. We denote by $\{g_i; 1 \leq i \leq s\}$ the set of representatives of the double coset $H\backslash G/H$. Put $D_i = Hg_iH$. Let V_i^H be an H-invariant subspace of V_i. Using the Frobenius reciprocity we have;

$$\dim V_i^H = \langle V_i, 1_H \rangle_H = \langle V_i, 1_H^G \rangle_G = 1.$$

Here $\langle V, W \rangle_G$ denotes the intertwining number. Let $[*|*]$ be a G-invariant Hermitian scalar product on V_i. We assume that $\dim V_i = n$. Now we can choose $\{v_1^i, \cdots, v_n^i\}$ as an orthonormal basis of V_i and $v_1^i \in V_i^H$. Let $(\rho_{k\ell}^i)_{1 \leq k,\ell \leq n}$ be a matrix representation of G afforded by V_i. We denote by $C(G/H)$ the set of functions which have constant value on each right coset, i. e.,

$$C(G/H) := \{f : G \to \mathbb{C}; \; f(xh) = f(x) \; \forall x \in G, \; \forall h \in H\}.$$

It is clear that $\dim C(G/H) = [G : H]$. Define a linear map

$$\varphi_i : V_i \longrightarrow C(G/H)$$

by

$$\varphi_i(v)(g) = [v|gv_1^i]$$

for $g, h \in G$ and $v \in V_i$. Since

$$\varphi_i(gv)(k) = [gv|kv_1^i] = [v|g^{-1}kv_1^i] = \varphi_i(v)(g^{-1}k) = (g\varphi_i(v))(k)$$

and $\varphi \not\equiv 0$, φ is an injective G-linear map. Now we obtain the following

$$C(G/H) = \bigoplus_{i=1}^{s} \varphi_i(V_i).$$

We define $\omega_i \in \varphi_i(V_i)$ to be a function such that $\omega_i(g) = [v_1^i|gv_1^i] = \overline{\rho_{11}^i(g)}$ for any element $g \in G$. As can be seen in the argument above we see

$$\varphi_i(V_i)^H = \mathbb{C}\omega_i.$$

Definition 3.2. The functions ω_i are called *zonal spherical functions* of Gelfand pair (G, H).

We list some easy cosequences from definition of zonal spherical functions.

Proposition 3.3. (1) *$\omega_i(h_1gh_2) = \omega_i(g)$ for any $g \in G$ and $h_1, h_2 \in H$.*

(2) *$\omega_i(1) = 1$ and $\omega_i(g^{-1}) = \overline{\omega_i(g)}$ for any $g \in G$.*

Proposition 3.4. *If we write* $\omega_i(D_k)=\omega_i(g_i)$ *for* $g\in D_k$, *then*

$$\frac{1}{|G|}\sum_{k=1}^{s}|D_k|\omega_i(D_k)\overline{\omega_j(D_k)}=\delta_{ij}\dim V_i^{-1}.$$

4. Zonal Spherical Functions of $(G(r,1,n),S_n)$

Fix $r\in\mathbb{Z}_+$ and $n\in\mathbb{N}_0=\{0,1,2,...\}$. Put $\xi=\exp(2\pi\sqrt{-1}/r)$. Let $C^n=\langle\xi\rangle\times\cdots\times\langle\xi\rangle$ denote the n-fold direct product of the cyclic group $C=\langle\xi\rangle$. The symmetric group S_n acts on C^n by:

$$\sigma(\xi_1,\xi_2,...,\xi_n)=(\xi_{\sigma^{-1}(1)},\xi_{\sigma^{-1}(2)},...,\xi_{\sigma^{-1}(n)}),\ (\xi_1,\xi_2,...,\xi_n)\in C^n,\ \sigma\in S_n.$$

The wreath product $C\wr S_n$ is the semidirect product of C^n with S_n defined by this action [11, 15]. Let denote $G(r,1,n)=C\wr S_n$. The conjugacy classes and the irreducible representations of $G(r,1,n)$ are determined by the r-tuples of partitions $(\nu^0,\dots,\nu^{r-1})$ such that $|\nu^0|+\cdots+|\nu^{r-1}|=n$. In this section we consider the pair of groups $G=G(r,1,n)$ and its subgroup $H=G(1,1,n)=S_n$.

Proposition 4.1. (1) *The representatives of double coset* $H\backslash G/H$ *are given by*

$$\Big\{(\underbrace{1,\cdots,1}_{\ell_0},\underbrace{\xi,\cdots,\xi}_{\ell_1},\cdots,\underbrace{\xi^{r-1},\cdots,\xi^{r-1}}_{\ell_{r-1}};e)\in G;\ \sum_{i=0}^{r-1}\ell_i=n\Big\},$$

weher e *is a unit element of* S_n.

(2)

$$|H(\underbrace{1,\cdots,1}_{\ell_0},\underbrace{\xi,\cdots,\xi}_{\ell_1},\cdots,\underbrace{\xi^{r-1},\cdots,\xi^{r-1}}_{\ell_{r-1}};e)H|=\binom{n}{\ell_0,\ell_1,\cdots,\ell_{r-1}}n!.$$

The group G acts on the ring of polynomials of n-variables as

$$(\xi_1,\xi_2,\cdots,\xi_n;\sigma)f(x_1,\cdots,x_n)=f(\xi_{\sigma(1)}^{-1}x_{\sigma(1)},\xi_{\sigma(2)}^{-1}x_{\sigma(2)},\cdots,\xi_{\sigma(n)}^{-1}x_{\sigma(n)}).$$

We define the map from $\mathbb{N}_0^r$ to the set of partitions Par as follows.

$$\psi:\mathbb{N}_0^r\ni(k_0,k_1,\cdots,k_{r-1})\mapsto(0^{k_0}1^{k_1}\cdots(r-1)^{k_{r-1}})\in Par.$$

Proposition 4.2. *The induced representation* $1_{S_n}^{G(r,1,n)}$ *is decomposed as the following.*

$$1_{S_n}^{G(r,1,n)}\cong\bigoplus_{\sum_{i=0}^{r-1}k_i=n}V^{(k_0,k_1,\cdots,k_{r-1})}.$$

Each $V^{(k_0,k_1,\cdots,k_{r-1})}$ *is an irreducible* $G(r,1,n)$*-module which is realized as follows;*

$$V^{(k_0,k_1,\cdots,k_{r-1})}=\bigoplus_{f\in M_n\left(\psi(k_0,k_1,\cdots,k_{r-1})\right)}\mathbb{C}f.$$

Here $M_n(\lambda)=\{x_{\sigma(1)}^{\lambda_1}x_{\sigma(2)}^{\lambda_2}\cdots x_{\sigma(n)}^{\lambda_n};\sigma\in S_n\}$ *for* $\lambda=(\lambda_1,\cdots,\lambda_n)$.

Since this decomposition is multiplicity free [4], we have the following proposition.

Proposition 4.3. *(G, H) is a Gelfand pair.*

Example 4.4. We take $G = G(3, 1, 4)$ and $H = S_4$. Then the induced representation 1_H^G is decomposed as follows:

$$\begin{aligned} 1_H^G = & V^{(4,0,0)} \oplus V^{(0,4,0)} \oplus V^{(0,0,4)} \oplus V^{(3,1,0)} \oplus V^{(3,0,1)} \\ & \oplus V^{(1,3,0)} \oplus V^{(1,0,3)} \oplus V^{(0,3,1)} \oplus V^{(0,1,3)} \oplus V^{(2,1,1)} \\ & \oplus V^{(1,2,1)} \oplus V^{(1,1,2)} \oplus V^{(2,2,0)} \oplus V^{(2,0,2)} \oplus V^{(0,2,2)}. \end{aligned}$$

We write down a basis of some irreducible components.

$$V^{(1,1,2)} = \bigoplus_{\{i_1,i_2,i_3\} \subset \{1,2,3,4\}} \mathbb{C} x_{i_1}^2 x_{i_2}^2 x_{i_3}, \quad V^{(0,4,0)} = \mathbb{C} x_1 x_2 x_3 x_4.$$

The S_4-invariant element of $V^{(1,2,2)}$ is a monomial symmetric function

$$m_{(2,2,1)}(x_1, x_2, x_3, x_4).$$

We define the inner product on 1_H^G as follows

$$[\alpha x^\lambda | \beta x^\mu] = \alpha \overline{\beta} \delta_{\lambda,\mu} \frac{1}{\binom{n}{k_0, k_1, \cdots, k_{r-1}}}.$$

Here α and β are complex numbers, k_i is the number of parts of λ which is equal to i, and $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_2}$. It is easy to see that this inner product is $G(r, 1, n)$-invariant, i.e.,

$$[(gf_1)(x) | (gf_2)(x)] = [f_1(x) | f_2(x)]$$

for $g \in G(r, 1, n)$, $f_1(x), f_2(x) \in V^{(k_0, k_1, \cdots, k_{r-1})}$. We recall the monomial symmetric functions for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) = (0^{k_0} 1^{k_1} 2^{k_2} \cdots (r-1)^{k_{r-1}})$. Clearly the monomial symmetric functions satisfy

$$[m_\lambda(x) | m_\mu(x)] = \delta_{\lambda\mu}.$$

Theorem 4.5. *The zonal spherical functions of Gelfand pair (G, H) are*

$$\omega^{(k_0, k_1, \cdots, k_{r-1})}(\xi_1, \xi_2, \cdots, \xi_n; \sigma) = m_\lambda(\xi_1, \xi_2, \ldots, \xi_n) / m_\lambda(1, \cdots, 1)$$

where $\lambda = (0^{k_0} 1^{k_1} 2^{k_2} \cdots (r-1)^{k_{r-1}})$ and $\sum_{i=0}^{r-1} k_i = n$.

For $\lambda = (0^{k_0} 1^{k_1} 2^{k_2} \cdots (r-1)^{k_{r-1}})$, we define

$$m_{(\ell_0, \ell_1, \ldots, \ell_{r-1})}^{(k_0, k_1, \ldots, k_{r-1})} = m_\lambda(\underbrace{1, \ldots, 1}_{\ell_0}, \underbrace{\xi, \ldots, \xi}_{\ell_1}, \cdots, \underbrace{\xi^{r-1}, \ldots, \xi^{r-1}}_{\ell_{r-1}}).$$

Proposition 4.6. *We assume that $\sum_{i=0}^{r-1} \ell_i = \sum_{i=0}^{r-1} k_i = n$.*

(1)

$$m^{(k_0,k_1,\cdots,k_{r-1})}_{(\ell_0,\ell_1,\cdots,\ell_{r-1})} = \sum_{a\in\mathcal{A}}\prod_{i=0}^{r-1}\binom{\ell_i}{a_{i0},a_{i1},\cdots,a_{ir-1}}\xi^{\sum_{0\le i,j\le r-1} ija_{ij}},$$

where

$$\mathcal{A} = \mathcal{A}^{(k_0,k_1,\cdots,k_{r-1})}_{(\ell_0,\ell_1,\cdots,\ell_{r-1})} = \{a=(a_{ij})\in M(r,\mathbb{N}_0);\ \sum_{i=0}^{r-1} a_{ij} = k_j,\ \sum_{j=0}^{r-1} a_{ij} = \ell_i\}.$$

(2) *The generating function is given by*

$$\sum_{k_0+\cdots+k_{r-1}=n} m^{(k_0,k_1,\cdots,k_{r-1})}_{(\ell_0,\ell_1,\cdots,\ell_{r-1})} t_0^{k_0}t_1^{k_1}\cdots t_{r-1}^{k_{r-1}} = \prod_{i=0}^{r-1}(\sum_{j=0}^{r-1}\xi^{ij}t_j)^{\ell_i}.$$

Example 4.7. We consider the case of $r=3$ and $n=4$. Put $(k_0,k_1,k_2) = (1,1,2)$ and $(\ell_0,\ell_1,\ell_2)=(1,2,1)$. A direct computation gives us

$$\begin{aligned}\omega^{(1,1,2)}_{(1,2,1)} &= \frac{1}{12}m_{2^21^1}(1,\xi,\xi,\xi^2)\\ &= \frac{1}{12}(2\xi^3+3\xi^4+2\xi^5+3\xi^6+2\xi^7) = -\frac{1}{4}\xi^2.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\mathcal{A}^{(1,1,2)}_{(1,2,1)} = \Bigg\{ &\begin{pmatrix}1&&\\&&2\\&1&\end{pmatrix}, \begin{pmatrix}&&1\\1&1&\\&&1\end{pmatrix}, \begin{pmatrix}1&&\\&1&1\\&&1\end{pmatrix}, \begin{pmatrix}&&1\\1&&1\\&1&\end{pmatrix},\\ &\begin{pmatrix}&1&\\1&&1\\&&1\end{pmatrix}, \begin{pmatrix}&&1\\&1&1\\1&&\end{pmatrix}, \begin{pmatrix}&1&\\&&2\\1&&\end{pmatrix}\Bigg\}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\omega^{(1,1,2)}_{(1,2,1)} = \frac{1}{12}m^{(1,1,2)}_{(1,2,1)} &= \frac{1}{12}(\xi^6+2\xi^5+2\xi^7+2\xi^4+2\xi^6+2\xi^3+\xi^4)\\ &= \frac{1}{12}(2\xi^3+3\xi^4+2\xi^5+3\xi^6+2\xi^7) = -\frac{1}{4}\xi^2.\end{aligned}$$

Theorem 4.8. *The zonal spherical functions of Gelfand pair $(G(r,1,n),S_n)$ have $(n+1,m+1)$-hypergeometric expressions*

$$\omega^{(k_0,k_1,\cdots,k_{r-1})}_{(\ell_0,\ell_1,\cdots,\ell_{r-1})} = F((-\ell_1,\cdots,-\ell_{r-1}),(-k_1,\cdots,-k_{r-1});-n;\tilde{\Xi}_r).$$

Here $\tilde{\Xi}_r = (1-\xi^{ij})_{1\le i,j\le r-1}$ *with* $\xi = \exp(2\pi\sqrt{-1}/r)$.

5. Zonal Spherical Functions of $(D(r,n),D(1,n))$

Let

$$D_r = \langle a,b; a^2=b^r=(ab)^2=1\rangle$$

be the dihedral group of order $2r$. We denote by $G=D(r,n)=D_r\wr S_n$. We define the subgroup H of G by

$$H = \langle a\rangle \wr S_n \cong D(1,n).$$

We consider the pair of groups (G, H).

We remark that $D(1,n) \cong W(B_n)$, where $W(B_n)$ is the Weyl group of type B and that $D(2,n) \cong V_4 \wr S_n$, where V_4 denotes by Kleinsche Vierergruppe . We define another subgroup K of G by

$$K = \langle b \rangle \wr S_n \cong G(r,1,n),$$

where $G(r,1,n)$ is the imprimitive complex reflection group.

Proposition 5.1. (1) *The representatives of double coset $H\backslash G/H$ are given by*

$$\Big\{(\underbrace{1,...,1}_{\ell_0},\underbrace{b,...,b}_{\ell_1},\dots,\underbrace{b^m,...,b^m}_{\ell_m};e) \in G;\ \sum_{i=0}^{m} \ell_i = n\Big\},$$

where $m = \frac{r-1}{2}$ if r is odd, $m = \frac{r}{2}$ if r is even, and e is a unit element of S_n.

(2)

$$|H(\underbrace{1,...,1}_{\ell_0},\underbrace{b,...,b}_{\ell_1},\dots,\underbrace{b^m,...,b^m}_{\ell_m};e)H| = \begin{cases} 2^{2n-\ell_0}\binom{n}{\ell_0,\cdots,\ell_m}n!, & \textit{if } r = 2m+1 \\ 2^{2n-\ell_0-\ell_m}\binom{n}{\ell_0,\cdots,\ell_m}n!, & \textit{if } r = 2m. \end{cases}$$

Proposition 5.2. *The induced representation 1_H^G is decomposed as follows.*

$$1_H^G \cong \bigoplus_{\sum_{i=0}^m k_i = n} W^{(k_0,k_1,\dots,k_m)}.$$

Each $W^{(k_0,k_1,\dots,k_m)}$ is an irreducible G-module which is realized as follows;

$$W^{(k_0,k_1,\dots,k_m)} = \bigoplus_{f \in M_n\left(\psi(k_0,k_1,\dots,k_m)\right)} \mathbb{C}f.$$

Here, in the case that $r = 2m+1$,

$$M_n(\lambda) = \left\{x_{\sigma(1)}^{\epsilon_1\lambda_1}x_{\sigma(2)}^{\epsilon_2\lambda_2}\cdots x_{\sigma(n)}^{\epsilon_n\lambda_n};\epsilon_i \in \{\pm 1\}, \sigma \in S_n\right\},$$

and if $r = 2m$,

$$M_n(\lambda) = \Big\{(x_{\sigma(1)}^{\lambda_1}+x_{\sigma(1)}^{-\lambda_1})(x_{\sigma(2)}^{\lambda_2}+x_{\sigma(2)}^{-\lambda_2})\cdots(x_{\sigma(k_m)}^{\lambda_{k_m}}+x_{\sigma(k_m)}^{-\lambda_{k_m}}) \\ \times x_{\sigma(k_m+1)}^{\epsilon_{k_m+1}\lambda_{k_m+1}}x_{\sigma(k_m+2)}^{\epsilon_{k_m+2}\lambda_{k_m+2}}\cdots x_{\sigma(n)}^{\epsilon_n\lambda_n};\epsilon_i \in \{\pm 1\}, \sigma \in S_n\Big\}$$

for $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda_i \geq \lambda_{i+1} \geq 0$.

By this proposition we can say that $(G,H) = (D(r,n), D(1,n))$ is a Gelfand pair. We define the inner product on each $W^{(k_0,\dots,k_m)}$ as follows;

$$[\alpha x_1^{\epsilon_1\lambda_1}\cdots x_n^{\epsilon_n\lambda_n}|\beta x_1^{\eta_1\mu_1}\cdots x_n^{\eta_n\mu_n}] = \frac{\alpha\overline{\beta}}{\binom{n}{k_0,k_1,\dots,k_m}2^{n-k_0}}\prod_{i=1}^{n}\delta_{\epsilon_i\lambda_i,\eta_i\mu_i}.$$

Here $\alpha, \beta \in \mathbb{C}, \epsilon_i, \eta_i \in \{\pm 1\}$, and $(\lambda_1, \lambda_2, ..., \lambda_n) = (0^{k_0}1^{k_1}\cdots m^{k_m})$. It is easy to see that this inner product is G-invariant on $W^{(k_0,...,k_m)}$. For $\lambda = (\lambda_1, ..., \lambda_n) = (0^{k_0}1^{k_1}\cdots m^{k_m})$, we define the polynomial of n variables by

$$f_\lambda(x) = \frac{2^{-k_0}}{k_0!k_1!\cdots k_m!}\sum_{\sigma\in S_n}(x_{\sigma(1)}^{\lambda_1} + x_{\sigma(1)}^{-\lambda_1})(x_{\sigma(2)}^{\lambda_2} + x_{\sigma(2)}^{-\lambda_2})\cdots(x_{\sigma(n)}^{\lambda_n} + x_{\sigma(n)}^{-\lambda_n}).$$

Note that f_λ satisfies

$$[f_\lambda(x)|f_\mu(x)] = \delta_{\lambda\mu}.$$

Let $g = (a^{s_1}b^{t_1}, ..., a^{s_n}b^{t_n}; \sigma) \in G$.

Theorem 5.3. *The zonal spherical function of the Gelfand pair* (G, H) *are*

$$\omega^{(k_0,k_1,...,k_m)}(a^{s_1}b^{t_1}, a^{s_2}b^{t_2}, ..., a^{s_n}b^{t_n}; \sigma) = f_\lambda(\xi^{t_1}, \xi^{t_2}, ..., \xi^{t_n})/f_\lambda(1, ..., 1),$$

where $\lambda = (0^{k_0}1^{k_1}2^{k_2}\cdots m^{k_m})$ *and* $\sum_{i=0}^m k_i = n$.

For $\lambda = (0^{k_0}1^{k_1}2^{k_2}\cdots m^{k_m})$, we define

$$f^{(k_0,k_1,...,k_m)}_{(\ell_0,\ell_1,...,\ell_m)} = \frac{1}{2^{n-k_0}}f_\lambda(\underbrace{1, ..., 1}_{\ell_0}, \underbrace{\xi, ..., \xi}_{\ell_1}, \cdots, \underbrace{\xi^m, ..., \xi^m}_{\ell_m}).$$

Proposition 5.4.

$$f^{(k_0,k_1,...,k_m)}_{(\ell_0,\ell_1,...,\ell_m)} = \sum_{a\in\mathcal{A}}\prod_{i=0}^m\binom{\ell_i}{a_{i0}, a_{i1}, ..., a_{im}}\prod_{0\le i,j\le m}\Big(\cos\Big(\frac{2\pi}{r}ij\Big)\Big)^{a_{ij}},$$

where

$$\mathcal{A} = \mathcal{A}^{(k_0,k_1,...,k_m)}_{(\ell_0,\ell_1,...,\ell_m)} = \Big\{a = (a_{ij}) \in M(m+1, \mathbb{N}_0); \sum_{i=0}^m a_{ij} = k_j, \sum_{j=0}^m a_{ij} = \ell_i\Big\}.$$

Theorem 5.5.

$$\omega^{(k_0,k_1,...,k_m)}_{(\ell_0,\ell_1,...,\ell_m)} = F\big((-\ell_1, ..., -\ell_m), (-k_1, ..., -k_m); -n; \tilde{\Theta}_r\big)$$

Here $\tilde{\Theta}_r = (1 - \cos(2\pi ij/r))_{1\le i,j\le m}$.

6. General Result

In this section we consider a generalization of our main results. We remark that, in Theorem 2.1,

$$\tilde{\Xi}_r = J_{r-1} - (\xi^{ij})_{1\le i,j\le r-1}.$$

Here $\Xi_r = (\xi^{ij})_{0\le i,j\le r-1}$ is a table of zonal spherical functions of Gelfand pair $(\mathbb{Z}/r\mathbb{Z}, 1)$ and, in Theorem 2.2,

$$\tilde{\Theta}_r = J_m - (\cos 2\pi ij/r)_{1\le i,j\le m}.$$

Here $\Theta_r = (\cos 2\pi ij/r)_{0\le i,j\le m}$ is a table of zonal spherical functions of Gelfand pair $(D_r, \langle a\rangle)$.

We assume that (G,H) is a Gelfand pair and the induced representation 1_H^G is decomposed as follows:

$$1_H^G \cong \bigoplus_{i=0}^{s-1} V_i, \ \dim V_i = d_i.$$

Let $Z(G,H)$ be a table of zonal spherical functions of (G,H). Then we have the Gelfand pair $(G \wr S_n, H \wr S_n)$. We obtain next theorem.

Theorem 6.1. *The zonal spherical functions of Gelfand pair $(G \wr S_n, H \wr S_n)$ have $(n+1,m+1)$-hypergeometric expressions*

$$\omega_{(\ell_0,\ell_1,\ldots,\ell_{s-1})}^{(k_0,k_1,\ldots,k_{s-1})} = F((-\ell_1,...,-\ell_{s-1}),(-k_1,...,-k_{s-1});-n;J_{s-1}-\tilde{Z}(G,H)).$$

Here J_{s-1} is a $s-1 \times s-1$ all-one-matrix and $\tilde{Z}(G,H)$ is a matrix which is obtained by removing 0th row and 0th column of $Z(G,H)$.

References

[1] H. Akazawa and H. Mizukawa, Orthogonal polynomials arising from the wreath products of dihedral group, Preprint 2002.

[2] E. Andrews, R. Askey and R. Roy *Special Functions,* Encyclopedia of Mathematics and its Applications, Cambridge, 1999

[3] K. Aomoto and M. Kita *Theory of Hypergeometric Functions(in Japanese),* Springer Tokyo, 1994

[4] S. Ariki, T. Terasoma and H. -F. Yamada, Higher Specht polynomials, Hiroshima Math. J. 27 (1997), no. 1, 177-188.

[5] E. Bannai and T. Ito, *Algebraic Combinatorics I. Association Schemes,* The Benjamin/Cummings Publishing Co. CA, 1984

[6] C. Dunkl, A Krawtchouk polynomial addition theorem and wreath products of symmetric groups, Indiana Univ. Math. J. 25 (1976), no. 4, 335-358.

[7] C. Dunkl, Cube group invariant spherical harmonics and Krawtchouk polynomials, Math. Z. 177 (1981), no. 4, 561-57

[8] C. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables,* Encyclopedia of Mathematics and its Applications, 81. Cambridge University Press, Cambridge, 2001.

[9] I. M. Gelfand, General theory of hypergeometric functions (in Russian), Dokl. Akad. Nauk SSSR 288 (1986), no. 1, 14-18.

[10] I. M. Gelfand and S. I. Gelfand, Generalized hypergeometric equations (in Russian), Dokl. Akad. Nauk SSSR 288 (1986), no. 2, 279-283

[11] G. James and A. Kerber, *The Representation Theory of the Symmetric Group,* Encyclopedia of Mathematics and its Applications, 16, 1981.

[12] M. Kita, On hypergeometric functions in several variables. II. The Wronskian of the hypergeometric functions of type $(n+1,m+1)$, J. Math. Soc. Japan 45 (1993), no. 4, 645-669.

[13] M. Kita and M. Ito, On the rank of the hypergeometric system $E(n+1,m+1;\alpha)$, Kyushu J. Math. 50 (1996), no. 2, 285-295.

[14] H. Koelink, q-Krawtchouk polynomials as spherical functions on the Hecke algebra of type B, Trans. Amer. Math. Soc. 352 (2000), no. 10, 4789-4813.

[15] I. G. Macdonald, *Symmetric Functions and Hall Polynomials, 2nd. ed.* , Oxford, 1995.

[16] H. Mizukawa, Zonal spherical functions of $(G(r,1,n),S_n)$ and $(n+1,m+1)$-hypergeometric functions, Preprint 2002.

[17] D. Stanton, Some q-Krawtchouk polynomials on Chevalley groups, Amer. J. Math. 102, 625-662 (1980), no. 4

[18] D. Stanton, Three addition theorems for some q-Krawtchouk polynomials, Geom. Dedicata 10 (1981), no. 1-4, 403-425

[19] M. Yoshida, *Hypergeometric Functions, My Love. Modular Interpretations of Configuration Spaces.* Aspects of Mathematics, E32. Friedr. Vieweg and Sohn, Braunschweig, 1997.

Hiroki Akazawa, Graduate School of Natural Science and Technology, Okayama University, Okayama 700-8530, Japan

E-mail address: akazawa@math.okayama-u.ac.jp

Hiroshi Mizukawa, Department of Mathematics, Okayama University, Okayama 700-8530, Japan

E-mail address: mzh@math.okayama-u.ac.jp