EXPECTED REFLECTION DISTANCE IN $G(r,1,n)$ AFTER A FIXED NUMBER OF REFLECTIONS

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ABSTRACT. Extending to $r > 1$ a formula of the authors, we compute the expected reflection distance of a product of t random reflections in the complex reflection group $G(r,1,n)$. The result relies on an explicit decomposition of the reflection distance function into irreducible $G(r,1,n)$-characters and on the eigenvalues of certain adjacency matrices.
RÉSUMÉ En tendant $r > 1$ une formule de l'auteur, nous calculons la distance de rflexion attendue d'un produit de t rflexions alatoires dans le groupe de rflexions complexe $G(r,1,n)$. Le rsultat s'appuie sur une dcomposition explicite de la fonction de distance de rflexion en irrductibles caractres $G(r,1,n)$ et sur des valeurs Eigen de certaines matrices adjacentes.

1. INTRODUCTION

Consider the graph G'_{1n} with the symmetric group $\mathfrak{S}_n$ as vertex set and an edge $\{\pi, \tau\}$ if and only if $\pi = \tau t$ for some transposition t. In [3], the authors computed the expected distance (in the graph-theoretic sense) from the identity after a random walk on G'_{1n} with a fixed number of steps. The motivation was that a random walk on G'_{1n} is a good approximation of a random walk on a graph originating from computational biology: its vertices are the genomes with n genes and its edges correspond to evolutionary events called reversals. Thus, a random walk on the latter graph is thought to simulate evolution. Solving the inverse problem, to compute the expected number of steps given a fixed distance, would then provide a measure for how closely related two taxa are.

In this paper we generalise the mathematical results of [3]. More precisely, we solve the problem described above with $\mathfrak{S}_n$ replaced by the complex reflection group $G(r,1,n)$ and the transpositions replaced by the set of reflections.

Our approach is analogous to that in [3]. We view the random walk as a Markov process with a certain transition matrix. This yields an expression for the expected distance which involves two unknown parts, namely the eigenvalues of the said matrices and the inner product of certain (virtual) $G(r,1,n)$-characters. The eigenvalues are computed using the Murnaghan-Nakayama type formula for $G(r,1,n)$ given by Ariki and Koike [1].

The inner product is computed using elements of the "symmetric functions" theory of $G(r,1,n)$-representations, thereby generalising the corresponding result in [3] to $G(r,1,n)$.

The paper is organised as follows. In Section 2 we review some material about the groups $G(r,1,n)$ and define the appropriate graphs. Thereafter, in Section 3, we give a brief sketch of the symmetric functions-like theory that governs $G(r,1,n)$-representations. We then describe the Markov chain approach and state the main theorem in Section 4. We do not prove it until in Section 7, though, since the proof relies on the computation of the eigenvalues and inner product described above; these computations take place in Sections 5 and 6, respectively.

2. The groups $G(r,1,n)$ and their reflection graphs

Choose positive integers r and n. Let ζ be a primitive rth root of unity in $\mathbb{C}$. We will view $G(r,1,n)$ as the group of permutations π of the set $\{\zeta^i j \mid i \in [r],\, j \in [n]\}$ such that $\pi(\zeta^i j) = \zeta^i \pi(j)$ for all $i \in [r]$, $j \in [n]$. The special cases $r=1$ and $r=2$ yield the symmetric group $\mathfrak{S}_n$ and the hyperoctahedral group B_n, respectively. Both are real reflection groups. In general, $G(r,1,n)$ is a complex reflection group, namely the symmetry group of the regular complex polytope known as the generalised cross-polytope β_n^r (see [6]). Also note that $G(r,1,n)$ is isomorphic to the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$.

An r**-partition** λ of n, written $\lambda \vdash_r n$, is an r-tuple of integer partitions $\lambda = (\lambda^1, \dots, \lambda^r)$ such that $n = \sum |\lambda^i|$. Consider $\pi \in G(r,1,n)$. It gives rise to an r-partition $\operatorname{type}(\pi) = (\lambda^1, \dots, \lambda^r) \vdash_r n$ as follows. Write down the disjoint cycle decomposition of π and consider only the absolute values of the entries. This causes some cycles to coincide; those that do belong to the same equivalence class called a **class cycle**. Each class cycle c corresponds to a part in λ^i, i being determined by the requirement that $\pi^k(j) = \zeta^{i-1} j$ for the smallest $k > 0$ such that $|\pi^k(j)| = |j|$, where j is any entry in (a representative of) c. The size of the part is the number of entries in c divided by r. It is straightforward to verify that π and τ are conjugate if and only if $\operatorname{type}(\pi) = \operatorname{type}(\tau)$. Thus, the r-partitions of n index the conjugacy classes (and, hence, the irreducible characters) of $G(r,1,n)$.

Example 2.1. *With $r = n = 4$ and $\zeta = i = \sqrt{-1}$, the element*

$$(1\ -2)(i\ -2i)(-1\ 2)(-i\ 2i)(3\ -4\ -3\ 4)(3i\ -4i\ -3i\ 4i) \in G(r,1,n)$$

contains two class cycles and has type $(\square\!\square, \emptyset, \square\!\square, \emptyset)$.

The element π is a **reflection** if λ^1 has exactly $n-1$ parts. We let $R = R(n,r)$ denote the set of reflections. Note that $R(n,1)$ is just the set of transpositions in $\mathfrak{S}_n$.

Although an arbitrary $t \in R$ is not in general conjugate to t^{-1}, we still have $t^{-1} \in R$. Hence, there is no ambiguity in the definition we now give. We let G'_{rn} be the graph with the elements of $G(r,1,n)$ as vertices and an edge $\{x,y\}$ if and only if $x = yt$ for some $t \in R$. We call G'_{rn} the **reflection graph** of $G(r,1,n)$.

It is well-known that the reflection distance $w_{rn}(\pi)$, i.e. the graph-theoretic distance between the identity and π in G'_{rn}, is given by n minus the number of class cycles of π that contain exactly r cycles. In other words,

$$w_{rn}(\pi) = n - \ell(\lambda^1). \tag{1}$$

Remark 2.2. In case $r \in \{1,2\}$, the reflection graph is just the undirected version of the Bruhat graph on the Coxeter group $G(r,1,n)$ defined by Dyer [2].

3. Irreducible characters and "symmetric functions"

In this section, we briefly review some of the theory of $G(r,1,n)$-representations, which in many ways resembles the theory of symmetric functions. We refer to Macdonald [5, Ch. I, App. B] for more details. Some knowledge of "ordinary" symmetric functions will be assumed, see e.g. Stanley [7, Ch. 7] or [5].

The irreducible characters of $G(r,1,n)$ are indexed by the r-partitions of n; we write χ^λ for the character indexed by $\lambda \vdash_r n$. They form an orthonormal basis of the $\mathbb{C}$-vector space $R^n(r)$ of class functions on $G(r,1,n)$ with respect to the inner product

$$\langle f, g\rangle = \sum_{\lambda \vdash_r n} \frac{f(\lambda)\overline{g(\lambda)}}{Z_\lambda},$$

where

$$Z_\lambda = z_{\lambda^1} \dots z_{\lambda^r} r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)}.$$

For $i \in [r]$, let $x_i = (x_{i1}, x_{i2}, \dots)$. Given $\lambda \vdash_r n$, we define

$$P_\lambda = \prod_{i=1}^{r} p_{\lambda^i}(x_i) \in \mathbb{C}[x_1, \dots, x_r],$$

where the p_μ are the ordinary power sum functions. Let $\Lambda^n(r)$ denote the $\mathbb{C}$-span of $\{P_\lambda\}_{\lambda\vdash_r n}$. It turns out that the **characteristic map** $\mathrm{ch}^n : R^n(r) \longrightarrow \Lambda^n(r)$ given by $f \mapsto \sum_{\lambda \vdash_r n} \frac{P_\lambda}{Z_\lambda} f(\lambda)$ is a vector space isomorphism.

Polynomial multiplication turns $\Lambda(r) = \bigoplus_{n\geq 0} \Lambda^n(r)$ into a graded algebra. The same holds for $R(r) = \bigoplus_{n\geq 0} R^n(r)$ (under a suitably defined multiplication whose nature needs not concern us here). Taking the characteristic map on each component then yields an isomorphism of graded algebras $\mathrm{ch} : R(r) \longrightarrow \Lambda(r)$ which we also call the characteristic map.

Now, consider another set of variables: for $i \in [r]$, put $\tilde{x}_i = (\tilde{x}_{i1}, \tilde{x}_{i2}, \dots)$. The connection between the x_i and the $\tilde{x}_i$ is governed by the transformation rules

$$p_m(\tilde{x}_i) = \sum_{j\in[r]} \frac{1}{r} T_{i,j} p_m(x_j)$$

and

$$p_m(x_i) = \sum_{j\in[r]} \overline{T_{j,i}} p_m(\tilde{x}_j),$$

where T is the character table of $\mathbb{Z}_r$. In particular, adopting the convention that the trivial $\mathbb{Z}_r$-character corresponds to the first row in T and the (conjugacy class consisting of the) identity element corresponds to the first column, we have $T_{i1} = T_{1i} = 1$ for all i. Hence,

$$p_m(\tilde{x}_1) = \sum_{j\in[r]} \frac{1}{r} p_m(x_j) \tag{2}$$

and

$$p_m(x_1) = \sum_{j\in[r]} p_m(\tilde{x}_j). \tag{3}$$

The main reason to care about this second set of variables is the following. For $\lambda \vdash_r n$, define

$$\widetilde{S}_\lambda = \prod_{i\in[r]} s_{\lambda^i}(\tilde{x}_i) \in \mathbb{C}[\tilde{x}_1, \tilde{x}_2, \ldots],$$

where the s_μ are the ordinary Schur functions. Then $\widetilde{S}_\lambda$ is the image of χ^λ under the characteristic map.

4. The Markov chain

We wish to view the walk on G'_{rn} as a Markov process. We can then use the properties of the transition matrix to compute the expected reflection distance. Our approach is analogous to the approach in [3].

Associated with the Cayley graph G'_{rn} is its adjacency matrix M'_{rn} with rows and columns indexed by the vertices in G'_{rn} and with entries indicating the number of edges (one or zero) between the corresponding vertices. The probability that a random walk on G'_{rn} starting in the identity ends up in π depends only on the type of π. Hence, to reduce the size of the problem, we may group the permutations into conjugacy classes, each indexed by its type. We then get the corresponding (multi-)graph G_{rn} with adjacency matrix $M_{rn} = (m_{ij})$, the number m_{ij} denoting the number of edges from some permutation of type i to any permutation of type j.

Example 4.1. *The group $G(2,1,2)$ has 8 elements of 5 different types. If the latter are ordered according to*

$$(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \emptyset), (\square\square, \emptyset), (\square, \square), (\emptyset, \begin{smallmatrix}\square\\\square\end{smallmatrix}), (\emptyset, \square\square),$$

we get

$$M_{22} = \begin{pmatrix} 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 & 2 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{pmatrix}$$

We view this adjacency matrix as a transition matrix in a Markov chain (after normalising by the common row sum $|R|$). It is easy to see that the expected reflection distance after t reflections taken from a uniform distribution is given by (see for instance [3]) $e_1 M_{rn}^t w_{rn}^T / |R|^t$, where w_{rn} is a vector containing the reflection distances from the different types to the identity. The vector e_1 has 1 in the first position and zeroes everywhere else.

In order to compute $e_1 M_{rn}^t w_{rn}^T$, we wish to diagonalise M_{rn}. It follows from Ito [4] that its eigenvalues are given by

$$\mathrm{eig}(M_{rn}, \lambda) = \sum_i \frac{n_i \chi^\lambda(i)}{\chi^\lambda(1^n, \emptyset, \ldots, \emptyset)}, \tag{4}$$

for $\lambda \vdash_r n$. Here, n_i is the number of elements of type i in $G(r,1,n)$, and the sum is taken over all reflection types i. For $r = 1$, the eigenvalues equal the **contents**

$$c(\lambda) = \frac{\binom{n}{2}\chi^\lambda(2, 1^{n-2})}{\chi^\lambda(1^n)} = \sum_{i=1}^{\ell(\lambda)} \left(\binom{\lambda_i}{2} - (i-1)\lambda_i \right) \tag{5}$$

(see [3, 4]). We will compute the other eigenvalues in Section 5.

The eigenvector corresponding to $\mathrm{eig}(M_{rn}, \lambda)$ is given by the values of χ^λ on the various conjugacy classes, see [4]. Hence, viewing the character table C as a matrix, we can diagonalise: $M_{rn} = C^T D (C^T)^{-1}$, where D is the diagonal matrix with the eigenvalues on the diagonal. Using the orthogonality of irreducible characters, we compute $(C^T)^{-1}$; it is obtained from C by dividing each column by its corresponding Z_μ. We obtain

$$e_1 M_{rn}^t w_{rn}^T = \sum_{\lambda \vdash_r n} \chi^\lambda((1^n), \emptyset, \ldots, \emptyset)(\mathrm{eig}(M_{rn}, \lambda))^t \sum_{\mu \vdash_r n} \frac{\chi^\lambda(\mu) w_{rn}(\mu)}{Z_\mu}.$$

In Section 6, we decompose $w_{rn}(\mu)$ into a linear combination of irreducible $G(r,1,n)$-characters, thus obtaining an expression for the second sum.

Combining all parts, we obtain the main theorem, thus extending the corresponding result for $r = 1$ in [3].

Theorem 4.2. *Assume $r, n \in \mathbb{N}$ and $rn > 1$. Then the expected reflection distance after t random reflections in $G(r,1,n)$ is given by*

$$n - \frac{1}{r}\sum_{k=1}^{n} \frac{1}{k} + \frac{1}{r} \sum_{p=1}^{n-1} \sum_{q=1}^{\min(p,n-p)} a_{pq} \left(\frac{r\left(\binom{p}{2} + \binom{q-1}{2} - \binom{n-p-q+2}{2} + n\right) - n}{r\binom{n+1}{2} - n} \right)^t$$
$$+ \frac{r-1}{r} \sum_{p=0}^{n-1} \sum_{q=1}^{n-p} b_{pq} \left(\frac{r\left(\binom{p}{2} + \binom{q}{2} - \binom{n-p-q+1}{2} + p\right) - n}{r\binom{n+1}{2} - n} \right)^t,$$

where

$$a_{pq} = (-1)^{n-p-q+1} \frac{(p-q+1)^2}{(n-q+1)^2(n-p)} \binom{n}{p} \binom{n-p-1}{q-1}$$

and

$$b_{pq} = \frac{(-1)^{n-p-q+1}}{n-p} \binom{n}{p} \binom{n-p-1}{q-1}.$$

The proof of this theorem is postponed until Section 7, since it uses the material derived in the following two sections.

Example 4.3. *Returning to the case $n = r = 2$, M_{22} diagonalises as*

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -2 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 1 & 2 & 2 & 1 & 2 \\ 1 & -2 & 2 & 1 & -2 \\ 2 & 0 & 0 & -2 & 0 \\ 1 & 2 & -2 & 1 & -2 \\ 1 & -2 & -2 & 1 & 2 \end{pmatrix}.$$

We recognise the leftmost matrix in this expression; it is the transpose of the character table of $G(2,1,2)$.

Plugging $n = r = 2$ into Theorem 4.2, we get

$$2 - \frac{3}{4} + \frac{1}{2}\frac{(-1)}{2}0^t + \frac{1}{2}\left(\frac{1}{2} \cdot (-1)^t - \frac{1}{2} \cdot 0^t - 2 \cdot 0^t\right) = \frac{5}{4} + \frac{(-1)^t}{4} - \frac{3}{2}0^t.$$

The asymptotics are now fairly easy to deal with. For $r = 1, 2$ we have a dependence on the parity of t reflecting the bipartite nature of G_{rn}. For larger r, G_{rn} is no longer bipartite, and this behaviour disappears.

Corollary 4.4. *As t goes to infinity, the expected reflection distance in $G(r,1,n)$ approaches*

$$n - \frac{1}{r} \sum_{k=1}^{n} \frac{1}{k} + \delta,$$

where

$$\delta = \begin{cases} \frac{(-1)^{n-1}}{n(n-1)} & \text{if } r = 1 \text{ and } t \text{ is even,} \\ \frac{(-1)^{n}}{n(n-1)} & \text{if } r = 1 \text{ and } t \text{ is odd,} \\ \frac{(-1)^{n}}{2n} & \text{if } r = 2 \text{ and } t \text{ is even,} \\ \frac{(-1)^{n+1}}{2n} & \text{if } r = 2 \text{ and } t \text{ is odd,} \\ 0 & \text{if } r \geq 3. \end{cases}$$

Proof. The case $r = 1$ was carried out in [3], so suppose $r \geq 2$. Consider the expressions inside the large brackets preceded by a_{pq} and b_{pq} in Theorem 4.2; call them B_1 and B_2, respectively. It is easily checked that $|B_1| < 1$ for all p, q. It is equally simple to verify that $|B_2| < 1$ unless $p = 0$, $q = 1$, $r = 2$ in which case we have $B_2 = -1$. Noting that $b_{01} = (-1)^n/n$ proves the corollary.

□

5. The eigenvalues of the adjacency matrices

To compute the eigenvalues of M_{rn}, we use the following $G(r,1,n)$-version of the Murnaghan-Nakayama formula which can be found in Ariki and Koike [1].

Theorem 5.1 ([1]). *For fixed i and j, let μ^i_j be the jth part of μ^i in $\mu = (\mu^1, \ldots, \mu^r)$, and let ζ be a primitive rth root of unity in $\mathbb{C}$. Then*

$$\chi^\lambda(\mu) = \sum_{p=1}^{r} \sum_{|\Gamma|=\mu^i_j} (-1)^{\mathrm{ht}(\Gamma)} \zeta^{(i-1)(p-1)} \chi^{(\lambda^1,\ldots,\lambda^p-\Gamma,\ldots,\lambda^r)}(\mu - \mu^i_j),$$

where the second sum runs over all rim hooks Γ of size μ^i_j in λ^p and $\mathrm{ht}(\Gamma)$ *is one less than the number of rows in Γ.*

From (4), it follows that the eigenvalue corresponding to $\lambda \vdash_r n$ is

$$\frac{r\binom{n}{2}\chi^\lambda((2,1^{n-2}),\emptyset,\ldots,\emptyset)}{\chi^\lambda(1^n,\emptyset,\ldots,\emptyset)} + \sum_{i=2}^{r} \frac{n\chi^\lambda(1^{n-1},\emptyset,\ldots,\emptyset,1,\emptyset,\ldots,\emptyset)}{\chi^\lambda(1^n,\emptyset,\ldots,\emptyset)}, \tag{6}$$

where, in the second sum, the ith argument of χ^λ is 1.

We thus need to compute some entries in the character table of $G(r,1,n)$.

Lemma 5.2. *For any $\lambda = (\lambda^1, \ldots, \lambda^r) \vdash_r n$, we have*

$$\chi^\lambda(1^n,\emptyset,\ldots,\emptyset) = \binom{n}{|\lambda^1|,\ldots,|\lambda^r|} \prod_{k=1}^{r} \chi^{\lambda^k}(1^{|\lambda^k|}),$$

$$\chi^\lambda((2,1^{n-2}),\emptyset,\ldots,\emptyset) = \sum_{p=1}^{r} \binom{n-2}{|\lambda^1|,\ldots,|\lambda^p|-2,\ldots,|\lambda^r|} \chi^{\lambda^p}(2,1^{|\lambda^p|-2}) \prod_{k\neq p} \chi^{\lambda^k}(1^{|\lambda^k|})$$

and

$$\chi^\lambda(1^{n-1},\emptyset,\ldots,\emptyset,1,\emptyset,\ldots,\emptyset) = \sum_{p=1}^{r} \binom{n-1}{|\lambda^1|,\ldots,|\lambda^p|-1,\ldots,|\lambda^r|} \zeta^{(i-1)(p-1)} \prod_{k=1}^{r} \chi^{\lambda^k}(1^{|\lambda^k|}).$$

(In the last equation, the ith argument of χ^λ is 1.)

Proof. For $\mu = (1^n, \emptyset, \ldots, \emptyset)$, the Murnaghan-Nakayama rule becomes

$$\chi^\lambda(1^n,\emptyset,\ldots,\emptyset) = \sum_{p=1}^{r} \sum_{\square} \chi^{(\lambda^1,\ldots,\lambda^p-\square,\ldots,\lambda^r)}(1^{n-1},\emptyset,\ldots,\emptyset),$$

where the inner sum runs over all outer squares of λ^p. Thus, the character equals the number of ways to remove one outer square at a time from the Ferrers' diagrams of λ. This number is $\binom{n}{|\lambda^1|,\dots,|\lambda^r|}\prod_{k=1}^r \chi^{\lambda^k}(1^{|\lambda^k|})$, since $\chi^{\lambda^k}(1^{|\lambda^k|})$ is the number of ways to remove one outer square at a time from λ^k.

The two other equations follow similarly. When $\mu = ((2,1^{n-2}),\emptyset,\dots,\emptyset)$, we first remove a rim hook of size 2 from some λ^p, whereas for $\mu = (1^{n-1},\emptyset,\dots,\emptyset,1,\emptyset,\dots,\emptyset)$, we start by removing the square corresponding to μ^j.

□

We are now ready to compute the eigenvalues.

Theorem 5.3. *Let $\lambda = (\lambda^1,\dots,\lambda^r) \vdash_r n$. The eigenvalue of M_{rn} corresponding to λ is given by*

$$\operatorname{eig}(M_{rn},\lambda) = r\sum_{p=1}^{r} c(\lambda^p) + r|\lambda^1| - n.$$

Proof. Combining equation (6) and Lemma 5.2, the eigenvalue becomes

$$r\sum_{p=1}^{r}\binom{|\lambda^p|}{2}\frac{\chi^{\lambda^p}(2,1^{|\lambda^p|-2})}{\chi^{\lambda^p}(1^{|\lambda^p|})} + \sum_{p=1}^{r}|\lambda^p|\sum_{j=2}^{r}\zeta^{(j-1)(p-1)}.$$

But

$$\binom{|\lambda^p|}{2}\frac{\chi^{\lambda^p}(2,1^{|\lambda^p|-2})}{\chi^{\lambda^p}(1^{|\lambda^p|})} = \operatorname{eig}(M_{1|\lambda^p|},\lambda^p) = c(\lambda^p),$$

and

$$\sum_{j=2}^{r}\zeta^{(j-1)(p-1)} = \sum_{j=1}^{r}\zeta^{(j-1)(p-1)} - 1 = \begin{cases} r-1 & \text{if } p=1,\\ -1 & \text{otherwise.}\end{cases}$$

□

6. Decomposing the distance function

Recall from (1) the distance function w_{rn} in the reflection graph of $G(r,1,n)$. Being a class function, it can be written as a linear combination of the irreducible $G(r,1,n)$-characters. In this section, we will make this decomposition explicit using the material reviewed in Section 3. In [3], the symmetric group case ($r=1$) was settled using a similar approach. However, the fact that $x_i \neq \tilde{x}_i$ for larger r calls for greater care.

Before stating the main theorem we need some preliminary results. We feel that the first is of independent interest.

Proposition 6.1. *The complete symmetric functions satisfy*

$$\prod_{i=1}^{r}\sum_{n\geq 0} h_n(x_i) = \left(\sum_{n\geq 0} h_n(\tilde{x}_1)\right)^r.$$

Proof. Throughout the proof, lower case Greek letters with or without superscripts, such as μ and μ^i, will denote ordinary integer partitions.

First, we manipulate the left hand side a little to obtain

$$\prod_{i=1}^{r}\sum_{n\geq 0} h_n(x_i) = \prod_{i=1}^{r}\sum_{\mu}\frac{p_\mu(x_i)}{z_\mu} = \sum_{(\mu^1,\ldots,\mu^r)}\frac{p_{\mu^1}(x_1)\ldots p_{\mu^r}(x_r)}{z_{\mu^1}\ldots z_{\mu^r}}.$$

Turning to the right hand side, we get

$$\left(\sum_{n\geq 0} h_n(\tilde{x}_1)\right)^r = \left(\sum_{\mu}\frac{p_\mu(\tilde{x}_1)}{z_\mu}\right)^r = \left(\sum_{\mu}\frac{p_{\mu_1}(\tilde{x}_1)\ldots p_{\mu_{\ell(\mu)}}(\tilde{x}_1)}{z_\mu}\right)^r$$

$$= \left(\sum_{\mu}\frac{1}{z_\mu}\prod_{i=1}^{\ell(\mu)}\frac{p_{\mu_i}(x_1)+\cdots+p_{\mu_i}(x_r)}{r}\right)^r$$

$$= \sum_{(\mu^1,\ldots,\mu^r)}\frac{1}{z_{\mu^1}\ldots z_{\mu^r} r^{\ell(\mu^1)+\cdots+\ell(\mu^r)}}\prod_{j=1}^{r}\prod_{i=1}^{\ell(\mu^j)}(p_{\mu_i^j}(x_1)+\cdots+p_{\mu_i^j}(x_r)).$$

For appropriate coefficients $K_{\mu^1,\ldots,\mu^r}$, this expression can be written as

$$\sum_{(\mu^1,\ldots,\mu^r)} K_{\mu^1,\ldots,\mu^r} p_{\mu^1}(x_1)\ldots p_{\mu^r}(x_r).$$

Fix $\lambda^1,\ldots,\lambda^r$. We must show that $K_{\lambda^1,\ldots,\lambda^r} = (z_{\lambda^1}\ldots z_{\lambda^r})^{-1}$.

Consider the last expression for the right hand side above. For a term indexed by $(\mu^1,\ldots,\mu^r)$, let f_i^j be the number of parts that equal i in μ^j. Similarly, let e_i^j be the number of parts that equal i in λ^j and put $N_i = \sum_j e_j^i$. Clearly, the only terms that contribute to $K_{\lambda^1,\ldots,\lambda^r}$ are those for which $\sum_j f_i^j = N_i$ for all i. Below, the sums are over all such $\mu^1,\ldots,\mu^r$ (so that, in particular, $\ell(\lambda^1)+\cdots+\ell(\lambda^r) = \ell(\mu^1)+\cdots+\ell(\mu^r)$). We get

$$K_{\lambda^1,\ldots,\lambda^r} = \frac{1}{r^{\ell(\lambda^1)+\cdots+\ell(\lambda^r)}}\sum_{(\mu^1,\ldots,\mu^r)}\frac{1}{z_{\mu^1}\ldots z_{\mu^r}}\prod_{i\geq 1}\binom{N_i}{e_i^1,\ldots,e_i^r}$$

$$= \frac{1}{r^{\ell(\lambda^1)+\cdots+\ell(\lambda^r)}}\sum_{(\mu^1,\ldots,\mu^r)}\frac{\prod_{k=1}^{r}(\prod_j e_j^k!)}{z_{\lambda^1}\ldots z_{\lambda^r}\prod_{k=1}^{r}(\prod_j f_j^k!)}\prod_{i\geq 1}\binom{N_i}{e_i^1,\ldots,e_i^r}$$

$$= \frac{1}{r^{\ell(\lambda^1)+\cdots+\ell(\lambda^r)} z_{\lambda^1}\ldots z_{\lambda^r}}\sum_{(\mu^1,\ldots,\mu^r)}\prod_{i\geq 1}\binom{N_i}{f_i^1,\ldots,f_i^r}.$$

To simplify this sum, consider the following situation: we have r boxes of distinguishable balls, the ith box containing N_i balls, and we wish to paint the balls using (at most) r colours. Of course, colouring the balls one by one, this can be done in $r^{\sum N_i} = r^{\ell(\lambda^1)+\cdots+\ell(\lambda^r)}$ ways. Another way to colour the balls is this: first choose

$(\mu^1,\dots,\mu^r)\vdash_r n$ with $\sum_j f_i^j = N_i$ for all i. In box i, there will be f_i^j balls with colour j; this box can be coloured in $\binom{N_i}{f_i^1,\dots,f_i^r}$ ways. Thus,

$$K_{\lambda^1,\dots,\lambda^r} = \frac{1}{r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)} z_{\lambda^1}\dots z_{\lambda^r}} r^{\ell(\lambda^1)+\dots+\ell(\lambda^r)} = \frac{1}{z_{\lambda^1}\dots z_{\lambda^r}},$$

as desired. □

Let $L \in R(r)$ be the function $\lambda \mapsto \ell(\lambda^1)$. Sometimes, by abuse of notation, we will let L denote its restriction to $R^n(r)$.

Lemma 6.2. *We have*

$$\mathrm{ch}(L) = \frac{1}{r}\sum_{n\geq 1}\frac{1}{n}p_n(x_1)\prod_{i=1}^{r}\left(\sum_{m\geq 0} h_m(x_i)\right)^{\frac{1}{r}}.$$

Proof. Again, in this proof symbols such as μ and μ^i will denote ordinary integer partitions.

Letting the first t y-variables equal one and the rest be zero in [7, 7.20] yields

$$\sum_{\mu}\frac{p_\mu}{z_\mu}t^{\ell(\mu)} = \exp\left(\sum_{n\geq 1}\frac{t}{n}p_n\right).$$

Hence, letting t_i be independent indeterminates,

$$\prod_{i=1}^{r}\sum_{\mu}\frac{p_\mu(x_i)}{z_\mu}t_i^{\ell(\mu)} = \exp\left(\sum_{i=1}^{r}\sum_{n\geq 1}\frac{t_i}{n}p_n(x_i)\right).$$

Differentiating with respect to t_1, we obtain

$$\left(\sum_{\mu}\frac{p_\mu(x_1)}{z_\mu}\ell(\mu)t_1^{\ell(\mu)-1}\right)\prod_{i=2}^{r}\sum_{\nu}\frac{p_\nu(x_i)}{z_\nu}t_i^{\ell(\nu)} = \sum_{n\geq 1}\frac{p_n(x_1)}{n}\exp\left(\sum_{i=1}^{r}\sum_{m\geq 1}\frac{t_i}{m}p_m(x_i)\right),$$

which, after putting all $t_i = \frac{1}{r}$, becomes

$$r\sum_{(\mu^1,\dots,\mu^r)}\frac{p_{\mu^1}(x_1)\dots p_{\mu^r}(x_r)}{z_{\mu^1}\dots z_{\mu^r}r^{\ell(\mu^1)+\dots+\ell(\mu^r)}}\ell(\mu^1) = \sum_{n\geq 1}\frac{p_n(x_1)}{n}\exp\left(\frac{1}{r}\sum_{i=1}^{r}\sum_{m\geq 1}\frac{p_m(x_i)}{m}\right).$$

Now, the left hand side is in fact $r\mathrm{ch}(L)$. The fact that $\exp\left(\sum_{m\geq 1}\frac{p_m}{m}\right) = \sum_{m\geq 0} h_m$ concludes the proof. □

We are now in position to state and prove the main result of this section.

Theorem 6.3. *For all $\lambda \vdash_r n$, $L(\lambda) = \sum_{\mu \vdash_r n} c_\mu \chi^\mu(\lambda)$, where*

$$c_\mu = \begin{cases} \sum_{k=1}^n \frac{1}{rk} & \text{if } \mu^1 = (n), \\ (-1)^{n-p-q} \frac{p-q+1}{r(n-q+1)(n-p)} & \text{if } \mu^1 = (p, q, 1^{n-p-q}), \\ \frac{(-1)^{n-p-q}}{r(n-p)} & \text{if } \mu^1 = (p) \text{ and } \mu^i = (q, 1^{n-p-q}) \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Passing to $\Lambda^n(r)$, we want to compute the coefficients c_μ in the expression $\mathrm{ch}^n(L) = \sum_{\mu \vdash_r n} c_\mu \widetilde{S}_\mu$.

Combining Lemma 6.2 and Proposition 6.1 yields

$$\mathrm{ch}(L) = \frac{1}{r} \sum_{n \geq 1} \frac{1}{n} p_n(x_1) \sum_{m \geq 0} h_m(\tilde{x}_1),$$

which, with the aid of (3), becomes

$$\mathrm{ch}(L) = \frac{1}{r} \sum_{n \geq 1} \frac{1}{n} \sum_{i=1}^r p_n(\tilde{x}_i) \sum_{m \geq 0} h_m(\tilde{x}_1).$$

Now, define coefficients c^i_μ by writing

$$\sum_{m \geq 0} h_m(\tilde{x}_1) \sum_{n \geq 1} \frac{1}{n} p_n(\tilde{x}_i) = \sum_n \sum_{\mu \vdash_r n} c^i_\mu \widetilde{S}_\mu,$$

so that $r c_\mu = \sum_{i=1}^r c^i_\mu$. For the rest of this proof, we let $\mu \vdash_r n$ be fixed.

First, we consider the case $i = 1$. Using [7, 7.72], it is not difficult to show that

$$c^1_\mu = \begin{cases} \sum_{k=1}^n \frac{1}{k} & \text{if } \mu^1 = (n), \\ (-1)^{n-p-q} \frac{p-q+1}{(n-q+1)(n-p)} & \text{if } \mu^1 = (p, q, 1^{n-p-q}), \\ 0 & \text{otherwise;} \end{cases}$$

see the proof of [3, Thm. 3] for the details.

Now, pick $i > 1$. It is well-known that

$$p_n = \sum_{q=1}^n (-1)^{n-q} s_{(q, 1^{n-q})}.$$

Hence, recalling that $s_m = h_m$, we obtain

$$c^i_\mu = \begin{cases} \frac{(-1)^{n-p-q}}{n-p} & \text{if } \mu^1 = (p) \text{ and } \mu^i = (q, 1^{n-p-q}), \\ 0 & \text{otherwise.} \end{cases}$$

The result follows. □

What we really need is the decomposition of the distance function w_{rn}, not L. It is now easily obtained.

Corollary 6.4. *For all* $\lambda \vdash_r n$, $w_{rn}(\lambda) = \sum_{\mu \vdash_r n} d_\mu \chi^\mu(\lambda)$, *where*

$$d_\mu = \begin{cases} n - \sum_{k=1}^{n} \frac{1}{rk} & \text{if } \mu^1 = (n), \\ -c_\mu & \text{otherwise.} \end{cases}$$

Here, c_μ *is as in Theorem 6.3.*

Proof. We know that $w_{rn} = n\chi_{\text{triv}} - L$, where χ_{triv} is the trivial character. Since the trivial character is indexed by $(n, \emptyset, \dots, \emptyset)$, the result follows. □

7. Proof of Theorem 4.2

We now turn to the proof of our main theorem. We have already shown that the expected reflection distance after t random reflections is given by

$$\sum_{\lambda \vdash_r n} \chi^\lambda(1^n, \emptyset, \dots, \emptyset) \left(\frac{\text{eig}(M_{rn}, \lambda)}{|R|} \right)^t \sum_{\mu \vdash_r n} \frac{\chi^\lambda(\mu) w_{rn}(\mu)}{Z_\mu}$$
$$= \sum_{\lambda \vdash_r n} \chi^\lambda(1^n, \emptyset, \dots, \emptyset) \left(\frac{\text{eig}(M_{rn}, \lambda)}{|R|} \right)^t \langle \chi^\lambda, w_{rn} \rangle.$$

If we decompose $w_{rn}(\mu) = \sum_{\lambda \vdash_r n} d_\lambda \chi^\lambda(\mu)$ and use that the number of reflections is $|R| = r\binom{n+1}{2} - n$, we obtain

$$\sum_{\lambda \vdash_r n} d_\lambda \chi^\lambda(1^n, \emptyset, \dots, \emptyset) \left(\frac{\text{eig}(M_{rn}, \lambda)}{r\binom{n+1}{2} - n} \right)^t.$$

The coefficients d_λ are zero for most λ, the exceptions being $\lambda^1 = (n)$, $\lambda^1 = (p, q, 1^{n-p-q})$ and $\lambda = (p, \emptyset, \dots, \emptyset, (q, 1^{n-p-q}), \emptyset, \dots, \emptyset)$.

If $\lambda^1 = (n)$, we have $d_\lambda = n - \sum_{k=1}^{n} \frac{1}{rk}$, $\chi^\lambda(1^n, \emptyset, \dots, \emptyset) = 1$ (since χ^λ is the trivial character) and $\text{eig}(M_{rn}, \lambda) = r\binom{n}{2} + (r-1)n = r\binom{n+1}{2} - n$, so we get

$$d_\lambda \chi^\lambda(1^n, \emptyset, \dots, \emptyset) \left(\frac{\text{eig}(M_{rn}, \lambda)}{r\binom{n+1}{2} - n} \right)^t = n - \sum_{k=1}^{n} \frac{1}{rk}.$$

Similarly, if $\lambda^1 = (p, q, 1^{n-p-q})$, we obtain $d_\lambda = (-1)^{n-p-q+1} \frac{p-q+1}{r(n-q+1)(n-p)}$,

$$\chi^\lambda(1^n, \emptyset, \dots, \emptyset) = \frac{n!(p-q+1)}{(q-1)!(n-p-q)!(n-p)(n-q+1)p!}$$
$$= \frac{(p-q+1)}{(n-q+1)} \binom{n-p-1}{q-1} \binom{n}{p}$$

and $\text{eig}(M_{rn}, \lambda) = rc(p, q, 1^{n-p-q}) + (r-1)n$.

Finally, if $\lambda^1 = (p)$ and $\lambda^i = (q, 1^{n-p-q})$, we have $d_\lambda = \frac{(-1)^{n-p-q+1}}{r(n-p)}$,

$$\chi^\lambda(1^n, \emptyset, \ldots, \emptyset) = \binom{n}{p}\chi^{(p)}(1^p)\chi^{(q,1^{n-p-q})}(1^{n-p}) = \binom{n}{p}\binom{n-p-1}{q-1}$$

and

$$\operatorname{eig}(M_{rn}, \lambda) = r\left(\binom{p}{2} + \binom{q}{2} - \binom{n-p-q+1}{2} + p\right) - n.$$

Putting it all together yields the theorem.

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