RECENT PROGRESS IN ALGEBRAIC COMBINATORICS

RICHARD P. STANLEY

Abstract. Significant progress continues to be made in algebraic combinatorics. We will survey three topics representative of this development.

Résumé: La combinatoire algébrique est un domaine en plein essor. Nous donnerons ici un aper cu de trois sujets typiques de ces derniers développements.

1. **The Laurent phenomenon.** There are many examples of rational functions with recursive definitions which turn out unexpectedly to be Laurent polynomials. The protypical example is the (generic) *Somos-4 sequence* [9] defined by

$$a_0 = a,\ a_1 = b,\ a_2 = c,\ a_3 = d,\ a_{n-4}a_n = a_{n-1}a_{n-3} + a_{n-2}^2 \text{ for } n \geq 4,$$

where a, b, c, d are independent indeterminates. A priori a_n is a rational function of a, b, c, d with a complicated denominator, but in fact when a_n is reduced to lowest terms the denominator is a monomial. In particular, when $a = b = c = d = 1$, a_n is an integer. A breakthrough in understanding the Laurent phenomenon algebraically was made by Fomin and Zelevinsky [3] as a consequence of their theory of cluster algebras. The integrality of the Somos-4 sequence itself (and a natural extension known as Somos-5) when $a = b = c = d = 1$ was elucidated by a combinatorial interpretation of a_n as the number of matchings of a certain graph by a team of undergraduate students at Harvard University supervised by Jim Propp [10] and independently by M. Bousquet-Mélou and J. West [1].

2. **Toric Schur functions and Gromov-Witten invariants.** Let Gr_{kn} denote the Grassmann variety (or Grassmannian) of all k-dimensional subspaces of the n-dimensional complex vector space $\mathbb{C}^n$. The cohomology ring $H^*(\mathrm{Gr}_{kn})$ (say over $\mathbb{Q}$) is the fundamental object for the development of classical *Schubert calculus*, which is concerned, at the enumerative level, with counting the number of linear subspaces that satisfy certain geometric conditions. For an introduction to Schubert calculus see [4][5], and for connections with combinatorics see [12]. In particular, $H^*(\mathrm{Gr}_{kn})$ has a basis of *Schubert cycles* Ω_λ indexed by

partitions λ whose shape is contained in a $k \times (n-k)$ rectangle (denoted $\lambda \subseteq k \times (n-k)$). Multiplication in the ring $H^*(\mathrm{Gr}_{kn})$ is given by

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq k \times (n-k)} c_{\mu\nu}^\lambda \sigma_\lambda, \tag{1}$$

where $c_{\mu\nu}^\lambda$ is a Littlewood-Richardson coefficient. Thus $c_{\mu\nu}^\lambda$ has a geometric interpretation as the intersection number of the Schubert varieties $\Omega_\mu, \Omega_\nu, \Omega_{\lambda^\vee}$, where $\lambda^\vee$ denotes the complement of λ (rotated 180°) in a $k \times (n-k)$ rectangle. More concretely,

$$c_{\mu\nu}^\lambda = \#\left(\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}\right), \tag{2}$$

the number of points of Gr_{kn} contained in the intersection $\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}$, where $\tilde{\Omega}_\sigma$ denotes a generic translation of Ω_σ. Equivalently, $c_{\mu\nu}^\lambda$ is the number of k-dimensional subspaces of $\mathbb{C}^n$ satisfying all of the geometric conditions defining $\tilde{\Omega}_{\lambda^\vee}$, $\tilde{\Omega}_\mu$, and $\tilde{\Omega}_\nu$.

Some recent results of Alexander Postnikov [8] deal with a quantum deformation of $H^*(\mathrm{Gr}_{kn})$. The cohomology ring $H^*(\mathrm{Gr}_{kn})$ can be deformed into a "quantum cohomology ring" $\mathrm{QH}^*(\mathrm{Gr}_{kn})$, which specializes to $H^*(\mathrm{Gr}_{kn})$ by setting $q = 0$. A basis for $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ remains those σ_λ for which λ fits in a $k \times (n-k)$ rectangle. Now, however, the usual multiplication $\sigma_\mu\sigma_\nu$ of Schubert classes has been deformed into a "quantum multiplication" $\sigma_\mu * \sigma_\nu$. It has the form

$$\sigma_\mu * \sigma_\nu = \sum_{d \geq 0} \sum_{\substack{\lambda \vdash |\mu|+|\nu|-dn \\ \lambda \subseteq k \times (n-k)}} q^d C_{\mu\nu}^{\lambda,d} \sigma_\lambda,$$

where $C_{\mu\nu}^{\lambda,d} \in \mathbb{Z}$. The geometric significance of the coefficients $C_{\mu\nu}^{\lambda,d}$ (and the motivation for defining $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ in the first place) is that they count the number of rational curves of degree d in Gr_{kn} that meet fixed generic translates of the Schubert varieties Ω_λ, Ω_μ, and $\Omega_{\nu^\vee}$. (Naively, a *rational curve* of degree d in Gr_{kn} is a set

$$C = \{(f_1(s,t), f_2(s,t), \ldots, f_{\binom{n}{k}}(s,t)) \in P^{\binom{n}{k}-1}(\mathbb{C}) \,:\, s,t \in \mathbb{C}\},$$

where $f_1(x,y), \ldots, f_{\binom{n}{k}}(x,y)$ are homogeneous polynomials of degree d such that $C \subset \mathrm{Gr}_{kn}$.) Since a rational curve of degree 0 in Gr_{kn} is just a point of Gr_{kn} we recover in the case $d = 0$ the geometric interpretation (2) of ordinary Littlewood-Richardson coefficients $c_{\mu\nu}^\lambda = C_{\mu\nu}^{\lambda,0}$. The numbers $C_{\mu\nu}^{\lambda,d}$ are known as (3-point) *Gromov-Witten invariants.*

We will discuss Postnikov's combinatorial description of a new generalization of (skew) Schur functions, called *toric Schur functions* and denoted $s_{\lambda/d/\mu}$, where ordinary (planar) Young diagrams are replaced by a toroidal analogue. The main result concerning toroidal Schur functions is the expansion

$$s_{\lambda/d/\mu}(x_1,\ldots,x_k) = \sum_{\nu\subseteq k\times(n-k)} C_{\mu\nu}^{\lambda,d} s_\nu(x_1,\ldots,x_k),$$

giving a new interpretation of the Gromov-Witten invariants $C_{\mu\nu}^{\lambda,d}$.

3. **Domino Schur functions and the imbalance of a standard Young tableaux.** Let T be a standard Young tableau (SYT) of shape $\lambda \vdash n$, such as

$$T = \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & 7 & 8 & \\ 5 & & & \end{array},$$

of shape $\lambda = (4,3,1)$. Read the entries of T in the usual reading order, obtaining a permutation $w_T \in \mathfrak{S}_n$. For the above example, $w_T = 12463785 \in \mathfrak{S}_8$. Let $\varepsilon_T = 1$ if w_T is an even permutation, and $\varepsilon_T = -1$ if w_T is odd. Define the *imbalance* I^λ of the partition λ by

$$I^\lambda = \sum_{\mathrm{sh}(T)=\lambda} \varepsilon_T,$$

where T ranges over all SYT of shape λ. The notion of the imbalance of λ first arose in the work of Frank Ruskey [11] in the case where $\lambda = (a^b)$ (i.e., the shape of λ is an $a \times b$ rectangle). A conjecture of Ruskey concerning this case was proved and refined by Dennis White [14]. Some further results appear in [13].

We will discuss connections between the imbalance of λ and such topics as domino Schur functions [6], shifted tableaux and Schur's Q-functions [7, §III.8], and the Wronski map on the real Grassmannian [2].

References

[1] M. Bousquet-Mélou and J. West, in preparation.

[2] A. Eremenko and A. Gabrielov, Degrees of real Wronski maps, *Discrete Comput. Geom.* **28** (2002), 331–347; `www.math.purdue.edu/~eremenko/papers.html`.

[3] S. Fomin and A. Zelevinsky, The Laurent phenomenon, *Advances in Applied Math.* **28** (2002), 119–144.; math.CO/0104241.

[4] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts **35**, Cambridge University Press, Cambridge, 1997.

[5] S. L. Kleiman and D. Laksov, Schubert calculus, *American Math. Monthly* **79** (1972), 1061–1082.

[6] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Ribbon tableaux, Hall-Littlewood symmetric functions, quantum affine algebras, and unipotent varieties, *J. Math. Phys.* **38**(2) (1997), 1041–1068; q-alg/9512031.

[7] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Oxford University Press, Oxford, 1995.

[8] A. Postnikov, Affine approach to quantum Schubert calculus, preprint; math.CO/0205165.

[9] J. Propp, The Somos Sequence Site (web page), `www.math.wisc.edu/~propp/somos.html`.

[10] J. Propp, Research Experiences in Algebraic Combinatorics at Harvard (web page), `www.math.harvard.edu/~propp/reach`.

[11] R. Ruskey, Research problem 90, *Discrete Math.* **70** (1989), 111–112.

[12] R. Stanley, Some combinatorial aspects of the Schubert calculus, in *Combinatoire et Réprésentation du Groupe Symétrique (Strasbourg, 1976)*, Lecture Notes in Math., No. 579, Springer-Verlag, Berlin, 1977, pp. 217–251.

[13] R. Stanley, Some remarks on sign-balanced and maj-balanced posets, preprint; `www-math.mit.edu/~rstan/papers.html#signbal`.

[14] D. White, Sign-balanced posets, *J. Combinatorial Theory Ser. A* **95** (2001), 1–38; `www.math.umn.edu/~white/papers`.

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