

A New Representation of Formal Power Series

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Abstract. *This paper is dedicated to the genesis arising at the boundary between the theory of formal power series (FPS) and combinatorics.*

Similarly to combinatorics where any rational sequence of natural numbers $\{r_k\}_{k \geq 0}$ is representable for all k in the form

$$(1.1) \quad r_{k+n} = \sum_{i=1}^n r_{k+n-i} X_{n-i}$$

where X_j – are, generally speaking, complex numbers (Berstel, Reutenauer, [BR]), we prove that any rational FPS r is representable in the form (1) where $r_s = \sum_{|w|=s} (r, w)w$, and X_j are elements of some special skew field. As a trivial consequence of such a representation were obtained: 1) truthfulness of Eilenberg’s Equality Theorem [E], decidability of the equivalence problem of finite multitape deterministic automata (Rabin, Scott [RS]) and decidability of problem of whether two given morphisms are equivalent on regular language, (Culik, Salomaa [CS]); 2) more simply formulated and proved the results from monographs on FPS (Salomaa, Soittola [SS], Berstel, Reutenauer [BR], Kvich, Salomaa [KS]); 3) solved partial cases of the problem of existence for an inverse element of Hadamard product and others; 4) provided 3 Conjectures and 10 Open problems.

The conclusion contains a complete comparative analysis of the attempts to utilize linear recurrence in theory of FPS by other authors.

RÉSUMÉ.

We use the standard notations from monographs Berstel, Reutenauer [BR] and Cohn [Coh]. In particular, it will be assumed that $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_t\}$ is a finite alphabet, $\Sigma^{-1} = \{\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_t^{-1}\}$, ε is empty word and unity in semigroup Σ^* and group G , generated by Σ , \emptyset is empty set and zero in semirings and fields, generated by Σ , $\underline{\varepsilon}, \underline{\sigma}_i, \underline{\sigma}_i^{-1}$ are corresponding characteristic FPS, \mathbf{k} is commutative zero-divisor-free semiring embeddable in commutative field \mathbf{K} (this includes the semirings $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$).

According to Salomaa, Soittola [SS], every FPS $r \in \mathbf{k}^{rat} \ll \Sigma^* \gg$ can be represented as a behaviour of $\mathbf{k} - \Sigma^*$ -automaton

$$\mathfrak{A} = \langle \{q_1, q_2, \dots, q_n\}, A, q_1, F \rangle$$

where $A \in \mathbf{k}^{n \times n} < \Sigma >$ - transition matrix, q_1 -initial state, $F \in \mathbf{k}^{n \times 1} < \{\underline{\varepsilon}, \emptyset\} >$ - final states:

$$r_{\mathfrak{A}} = \sum_{i=0}^{\infty} (A^i F)_1.$$

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We denote $q_i^{(j)} = (A^j F)_i$, then $r_{\mathfrak{A}} = \sum_{i=0}^{\infty} q_1^{(i)}$ and

$$(1.2) \quad q_i^{(j)} = \sum_{s=1}^n A_{is} q_s^{(j-1)}$$

Let us consider a system of n equations with $(n+1)$ unknowns X_0, X_1, \dots, X_n :

$$(1.3) \quad \begin{cases} q_i^{(n)} X_n = \sum_{j=1}^n q_i^{(n-j)} X_{n-j}, & i = \overline{1, n} \end{cases}$$

and show that it always has a non-zero solution in Malcev-Neumann skew field $\mathbf{K}((G))$ of FPS with well-ordered support.

We solve the system (3) following the usual Gauss algorithm by successive excluding unknowns X_0, X_1, \dots, X_n . On step 0 all coefficients of unknowns are $q_i^{(j)} \in \mathbf{k} \ll \Sigma^* \gg$ and of course are elements of $\mathbf{K}((G))$. Let us assume that on step i an equation for X_i has the form of

$$(1.4) \quad q_{in} X_n = q_{i(n-1)} X_{n-1} + \dots + q_{ii} X_i, \quad q_{is} \in \mathbf{K} \ll G \gg, \quad s = \overline{n, i}$$

and we can compute a leading term for every q_{is} .

Suppose $q_{ii} \neq \emptyset$ (otherwise we can exchange i -th column with one of $n-1, \dots, i+1$; if all $q_{is} = \emptyset$ for $s = \overline{i, n-1}$ then assume $X_n = \emptyset$ and go to an equation for X_{i+1}). Then for all non-zero q_{is} denote $q_{is} = \alpha_{is} + q'_{is}$ where α_{is} is a leading term in q_{is} . It follows that after multiplying both parts of the equation on α_{ii}^{-1} and solving it for X_i we obtain

$$(1.5) \quad X_i = (-\alpha_{ii}^{-1} q'_{ii})^* (\alpha_{ii}^{-1} \alpha_{in} + \alpha_{ii}^{-1} q'_{in}) X_n - \dots - (\dots) X_{i+1}$$

where the bracket content (\dots) is analogous to the coefficient of X_n and is not provided for the sake of simplicity. Substitute equation (5) into the remaining equations for X_j , $j = \overline{1, i-1}$:

$$(1.6) \quad \begin{aligned} (\alpha_{jn} + q'_{jn}) &= (\alpha_{ji} + q'_{ji}) (-\alpha_{ii}^{-1} q'_{ii})^* (\alpha_{ii}^{-1} \alpha_{in} + \alpha_{ii}^{-1} q'_{in}) X_n = \\ &= (\dots) X_{n-1} + \dots + (\dots) X_{i+1} \end{aligned}$$

Since $\text{supp}(\alpha_{ii}^{-1} q'_{ii}) > \varepsilon$ then leading term of the coefficient of X_n should be searched for in $\alpha_{jn} - \alpha_{ji} \alpha_{ii}^{-1} \alpha_{in}$. If the coefficient equals \emptyset then we take next in ascending order elements from $\text{supp}(q'_{jn})$, $\text{supp}(q'_{ji})$, $\text{supp}(\alpha_{ii}^{-1} q'_{in})$ and so on. This process is constructive (see Lewin [L], Cohn [Coh]), so the inductive hypothesis holds true – at the beginning of next step of Gauss algorithm all coefficients of unknowns $X_n, X_{n-1}, \dots, X_{i+1}$ will be again from $\mathbf{K}((G))$ with known leading terms.

At the last step for the equation $q_{nn} X_n = q_{n(n-1)} X_{n-1}$ we have:

(i) if $q_{n(n-1)} \neq \emptyset$ that is $q_{n(n-1)} = \alpha_{n(n-1)} + q'_{n(n-1)}$ then assume

$$X_n = \underline{\varepsilon}, X_{n-1} = \left(-\alpha_{n(n-1)}^{-1} q'_{n(n-1)} \right)^* \alpha_{n(n-1)}^{-1} q_{nn};$$

(ii) if $q_{n(n-1)} = \emptyset$, then assume $X_n = \emptyset, X_{n-1} = \underline{\varepsilon}$. So we proved

Theorem 1.1. *A solution of the system (3) in $\mathbf{K}((G))$ exists always in the form:*

$$(1.7) \quad (\tilde{X}_0, \tilde{X}_1, \dots, \tilde{X}_p, \underline{\varepsilon}, \emptyset, \dots, \emptyset), \quad 1 \leq p \leq n-1$$

while some \tilde{X}_i also can be \emptyset . ■

We prove the main theorem of the paper.

Theorem 1.2. For all $k \in \mathbf{N}$ holds

$$(1.8) \quad q_i^{(n+k)} = \sum_{j=1}^n q_i^{(n+k-j)} \tilde{X}_{n-j}, \quad i = \overline{1, n}$$

Proof. For $k = 0$ the statement is proved - suppose it is true for k . Then

$$\begin{aligned} q_i^{(n+k+1)} &\stackrel{(2)}{=} \sum_{j=1}^n A_{ij} q_j^{(n+k)} = \sum_{j=1}^n A_{ij} \sum_{l=1}^n q_j^{(n+k-l)} \tilde{X}_{n-l} = \\ &= \sum_{l=1}^n \left(\sum_{j=1}^n A_{ij} q_j^{(n+k-l)} \right) \tilde{X}_{n-l} \stackrel{(2)}{=} \sum_{l=1}^n q_i^{(n+k+1-l)} \tilde{X}_{n-l}. \blacksquare \end{aligned}$$

Definition 1. We call a representation of FPS q_i in the form (8) a *linear recurrence representation* (further referred shortly as LRR), vector-solution (7) - a *stencil*, $q_i^{(j)}$ - j -th *layer* of FPS q_i . ■

Example 1.4. Following the considerations about the solving of system (3) one can find that FPS $s = (\underline{a}^2(\underline{ab})^* \underline{b}^2(\underline{ab})^*)^*$ has LRR

$$s_4 = \underline{a}^2 \underline{b}^2, s_3 = s_2 = s_1 = \emptyset, s_0 = \underline{\varepsilon},$$

$$s_{n+5} = s_{n+4} \cdot \emptyset + s_{n+3}(\underline{b}^{-2} \underline{ab}^3 + \underline{ab}) + s_{n+2} \cdot \emptyset + s_{n+1}(\underline{a}^2 \underline{b}^2 - \underline{ab}^{-1} \underline{ab}^3) + s_n \cdot \emptyset. \blacksquare$$

Having analyzed the process of solving system (3) it is not difficult to prove:

Theorem 1.3. There exists a stencil with coefficients from the set $\{-1, 0, 1\}$ for a characteristic series of an arbitrary given rational languages. ■

The opposite is interesting:

Open problem 1. ("Fatou extension") If: 1) stencil of FPS r has all the coefficients from the set $\{-1, 0, 1\}$, 2) layers of r have coefficients from the set $\{0, 1\}$, then: r is \mathbf{N} -rational? And \mathbf{Z} -rational? And K -algebraic?

(we point out to the relationship of this problem with the counter-example Reutenauer [R]). ■

Corollary 1 (Eilenberg's Equality Theorem [E]). Let $\mathfrak{A} = \langle \{q_1, \dots, q_n\}, A, q_1, F_1 \rangle$ and $\mathfrak{B} = \langle \{p_1, \dots, p_m\}, B, p_1, F_2 \rangle$ be $\mathbf{k} - \Sigma^*$ -automata. Then $r_{\mathfrak{A}} = r_{\mathfrak{B}}$ iff $(r_{\mathfrak{A}}, w) = (r_{\mathfrak{B}}, w)$ for all $w \in \Sigma^*$ of length at most $(n+m-1)$.

Proof. Consider a system of equations:

$$\begin{cases} q_i^{n+m} = \sum_{j=1}^{n+m} q_i^{(n+m-j)} X_{n+m-j}, & i = \overline{1, n} \\ p_i^{n+m} = \sum_{j=1}^{n+m} p_i^{(n+m-j)} X_{n+m-j}, & i = \overline{1, m} \end{cases} \quad \blacksquare \quad (9)$$

Definition 2. We call the solution of the system (9) and the system itself a *common stencil for automata* \mathfrak{A} and \mathfrak{B} . Common stencil exists for any finite number of $\mathbf{k} - \Sigma^*$ -automata. ■

Corollary 2. (Equivalence Problem for Multitape Deterministic Finite Automata, Rabin, Scott [RS]) Two automata $\mathfrak{A}_1 = \langle \{q_1, \dots, q_n\}, \Sigma_1 \cup \Sigma_2 \cup \dots \cup \Sigma_k, \delta_1, q_1, F_1 \rangle$ and $\mathfrak{A}_2 = \langle \{p_1, \dots, p_m\}, \Sigma_1 \cup \Sigma_2 \cup \dots \cup \Sigma_k, \delta_2, p_1, F_2 \rangle$ are equivalent iff the sets of their acceptable words of length at most $(n+m-1)$ are equal.

Proof. Consider common stencil for automata \mathfrak{A} and \mathfrak{B} . As the direct product of fully ordered groups equipped with lexicographic order $\Sigma_1 < \Sigma_2 < \dots < \Sigma_k$ is still fully ordered group G_k (Passman [Pa]), hence this common stencil exists in form of solution for system (9) in $\mathbf{Z}((G_k))$. ■

Remark 1. Harju, Karhumaki [HK] result is also in checking up of all words of length at most $(n+m-1)$. To check the equivalence of two finite multitape deterministic automata an exponential time, therefore, is required. At the same time there exist polynomial algorithms for the checking of the equivalence of $\mathbf{k} - \Sigma^*$ -automata ($O(n^4)$ - Tzeng [T], $O(n^3)$ - Archangelsky [A1]). This provides a hint that should exist a polynomial algorithm. Indeed not every initial set of layers should be checked up because not all of them in combination with stencil would generate only 'clean' noncommutative polynomials - the ones without σ_i^{-1} . ■

Open problem 2. How many 'clean' tuples of layers there exist for a given stencil? ■

Remark 2. *Corollary 2 could have been proven more simpler by leaving out the process of finding of stencil and the proof of its existence. According to Hebish, Weinert [HW] the semiring of FPS on partially commutative monoids over \mathbf{Z} is zero-divisor-free and additively-cancellative and multiplicatively-left-cancellative. This means that a solution of the system (9) exists over some partially commutative skew field. ■*

Corollary 3. *(Equivalence Problem for Morphisms on Regular Languages, Culik, Salomaa [CS]) Let L be a regular language, defined by the minimal deterministic automaton $\mathfrak{A} = \langle \{q_1, \dots, q_n\}, \Sigma, \delta, q_1, F \rangle$, and $h, g : \Sigma^* \rightarrow \Delta^*$ be morphisms. If $h(w) = g(w)$ for all $w \in L$ of length at most $(2n - 1)$, then $h(w) = g(w)$ for all $w \in L$.*

Proof. One may assume that $\Sigma \cap \Delta = \emptyset$ and letters from Σ and Δ commute. We define the transition function δ_h in 2-tape automaton $\mathfrak{A}_h = \langle \{q_1, \dots, q_n\}, \Sigma \cup \Delta, \delta_h, q_1, F \rangle$ as follows $\delta_h(q_i, \sigma_j h(\sigma_j)) = q_k \Leftrightarrow \delta(q_i, \sigma_j) = q_k$. Similarly define δ_g and \mathfrak{A}_g . Until common stencil of \mathfrak{A}_h and \mathfrak{A}_g is being built we assume for convenience each $\sigma_j h(\sigma_j)$ and $\sigma_j g(\sigma_j)$ to be one unique letter. Thus the length of common stencil will be $2n$. ■

Remark 3. *Proof of Corollary 3 does not use unlike Karhumaki [K] an Eihrengucht's conjecture. Actually we have proven a more stronger result – the decidability of morphism equivalence on regular language with multiplicities of words are taken into consideration. ■*

Corollary 4. *Let $r \in \mathbf{k}^{\text{rat}} \ll \Sigma^* \gg$ and p be number of first nonzero element in stencil of r , i.e. $\tilde{X}_0 = \dots = \tilde{X}_{p-1} = \emptyset, \tilde{X}_p \neq \emptyset, 0 \leq p \leq n$. Then*

(i) *r is identecically zero iff $r_i = \emptyset$ for all $i = \overline{0, (n-1)}$;*

(ii) *r is polynomial iff $r_i = \emptyset$ for all $i = \overline{p, (n-1)}$;*

(iii) *r is ultimately constant iff $r_i = c \underline{\Sigma}^i$ for all $i = \overline{p, (n-1)}$;*

(iv) *r is identically constant iff $r_i = c \underline{\Sigma}^i$ for all $i = \overline{0, (n-1)}$.*

Proof. Trivial combinatorical considerations. ■

Remark 4. *Proof of Corollary 4 does not use, unlike Salomaa, Soittola [SS], Kuich, Salomaa [KS] Hadamard product and morphisms. ■*

Let us investigate more scrupulously how of the summands with negative powers of letters in $\sum r_i \tilde{X}_i$ annihilate. In the first approximation it can be done by tracing down step-by-step how only ‘clean’ non-commutative polynomials are left in the following examples.

Example 1.5 (Berstel, Reutenauer [BR]). *FPS $s = \sum_w |w|_a w = \underline{\Sigma}^* \underline{a} \underline{\Sigma}^*$ has the follows LRR:*

$$s_0 = \emptyset, s_1 = \underline{a}$$

$$s_{n+2} = s_{n+1}(2\underline{a} + \underline{b} + \underline{a}^{-1} \underline{b} \underline{a}) + s_n(-\underline{a}^2 - 2\underline{b} \underline{a} - \underline{b} \underline{a}^{-1} \underline{b} \underline{a}) \quad \blacksquare$$

Example 1.6 (Berstel, Reutenauer [BR]). *FPS*

$s = \sum_w (|w|_a - |w|_b) w = \underline{\Sigma}^ (\underline{a} - \underline{b}) \underline{\Sigma}^*$ has the follows LRR:*

$$s_0 = \emptyset, s_1 = \underline{a} - \underline{b},$$

$$s_{n+2} = s_{n+1} \cdot 2(\underline{a}^{-1} \underline{b})^* (\underline{a} - \underline{a}^{-1} \underline{b}^2) + s_n (\underline{a} + \underline{b}) (\underline{a} + \underline{b} - 2(\underline{a}^{-1} \underline{b})^* (\underline{a} - \underline{a}^{-1} \underline{b}^2)) \quad \blacksquare$$

Example 1.7 (Reutenauer [R]). *FPS*

$$s = \sum_w (\alpha^{2(|w|_x - |w|_y)} + \alpha^{2(|w|_y - |w|_x)}) w = (\alpha^2 \underline{x} + \alpha^{-2} \underline{y})^* + (\alpha^{-2} \underline{x} + \alpha^2 \underline{y})^* \\ \alpha = \frac{1}{2}(\sqrt{5} + 1),$$

has the follows LRR: $s_0 = 2\underline{x}, s_1 = 3\underline{x} + 3\underline{y}$,

$$s_{n+2} = s_{n+1} \cdot 3(\underline{x}^{-1}\underline{y})^*(\underline{x} - \underline{x}^{-1}\underline{y}^2) + s_n(\alpha^{-2}\underline{x} + \alpha^2\underline{y})(\alpha^{-2}\underline{x} + \alpha^2\underline{y} - 3(\underline{x}^{-1}\underline{y})^*(\underline{x} - \underline{x}^{-1}\underline{y}^2)) \blacksquare$$

Let us consider arbitrary sequential n-tuple of layers of $r \in \mathbf{k}^{rat} \ll \Sigma^* \gg$. One can say that they are n-inert in ring $\mathbf{K}((G))$ in several weak sense because $r_{k+n-i} \in \mathbf{k} \langle \Sigma^* \rangle$ (and of course, $r_{k+n-i} \in \mathbf{K}((G))$), $\tilde{X}_{n-i} \in \mathbf{K}((G))$, but $\sum_{i=1}^n r_{k+n-i}\tilde{X}_{n-i} \in \mathbf{k} \langle \Sigma^* \rangle$. And while the inertia theorem is proved (Bergman [Ber], Cohn [Coh]) also for ring $\mathbf{k} \langle \Sigma^* \rangle$ in ring $\mathbf{K} \ll \Sigma^* \gg$, but not in ring $\mathbf{K}((G))$, the following analogue seems to be the case.

Conjecture 1. $\mathbf{k} \langle \Sigma^* \rangle$ is inert in the $\mathbf{K}((G))$. \blacksquare

Conjecture 2. Assuming Conjecture 1 is true - would matrix-trivializer M exist such that $M, M^{-1} \in \mathbf{K}^{n \times n} \ll \Sigma^* \gg$? \blacksquare

Formulae (8) implies the following formulae for computing the coefficients in LRR:

$$(r_{k+n}, w) = \sum_{\substack{(1) 1 \leq i \leq n \\ (2) w_{i_s} \tilde{w}_{i_s} = w, w_{i_s} \in \Sigma^*, \tilde{w}_{i_s} \in G}} (r_{k+n-i}, w_{i_s})(\tilde{X}_{n-i}, \tilde{w}_{i_s}) \quad (10)$$

The second condition of summing means that $w_{i_s} = \alpha_{i_s}\beta_{i_s}, \beta_{i_s}^{-1}\gamma_{i_s} = \tilde{w}_{i_s}, \alpha_{i_s}, \beta_{i_s}, \gamma_{i_s} \in \Sigma^*$. Therefore $|\beta_{i_s}| \leq |w_{i_s}| = k + n - i$ and number of summands in (10) is limited.

Conjecture 3. Would the length of canceling suffixes and prefixes (like a β_{i_s}) be limited too for each LRR? \blacksquare

Open problem 3 (Archangelsky [A2]). For a given $\tilde{r} \in \mathbf{K}((G))$ determine whether the lengths of all negative subwords of words in $\text{supp}(\tilde{r})$ are limited (i.e., subwords in alphabet Σ^{-1} only). \blacksquare

We apply rule (10) for examining coefficients in the inverse element of Hadamard product. We mean FPS p is Hadamard inverse of FPS r iff $r \odot p = \sum_w 1 \cdot w = \underline{\Sigma}^*$. The problem of existence of such an element is still open. All papers on the issue either study FPS on cyclic/commutative semigroups (Cori [Cor], Benzaghou [Ben1, Ben2], Benzaghou, Bezivin [BB], Anselmo, Bertoni [AB], Poorten [Po]) or simple samples of inversable FPS on $\Sigma^*, |\Sigma| \geq 2$ (Gerardin [G]).

Theorem 1.4. Let Σ be alphabet, $|\Sigma| \geq 2$, $\mathfrak{A}_r, \mathfrak{A}_p$ be $\mathbf{Q}^+ - \Sigma^*$ - automata which behaviours are FPS r, p and let the coefficients in the common stencil of automata

$$\begin{cases} \mathfrak{A}_r \\ \mathfrak{A}_p \\ q = \underline{\Sigma}q + \underline{\varepsilon} \end{cases} \quad (11)$$

are in \mathbf{Q}^+ . Then $r \odot p = \underline{\Sigma}^*$ implies both r, p have a finite image.

Proof. Consider common stencil of automata (11) (t is the sum of states for automata \mathfrak{A}_r and \mathfrak{A}_p plus 1):

$$\begin{cases} r_{n+t} = \sum_{i=1}^t r_{n+t-i} \tilde{X}_{t-i} \\ p_{n+t} = \sum_{i=1}^t p_{n+t-i} \tilde{X}_{t-i} \\ \underline{\Sigma}^{n+t} = \sum_{i=1}^t \underline{\Sigma}^{n+t-i} \tilde{X}_{t-i}, \quad n \geq 0 \end{cases} \quad (12)$$

According to (12) and (10) a coefficient of the word $w \in \text{supp}(r_{n+t})$ in r_{n+t} satisfies the follows:

$$\alpha = \sum_{s \in S} \alpha_s x_s \quad (13)$$

where α_s - coefficients of $\text{supp}(r_{n+t-i})$ and x_s - coefficients of $\text{supp}(\tilde{X}_{t-i})$, and $|S|$ is finite. Respectively,

$$\frac{1}{\alpha} = \sum_{s \in S} \frac{1}{\alpha_s} x_s \quad (14)$$

$$1 = \sum_{s \in S} 1 \cdot x_s \quad (15)$$

Multiplying (13) and (14) we obtain

$$\begin{aligned}
& \alpha \cdot \frac{1}{\alpha} = 1 = \left(\sum_{s \in S} \alpha_s x_s \right) \left(\sum_{s \in S} \frac{1}{\alpha_s} x_s \right) = \\
& = \sum_{s \in S} x_s^2 + \sum_{i \neq j; i, j \in S} \left(\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \right) x_i x_j \geq \sum_{s \in S} x_s^2 + 2 \sum_{i \neq j; i, j \in S} x_i x_j = \\
& = \left(\sum_{s \in S} x_s \right)^2 = 1,
\end{aligned}$$

that is why all $\alpha_i = \alpha_j = \alpha$, i.e. new coefficients do not appear in r_{n+t} . ■

Open problem 4. *Positiveness of all coefficients in all stencils and layers is an essential part of the proof of Theorem 3. In general case this limitation would not exist – therefore one would require to solve (or describe the set of solutions for) the system of Diophantine equations $\{(13), (14), (15)\}$ ($\alpha_i \in \mathbf{N}^+, x_i \in \mathbf{Q}$). For small numbers of unknowns the system above indeed has only trivial solutions. It seems like the class of invertable by Hadamard rational FPS is very narrow. ■*

Method of Theorem 4 may be implemented for the obtaining a necessary condition for the solution of following

Open problem 5 (Restivo, Reutenauer [RR]). *Let s be a FPS with integer coefficients and p a prime number; if $\sum_w p^{(s,w)} w$ is rational, then so are s and $\sum_w p^{-(s,w)}$. ■*

Corollary 5. *If $s \in \mathbf{Q}^{\text{rat}} \ll \Sigma^* \gg$, $p \in \mathbf{N}$, $s_1 = \sum_w p^{(s,w)}$, $s_2 = \sum_w p^{-(s,w)}$, $s_1, s_2 \in (\mathbf{Q}^+)^{\text{rat}} \ll \Sigma^* \gg$ and the coefficients of the common stencil of automata*

$$\begin{cases} \mathfrak{A}_{s_1} \\ \mathfrak{A}_{s_2} \\ q = \Sigma q + \varepsilon \end{cases}$$

are in \mathbf{R}^+ , then s, s_1, s_2 have a finite image. ■

Let us try for a given LRR build a FPS, a representation of which the former is:

$$\begin{aligned}
r & = \sum_{i=0}^{n-1} r_i + \sum_{i=n}^{\infty} r_i = \sum_{i=0}^{n-1} r_i + \sum_{i=n}^{\infty} \sum_{j=1}^n r_{i-j} \tilde{X}_{n-j} = \\
& = \sum_{i=0}^n r_i + \sum_{j=1}^n \sum_{i=j-1}^{\infty} r_i \tilde{X}_{j-1} = \\
& = \sum_{i=0}^{n-1} r_i - \sum_{j=1}^{n-1} \sum_{s=0}^{j-1} r_s \tilde{X}_j + \sum_{j=1}^n \sum_{i=0}^{\infty} r_i \tilde{X}_{j-1} = \\
& = r_0 + \sum_{i=1}^{n-1} (r_i - \sum_{s=0}^{i-1} r_s \tilde{X}_i) + r \sum_{j=1}^n \tilde{X}_{j-1}
\end{aligned} \tag{16}$$

Solve this equation for r :

$$r = (r_0 + \sum_{i=1}^{n-1} (r_i - \sum_{s=0}^{i-1} r_s \tilde{X}_i)) (\sum_{j=1}^n \tilde{X}_{j-1})^* \tag{17}$$

Unarguably we took too much liberty when applying limit to both parts of identity (16). It still needs to be proved that the obtained expression is indeed the sum of r_i and only them. Because of size limit we would not do that but do illustrate using Example 2 that it is true:

$$\begin{aligned}
& (\underline{a} + \underline{b})^* \underline{a} (\underline{a} + \underline{b})^* = s \stackrel{(17)}{=} (s_1 + s_0 - s_0 \tilde{X}_1) (\tilde{X}_1 + \tilde{X}_2)^* = \\
& = \underline{a} (2\underline{a} + \underline{b} + \underline{a}^{-1} \underline{b} \underline{a} - \underline{a}^2 - 2\underline{b} \underline{a} - \underline{b} \underline{a}^{-1} \underline{b} \underline{a})^* = ((\underline{\varepsilon} - 2\underline{a} - \underline{b} - \underline{a}^{-1} \underline{b} \underline{a} + \\
& + \underline{a}^2 + 2\underline{b} \underline{a} + \underline{b} \underline{a}^{-1} \underline{b} \underline{a}) \underline{a}^{-1})^{-1} = ((\underline{a}^{-1} - \underline{\varepsilon} - \underline{b} \underline{a}^{-1}) (\underline{\varepsilon} - \underline{a} - \underline{b}))^{-1} = \\
& = ((\underline{\varepsilon} - \underline{a} - \underline{b}) \underline{a}^{-1} (\underline{\varepsilon} - \underline{a} - \underline{b}))^{-1} = (\underline{a} + \underline{b})^* \underline{a} (\underline{a} + \underline{b})^*. \blacksquare
\end{aligned}$$

Brzozowski, Cohen [BC] studied a decompositions of rational languages into star languages : $P = R^* S$. One may ask about such decomposition in $\mathbf{K}((G))$. Of course, arbitrary regular language R may be trivially decomposed into star FPS in $\mathbf{K}((G))$: $\underline{R} = \underline{P}^* (\underline{\varepsilon} - \underline{P}) \underline{R}$, where P is arbitrary regular language too.

It is interesting to study a nontrivial case. Consider a common stencil of two arbitrary FPS in the form (17). It implies

Theorem 1.5. *Each two FPS $r, p \in \mathbf{k}^{\text{rat}} \ll \Sigma^* \gg$ have a representation in $\mathbf{K}((G))$ with nontrivial common star factor : $r = \tilde{r}_1 \tilde{s}^*, p = \tilde{p}_1 \tilde{s}^*$. ■*

Judging by appearance the regular expression (17) does not represent FPS from $\mathbf{k} \ll \Sigma^* \gg$, since it contains inverse elements from Σ^{-1} and \mathbf{K} . The transition matrix for the corresponding $\mathbf{K} - (\Sigma \cup \Sigma^{-1})^*$ -automaton would contain elements from Σ^{-1} and \mathbf{K} too – although the behavior of this automaton would be exactly FPS r that is without Σ^{-1} and $\mathbf{K} \setminus \mathbf{k}$.

Open problem 6 ("Fatou extensions"). *Let $A \in \mathbf{Z}^{n \times n} \langle \Sigma \cup \Sigma^{-1} \rangle$ but all layers of FPS $r = \sum_{i=0}^{\infty} (A^i)_{1,n}$ are in $N \langle \Sigma^* \rangle$. Would $r \in N^{\text{rat}} \ll \Sigma^* \gg$ be true? And $\mathbf{Z}^{\text{rat}} \ll \Sigma^* \gg$? And $K^{\text{alg}} \ll \Sigma^* \gg$? ■*

Open problem 7 (Berstel etc. [BBCPP]). *Does a function $n \rightarrow r_n$ preserve a rationality? That is if $\{a_n\}_{n \geq 0}$ is a rational sequence of natural numbers, r is rational FPS then would $\sum_{i=0}^{\infty} r_{a_i}$ be rational? ■*

Open problem 8. *Based on given LRR of FPS p, q build LRR of : $p^*, p + q, pq, p \odot q, p \sqcup \sqcup q$. ■*

Open problem 9. *Describe the set of all stencils of given rational FPS. ■*

Open problem 10. *Stencils in their turn are rational FPS. One can be built their LRR and so on. What can be said about the process ? ■*

Conclusion

Many researchers guessed about the existence of a linear dependency between the current value of FPS and a limited number of previous ones, but have failed to express it in a convenient universal form that would allow to obtain trivially results above. Thus for example,

Restivo, Reutenauer [RR]: *FPS $s \in K \ll \Sigma^* \gg$ is rational iff for any word x there is a common linear recurrence relation over K satisfied by all the sequences $\{(s, ux^n v)\}_{n \geq 0}, u, v \in \Sigma^*$.*

The below listed authors used for stencil the same ring as for represented FPS, what undercut readability and applications:

Salomaa, Soittola [SS]: *Assume $r \in K^{\text{rat}} \ll \Sigma^* \gg$ and N is rank of r . Show that if $|w_0| = N$ then there are words w_1, \dots, w_N and elements c_1, \dots, c_N of K such that $|w_i| < N, i = \overline{1, N}$ and for all words w :*

$$(r, ww_0) = c_1(r, ww_1) + \dots + c_N(r, ww_N)$$

Berstel, Reutenauer [BR]: *For any rational series S of rank n there exist a prefix-closed set P of n elements, with an associated prefix set C , and coefficients $\alpha_{c,p} (c \in C, p \in P)$ such that, for all words w and all $c \in C$:*

$$(S, cw) = \sum_{p \in P} \alpha_{c,p} (S, pw).$$

or limited the domain of definition of the linear relation :

Eilenberg [E]: *$f = \sum a_n z^n$ is rational iff the following "recursion formula" holds for all t sufficiently large:*

$$a_{t+m} + c_1 a_{t+m} + \dots + c_m a_t = 0.$$

On the other hand Cohn [Coh] did not lost universality and convenience but to achieve that he had to 'maim' previous layers:

A series $r \in K((X; \alpha))$ is rational iff there exist integer m, n_0 and elements $c_1, \dots, c_m \in K$ such that for all $n > n_0$:

$$r_n = r_{n-1}^\alpha c_1 + r_{n-2}^{\alpha^2} c_2 + \dots + r_{n-m}^{\alpha^m} c_m$$

As for Varricchio [V] – he did not go beyond the statement of a linear dependency for initial interval of FPS:

Let $s \in K^{\text{rat}} \ll \Sigma^ \gg, \Sigma^{[N]}$ be the set of words of Σ whose length is less then or equal to N , μ be matrix interpretation of S . Then one can effectively compute an integer N , depending on S with the property that for any $u \in \Sigma^{[N+1]}$ there exist a set $T = \{\sigma_v\}_{v \in \Sigma^{[N]}} \subseteq K$ such that $\mu(u) = \sum_{\sigma_v \in T} \mu(v)$.*

It is very strange that author failed to discover the attempts to use linear recurrence in FPS on free commutative monoid $c(\Sigma^*)$, $|\Sigma| \geq 2$. According to Kuich, Salomaa [KS] $K^{alg} \ll c(\Sigma^*) \gg = K^{rat} \ll c(\Sigma^*) \gg$. Therefore many K - algebraic FPS can be studied with the help of LRR.

As one see the proposed approach contrary to the predecessors is systematic and handy. As a indirect proof of that fact is a large number of correlations between FPS and combinatorics collected by the author and left out the scope of this work.

References

- [BR] Berstel J., Reutenauer C., *Rational Series and Their Languages*, Springer-Verlag(1988).
- [E] Eilenberg S., *Automata, Languages, and Machines*, Vol.A, Acad. Press (1974).
- [RS] Rabin M.O., Scott D., *Finite Automata and their decision problems*, IBM J.Res.Develop. 3 (1959), 114-125. 7
- [CS] Culik K. II, Salomaa A., *On the decidability of homomorphism equivalence for languages*, J.Comput.Syst.Sci, 17(1978) , 163-175.
- [SS] Salomaa A., Soittola M., *Automata-Theoretic Aspects of Formal Power Series*, Springer-Verlag, 1978.
- [KS] Kuich W., Salomaa A., *Semirings, Automata, Languages*, 1985.
- [Coh] Cohn P.M., *Free Rings and Their Relations*, 2nd edn. , Acad. Press (1985).
- [L] Lewin J., *Fields of fractions for group algebras of free groups*, Trans.AMS , 192 (1974), 339-346.
- [R] Reutenauer C., *On a question of S.Eilenberg*, Theor.Comp.Sci., 5(1977), 219.
- [Pa] Passman D.S., *The Algebraic Structure of Group Rings*, Wiley (1977).
- [HK] Harju T., Karhumaki J., *The equivalence problem of multitape finite automata*, Theor.Comp.Sci., 78 (1991) 347-355.
- [T] Tzeng W.-G., *A polynomial-time algorithm the equivalence of probabilistic automata*, FOCS (1989).
- [A1] Archangelsky K.V., *Efficient algorithm for checking multiplicity equivalence for the finite zet-sigma-automata*, Lect.NotesComp.Sci., 2450 (2002) ,283-289.
- [HW] Hebish U., Weinert H.J., *Generalized semigroup semirings which are zero-divisor-free or multiplicatively left-cancellative*, Theor.Comp.Sci., 92 (1992), 269-289.
- [K] Karhumaki J., *Equations over finite sets of words and equivalence problems in automata theory*, Thepr.Comp.Sci., 106(1993), 103-118.
- [Ber] Bergman G.M., *Commuting elements in free algebras and related topics in ring theory*, Thesis, Harvard University, 1967.
- [A2] Archangelsky K.V., *The length of homogeneous subwords in rational formal power series*, Bull. Eur. Ass.Theor. Comp. Sci., 71(2000) 253.
- [Cor] Cori R., *Formules d'inversion de series formelles*, Ecole de Printemps d'Informatique Theretique "Series formelles en variables non commutatives et applications", Landes, (1977) 55-67.
- [Ben1] Benzaghoul B., *Sur les series rationnelles a une variable*, Ecole de Printemps d'Informatique Theretique "Series formelles en variables non commutatives et applications", Landes, (1977) 1-3.
- [Ben2] Benzaghoul B., *Algebres de Hadamard*, Bull. Soc. Math. France, 98 (1970), 209-252.
- [BB] Benzaghoul B., Bezzivin J.-P., *Propertes algebriques de suites differetielement finies*, Bull.Soc.math.France, 120 (1992), 327-346.
- [AB] Anselmo M., Bertoni A., *On 2pfa's and the Hadamard quotient of formal power series*, Bull. Belg.Math.Soc. 1 (1994), 165-173.
- [Po] Poorten van der A.J., *Solution de la conjecture de Pisot sur le quotient de Hadamard de deux fractions rationnelles*, C.R.Acad.Sc. Paris 306, serie I (1988), 97-102.
- [G] Gerardin K., *Quotient de Hadamard de series rationnelles en variables non commutatives*, Ecole de printemps d'Informatique Theretique "Series formelles en variables non commutatives et applications", Landes, (1977) ,119-134.
- [RR] Restivo A., Reutenauer C., *Rational languags and the Burnside problem*, Theor.Comp.Sci., 40(1985),13-30.
- [BC] Brzozowski J.A., Cohen R., *On decomposition of regular events*, J.ACM, 16(1969),132-144.
- [BCCPP] Berstel J., Boasson L., Carton O., Petazzoni B., Pin J.-E., *Operations preserving recognizable languages*, FCT'03.
- [V] Varricchio S., *Rational series with coefficients in commutative ring*, LITP 91.38, Institut Blaise Pascal, 1991.