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On a Conjecture Concerning Littlewood-Richardson Coefficients

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Abstract. *We prove that a conjecture of Fomin, Fulton, Li, and Poon, associated to ordered pairs of partitions, holds for many infinite families of such pairs. We also show that the generic bounded height case can be reduced to only checking that the conjecture holds for a finite number of pairs, for any given height. Moreover, we propose a natural generalization of the conjecture to the case of skew shapes.*

Résumé. *Nous démontrons qu'une conjecture de Fomin, Fulton, Li et Poon, associée aux couples de partages, se vérifie pour plusieurs classes infinies de tels couples. Nous montrons aussi que le cas générique, pour des partages de hauteurs bornés, se réduit à la vérification de la conjecture pour un nombre fini de couples, et ce pour chaque hauteur. De plus, nous présentons une généralisation naturelle de la conjecture au cas des couples de partages gauches.*

1. Introduction

In [**1**], Fomin, Fulton, Li, and Poon state a very interesting conjecture concerning the Schur-positivity of special differences of products of Schur functions of the form

$$s_{\mu^*}s_{\nu^*} - s_\mu s_\nu,$$

where μ^* and ν^* are partitions constructed from an ordered pair of partitions μ and ν through a seemingly strange procedure. In our presentation, their transformation $(\mu,\nu)\mapsto(\mu^*,\nu^*)$ on ordered pairs of partitions, will rather be denoted

$$(\mu,\nu)\longmapsto(\mu,\nu)^* = (\lambda(\mu,\nu),\rho(\mu,\nu)) \tag{1.1}$$

and will still be called the $*$-operation. As we shall see, this change of notation is essential in order to simplify the presentation of the many nice combinatorial properties of this operation. On the other hand, it makes clear that both entries, λ and ρ of the image $(\mu,\nu)^*$ of (μ,ν), depend on μ and ν.

With this slight change of notation, the original definition of the $*$-operation is as follows. Let $\mu = (\mu_1,\mu_2,\ldots,\mu_n)$ and $\nu=(\nu_1,\nu_2,\ldots,\nu_n)$ two partitions with the same number of parts, allowing zero parts. From these, two new partitions

$$\lambda(\mu,\nu) = (\lambda_1,\lambda_2,\ldots,\lambda_n) \qquad \text{and} \qquad \rho(\mu,\nu)=(\rho_1,\rho_2,\ldots,\rho_n)$$

are constructed as follows

$$\begin{aligned} \lambda_k &:= \mu_k - k + \#\{j \mid 1\le j\le n,\ \nu_j - j \ge \mu_k - k\}; \\ \rho_j &:= \nu_j - j + 1 + \#\{k \mid 1 \le k \le n,\ \mu_k - k > \nu_j - j\}. \end{aligned} \tag{1.2}$$

Although this definition does not make it immediately clear, both $\lambda(\mu,\nu)$ and $\rho(\mu,\nu)$ are truly partitions, and they are such that

$$|\lambda(\mu,\nu)| + |\rho(\mu,\nu)| = |\mu| + |\nu|,$$

where as usual $|\mu|$ denotes the sum of the parts of μ.

Recall that the product of two Schur functions can always be expanded as a linear combination

$$s_\mu s_\nu = \sum_\theta c^\theta_{\mu\,\nu} s_\theta,$$

of Schur functions indexed by partitions θ of the integer $n = |\mu| + |\nu|$, since these Schur functions constitute a linear basis of the homogeneous symmetric functions of degree n. It is a particularly nice feature of this expansion that the coefficients $c^\theta_{\mu\,\nu}$ are always non-negative integers. They are called the *Littlewood-Richardson coefficients.* More generally, we say that a symmetric function is *Schur positive* whenever the coefficients in its expansion, in the Schur function basis, are all non-negative integers. For more details on symmetric function theory see Macdonald's classical book [**2**], whose notations we will mostly follow. We can then state the following:

Conjecture 1.1 (Fomin-Fulton-Li-Poon). *For any pair of partitions* (μ,ν), *if*

$$(\mu,\nu)^* = (\lambda,\rho),$$

then the symmetric function

$$s_\lambda s_\rho - s_\mu s_\nu \tag{1.3}$$

is Schur-positive.

In other words, this says that $c^\theta_{\mu\,\nu} \le c^\theta_{\lambda\,\rho}$, for all θ such that s_θ appears in the expansion of $s_\mu s_\nu$.

For an example of one of the simplest case of the $*$-operation, let $\mu = (a)$ and $\nu = (b)$, with $a > b$, be two one-part partitions. In this case, we get

$$((a),(b))^* = (a-1, b+1),$$

so that Conjecture 1.1 corresponds exactly to an instance of the classical Jacobi-Trudi identity:

$$\begin{aligned} s_{a-1}s_{b+1} - s_a s_b &= \det\begin{pmatrix} s_{a-1} & s_a \\ s_b & s_{b+1} \end{pmatrix} \\ &= s_{a-1,b+1}. \end{aligned}$$

In this article we give a new recursive combinatorial description of the $*$-operation. This recursive description allows us to prove many instances of Conjecture 1.1 and to show that it reduces to checking a finite number of instances for any fixed ν, if we bound the number of parts of μ. Moreover we show how to naturally generalize the conjecture to pairs of skew partitions.

2. Combinatorial description of the $*$-operation.

We first derive some nice combinatorial properties of the transformation "$*$". To help in the presentation of these properties, let us introduce some further notations. We often identify a partition with its (Ferrers) diagram. Diagrams are drawn here using the "French" convention of ordering parts in decreasing order from bottom to top.

We write $\mu = \overrightarrow{\alpha}^\ell$, if the partition μ is obtained from the partition α by adding one cell in line ℓ; and $\mu = \alpha{\uparrow_k}$, if μ is obtained from α by adding one cell in column k. In other words, $\mu = \overrightarrow{\alpha}^i$ means that $\mu_i = \alpha_i$ for all $i \neq \ell$, and $\mu_\ell = \alpha_\ell + 1$. This is illustrated in Figure 1 in term of diagrams.

FIGURE 1.

Observe that,

$$\mu = \overrightarrow{\alpha}^{i} \quad \text{iff} \quad \mu' = \overrightarrow{\alpha'}^{\mu_i}$$
$$\text{iff} \quad \mu = \alpha{\uparrow}_{\mu_i}$$
$$\text{iff} \quad \mu' = \alpha'{\uparrow}_i$$

We can now state our recursive description of the $*$-operation.

Proposition 2.1 (Recursive formula). *For any partitions α and ν, if $(\lambda, \rho) = (\alpha, \nu)^*$, then we have*

$$(\overrightarrow{\alpha}^{i}, \nu)^* = \begin{cases} (\lambda, \overrightarrow{\rho}^{j}) & \text{where } j \text{ is such that } \nu_j - j = \alpha_i - i, \text{ if any,} \\ (\overrightarrow{\lambda}^{i}, \rho) & \text{otherwise.} \end{cases} \tag{2.1}$$

Moreover, when in the first case, we have $\overrightarrow{\rho}^{j} = \rho{\uparrow}_{\mu_i}$. In a similar manner, for given μ and β, if $(\lambda, \rho) = (\mu, \beta)^*$, then

$$(\mu, \overrightarrow{\beta}^{i})^* = \begin{cases} (\overrightarrow{\lambda}^{j}, \rho) & \text{where } j \text{ is such that } \mu_j - j = \nu_i - i, \text{ if any,} \\ (\lambda, \overrightarrow{\rho}^{i}) & \text{otherwise,} \end{cases} \tag{2.2}$$

and, when in the first case, we have $\overrightarrow{\lambda}^{j} = \lambda{\uparrow}_{\nu_i}$.

We can clearly use Proposition 2.1 to recursively compute $\lambda(\mu, \nu)$ and $\rho(\mu, \nu)$. Examples are given in Section 4. The computation of the $*$-operation can be simplified in view of the following property of the $*$-operation. For any pair of partitions (μ, ν), we have

$$(\mu, \nu)^* = (\lambda, \rho) \quad \text{iff} \quad (\nu', \mu')^* = (\lambda', \rho'), \tag{2.3}$$

where, as usual, μ' stands from the conjugate of μ. Using the fact that the involution ω (which is the linear operator that maps s_μ to $s_{\mu'}$) is multiplicative, it easily follows that

Proposition 2.2. *Conjecture 1.1 holds for the pair (μ, ν) if and only if it holds for the pair (ν', μ').*

In practice, there are many ways to describe the $*$-operation recursively, since we can freely choose how to make partitions grow. It is sometimes convenient to start from the pair $(0, \nu)$, whose image under the $*$-operation has a simple description.

Lemma 2.3. *Let ν be any partition. Then*

$$\rho(0, \nu) = (\nu_1, \nu_2 - 1, \cdots, \nu_k - (k-1))$$
$$\lambda'(0, \nu) = (\nu'_1 - 1, \nu'_2 - 2, \cdots, \nu'_k - k)$$

where $k = \max\{i : \nu_i - (i-1) \geq 1\}$.

We will sometimes denote respectively $\bar{\nu}$ and $\underline{\nu}$ the partitions $\lambda(0, \nu)$ and $\rho(0, \nu)$. For example if $\nu = (8, 6, 6, 5, 5, 4, 4, 2, 1)$, then

$$\bar{\nu} = (4, 4, 4, 3, 2, 2, 1, 1) \quad \text{and} \quad \underline{\nu} = (8, 5, 4, 2, 1)$$

as is illustrated in Figure 2. In Section 4 we elaborate on the various ways that Proposition 2.1 can be used to compute the $*$-operation. This gives rise to a $*$-operation on pairs of Young tableaux.

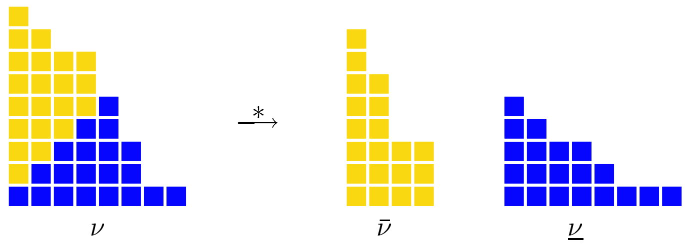

FIGURE 2.

Figure 3 illustrates the effect of the $*$-operation on some pairs of the form $((n), \nu)$.

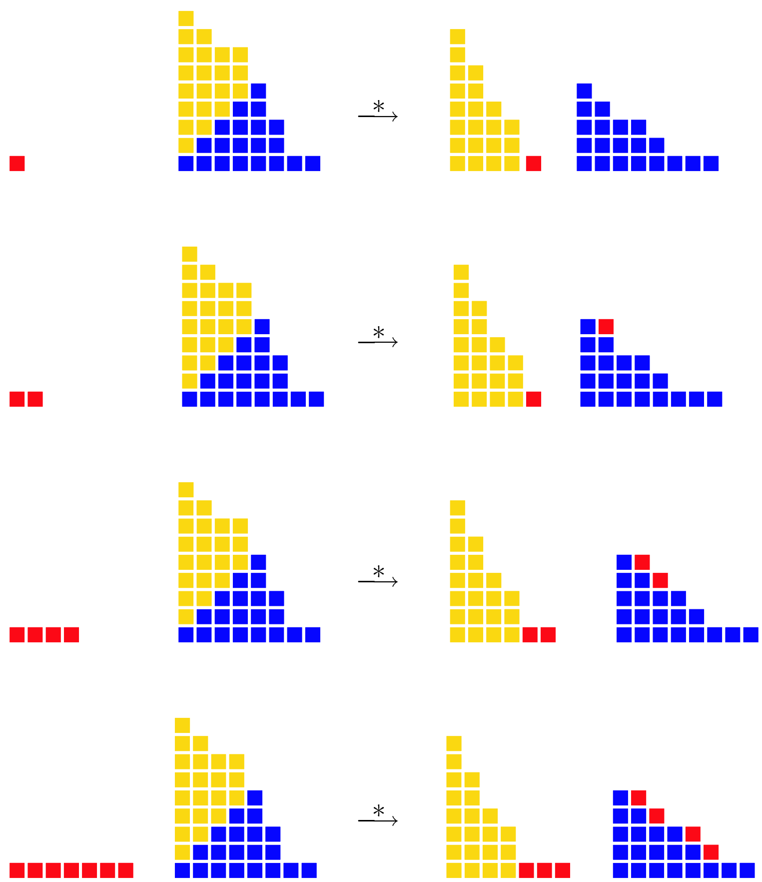

FIGURE 3.

Given partitions μ and ν, define the partition $\mu + \nu$ by

$$(\mu + \nu)_i := \mu_i + \nu_i.$$

and set

$$\mu \cup \nu := (\mu' + \nu')'.$$

For example, if $\mu = (3,3,2,2,1)$ and $\nu = (5,3,1,0,0)$, then $\mu \cup \nu = (5,3,3,3,2,2,1,1)$ and $\mu + \nu = (8,6,3,2,1)$. As usual, for μ and ν two partitions of n, μ is said to be *dominated* by ν, in formula $\mu \preceq \nu$, if for all $k \geq 1$:

$$\mu_1 + \mu_2 + \cdots + \mu_k \leq \nu_1 + \nu_2 + \cdots + \nu_k.$$

Another remarkable property of the $*$-operation is that its image behaves nicely under the dominance order. More precisely,

Lemma 2.4. *For any pair of partitions* (μ, ν)*, if* $(\lambda, \rho) = (\mu, \nu)^*$*, then we have*

$$\mu \cup \nu \succeq \lambda \cup \rho, \qquad \text{and equivalently} \tag{2.4}$$

$$\mu + \nu \preceq \lambda + \rho. \tag{2.5}$$

Observe that when s_θ appears in $s_\mu s_\nu$ with a nonzero coefficient, then

$$\mu \cup \nu \preceq \theta \preceq \mu + \nu,$$

thus (2.4) and (2.5) imply that

$$\lambda \cup \rho \preceq \theta \preceq \lambda + \rho,$$

which is compatible with Conjecture 1.1.

Lemma 2.4 immediately implies a statement very similar to that of Conjecture 1.1.

Proposition 2.5. *For any pair of partitions* (μ, ν)*, if* $(\lambda, \rho) = (\mu, \nu)^*$*, then*

$$h_\lambda h_\rho - h_\mu h_\nu$$

is Schur-positive.

Recalling that $h_\mu h_\nu = h_{\mu \cup \nu}$, this follows from the fact that a difference of two homogeneous symmetric functions $h_\alpha - h_\beta$ is Schur-positive, if and only if $\alpha \preceq \beta$ (see [**4**, Chapter 2]). A clear link between this proposition and Conjecture 1.1 is established through the classical identity:

$$h_\alpha = s_\alpha + \sum_{\beta \succeq \alpha} K_{\beta\alpha} s_\beta,$$

where as usual $K_{\beta\alpha}$, the *Kostka* numbers, count the number of semi-standard tableaux of shape β and content α.

The following results, shows that the $*$-operation is also compatible with "inclusion" of partitions. Here, we say that α is *included* in μ, if the diagram of α is included in the diagram of μ. We will simply write

$$(\alpha, \beta) \subseteq (\mu, \nu), \qquad \text{whenever} \qquad \alpha \subseteq \mu \quad \text{and} \quad \beta \subseteq \nu.$$

Lemma 2.6. *Let* α, β, μ *and* ν *be partitions such that* $(\alpha, \beta) \subseteq (\mu, \nu)$*. Then* $\lambda(\alpha, \beta) \subseteq \lambda(\mu, \nu)$ *and* $\rho(\alpha, \beta) \subseteq \rho(\mu, \nu)$*.*

An immediate, but interesting, consequence of this lemma is the following observation.

Observation 2.7. Let (α, β) and (γ, δ) be two fixed points such that $(\alpha, \beta) \subseteq (\gamma, \delta)$. Writing simply λ for $\lambda(\mu, \nu)$ and ρ for $\rho(\mu, \nu)$, we see (using Lemma 2.6) that

$$(\alpha, \beta) \subseteq (\mu, \nu) \subseteq (\gamma, \delta),$$

implies

$$(\alpha, \beta) \subseteq (\lambda, \rho) \subseteq (\gamma, \delta).$$

As is underlined in [**1**], a pair of partitions (α, β) is a fixed point of the $*$-operation if and only if

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \cdots \geq \beta_n \geq \alpha_n. \tag{2.6}$$

Let us underline here that, for any (μ, ν), it is easy to characterize the "largest" (resp. "smallest") fixed point contained in (resp. containing) the pair (μ, ν). We will see below how this observation can be used to link properties of λ and ρ to properties of μ and ν. Recall that an *horizontal strip* is a skew shape μ/α with no two squares in the same column, and that a *ribbon* is a connected skew shape with no 2×2 squares (see [**5**, Chapter 7], for more details). If we drop the condition of being connected in this last definition, we say that we have a *weak ribbon*.

Another striking consequence of Lemma 2.6 is that it allows a natural extension of the $*$-operation to skew partitions. Denoting $(\mu,\nu)/(\alpha,\beta)$ the pair of skew shapes $(\mu/\alpha,\nu/\beta)$, we can simply define

$$(\mu/\alpha,\nu/\beta)^* := (\mu,\nu)^*/(\alpha,\beta)^*. \tag{2.7}$$

In other words, we have

$$\lambda(\mu/\alpha,\nu/\beta) := \lambda(\mu,\nu)/\lambda(\alpha,\beta), \tag{2.8}$$

and

$$\rho(\mu/\alpha,\nu/\beta) := \rho(\mu,\nu)/\rho(\alpha,\beta). \tag{2.9}$$

The $*$-operation, or its extension as above, preserves (among others) the following families of pairs of (skew) shapes.

Proposition 2.8. *The "$*$" -operation preserves the families of*

- **(1)** *pairs of hooks;*
- **(2)** *pairs of two-rows partitions;*
- **(3)** *pairs of horizontal strips;*
- **(4)** *pairs of weak ribbon.*

Note that **(1)** and **(2)** follow directly from observation 2.7, and that the statements **(3)** and **(4)** are made possible in view of our extension of the $*$-operation.

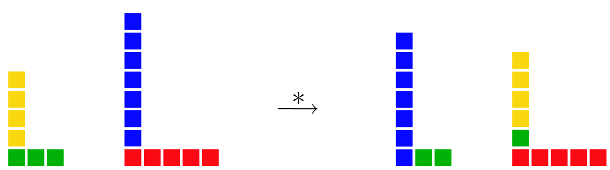

FIGURE 4. The effect of the $*$-operation on hooks.

Results outlined in the sequel, and extensive computer experimentations suggests that we have the following extension of Conjecture 1.1.

Conjecture 2.9. *For any skew partitions μ/α and ν/β, if*

$$(\lambda,\rho) = (\mu/\alpha,\nu/\beta)^*,$$

then the symmetric function

$$s_\lambda s_\rho - s_{\mu/\alpha} s_{\nu/\beta} \tag{2.10}$$

is Schur-positive.

This has yet to be understood in geometrical terms. On the other hand, it is clear that Proposition 2.2 extends to skew partitions. Our last combinatorial observation concerning the $*$-operation is the following. Let τ and ν be two fixed partitions, and consider all possible μ's such that $\rho(\mu,\nu)=\tau$. We claim that there is a minimal such μ, if any, and we denote it $\theta(\tau,\nu)$. More precisely, we easily show that

Proposition 2.10. *Given partitions τ and ν, for any μ such that $\rho(\mu,\nu)=\tau$, then*

$$\theta(\tau,\nu) \subseteq \mu.$$

Furthermore, $\theta = \theta(\tau, \nu)$ *is exactly the partition*

$$\theta = \rho_1^{b_1} \rho_2^{b_2 - b_1} \rho_3^{b_3 - b_2} \cdots,$$

with $b_j = \rho_j - \nu_j + j - 1$.

3. Main results

In this section we state our results concerning the validity of Conjecture 1.1 for certain families of pairs, as well as its reduction to a finite number of tests for other families. We show the following.

Theorem 3.1. *Conjecture 1.1 (or 2.9) holds*

- **(1)** *For any pair* (μ, ν) *of hook shapes.*
- **(2)** *For pairs of two-line and two-column partitions.*
- **(3)** *For skew pairs the form* $(\mu/\alpha, \nu/\beta)$*, with all of* μ*,* ν*,* α *and* β *are hooks.*
- **(4)** *For skew pairs of the form* $(0, \nu/\beta)$*, with* ν/β *a weak ribbon.*

Other families for which we have partial results correspond to Stembridge's (see [**6**]) classification of all multiplicity-free products of Schur functions. More precisely, these are all products of two Schur functions with Schur function expansion having no coefficient larger then 1. Thus, to show Conjecture 1.1 in these cases, we need only show that the coefficient $c_{\lambda\rho}^{\theta}$ of s_θ in the product of s_λ and s_ρ is nonzero, whenever $c_{\mu\nu}^{\theta} = 1$.

Stembridge uses the following notions for his presentation. A *rectangle* is a partition with at most one part size, i.e., empty, or of the form (c^r) for suitable $c, r > 0$; a *fat hook* is a partition with exactly two parts sizes, i.e., of the form $(b^r c^s)$ for suitable $b > c > 0$; and a *near-rectangle* is a fat hook such that it is possible to obtain a rectangle from it by deleting a single row or column. He shows that the product $s_\mu s_\nu$ is multiplicity-free if and only if

- **(a)** μ or ν is a one-line rectangle, or
- **(b)** μ is a two-line rectangle and ν is a fat hook or vice-versa, or
- **(c)** μ is rectangle and ν is a near rectangle or vice-versa, or
- **(d)** μ and ν are both rectangles.

Although we currently have proofs of the conjecture for cases **(a)** and **(d)** of Stembridge's pairs, proofs for cases **(b)** and **(c)** are still in the process of being completed. Since all these share a common approach, we have decide to postpone their presentation to an upcoming paper.

On another register, a careful study of the recursive construction of $\lambda(\mu, \nu)$ and $\rho(\mu, \nu)$ shows that, in a sense, Conjecture 1.1 follows, under some conditions, from a finite number of cases when ν is fixed and μ becomes large.

More precisely, we obtain the result below. As usual, the number of nonzero parts of μ is denoted by $\ell(\mu)$ and called the *height* of μ.

Theorem 3.2. *For any positive integer* p*, let* ν *be a fixed partition with at most* p *parts, i.e.* $\ell(\nu) \leq p$*. Then, the validity of Conjecture 1.1 for the infinite set of all pairs* (μ, ν)*, with* $\ell(\mu) \leq p$*, reduces to checking the validity of the conjecture for the finite set of pairs* (α, ν)*, with* α *having at most* p *parts, and largest part bounded as follows*

$$\alpha_1 \leq p\,(\nu_1 + p). \tag{3.1}$$

Theorem 3.2 can also be generalized in a straightforward manner to the set of skew shapes pairs $(\mu/\alpha, \nu/\beta)$ of bounded height, with ν and α fixed.

4. Final remarks

To study more consequences of the properties of "$*$", we consider the *double Young lattice*, $\mathcal{D}$, which is just the direct product of two copies of the usual Young lattice. This poset is naturally graded by $(\mu,\nu) \mapsto |\mu| + |\nu|$. A *standard (tableau) pair* of *shape* (μ,ν) is a maximal chain in this graded poset that starts at $(0,0)$, the pair of empty partitions, and ends at (μ,ν). For example, we have

$$(0,0) \subseteq (0,1) \subseteq (0,2) \subseteq (1,2) \subseteq (1\,1,2) \subseteq (2\,1,2) \subseteq (2\,1,3) \tag{4.1}$$

Clearly, as in the usual case, such a chain can be identified with a pair (t,r) of standard tableaux, of respective shapes μ and ν, with non-repeated entries from the the set $\{1,2,\ldots,n\}$, $n = |\mu| + |\nu|$. The number $f_{(\mu,\nu)}$ of standard pairs of shape (μ,ν) is thus

$$f_{(\mu,\nu)} = \binom{|\mu|+|\nu|}{|\mu|} f_\mu\, f_\nu \tag{4.2}$$

where f_μ and f_ν are both given by the usual hook formula.

In terms of tableaux, the standard pair (4.1) corresponds to:

$$\left(\begin{array}{|c|c|}\hline 4 & \\ \hline 3 & 5 \\ \hline\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline\end{array}\right).$$

The double Young lattice occurs naturally in the study of representations of the hyperoctahedral groups. This suggests that there might be a link between that subject and the study of properties of the transformation "$*$". A *semi-standard pair* is a chain

$$(0,0) = \pi_0 \subseteq \pi_1 \subseteq \cdots \subseteq \pi_k = (\mu,\nu)$$

in $\mathcal{D}$, such that π_{j+1}/π_j is an horizontal strip pair for each $1 \leq j \leq k-1$. For example, the pair of semi-standard tableaux

$$\left(\begin{array}{|c|c|c|}\hline 3 & & \\ \hline 2 & 3 & 3 \\ \hline\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 2 & 3 & \\ \hline 1 & 1 & 3 \\ \hline\end{array}\right),$$

corresponds to the path

$$(0,0) \subseteq (0,2) \subseteq (1,21) \subseteq (31,32).$$

Lemma 4.1. *The function $* : \mathcal{D} \longrightarrow \mathcal{D}$ is a level preserving increasing transformation that preserves both standard and semi-standard pairs.*

For example, for the standard pair

$$\left(\begin{array}{|c|c|c|c|c|}\hline 26 & & & & \\ \hline 22 & 23 & 24 & & \\ \hline 16 & 17 & 18 & 19 & 20 \\ \hline 9 & 10 & 11 & 12 & 13 \\ \hline\end{array}\ ,\ \begin{array}{|c|c|c|c|c|c|c|c|}\hline 25 & & & & & & & \\ \hline 21 & & & & & & & \\ \hline 14 & 15 & & & & & & \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline\end{array}\right)$$

the corresponding result is

$$\left(\begin{array}{|c|c|c|c|c|}\hline 26 & & & & \\ \hline 22 & 24 & & & \\ \hline 16 & 17 & 19 & 20 & \\ \hline 9 & 10 & 11 & 12 & 13 \\ \hline\end{array}\ ,\ \begin{array}{|c|c|c|c|c|c|c|c|}\hline 25 & & & & & & & \\ \hline 21 & 23 & & & & & & \\ \hline 14 & 15 & 18 & & & & & \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline\end{array}\right).$$

Observe that, in this example, the original right tableau is contained in the resulting right tableau. This is heavily dependent on the very particular labeling that has been chosen here in the original pair. Up to a careful choice of labeling this phenomenon becomes a frequent (but not systematic) occurrence. Fixed points, for standard pairs, are easily characterized as follows.

Lemma 4.2. *A standard pair (t, r), of shape (μ, ν), is fixed point of the $*$-operation, if and only if (μ, ν) is fixed, and the tableau, obtained by alternating rows of r and rows of t, is standard.*

We believe that to get a better understanding of the $*$-operation, the study of its effect on tableaux and semi-standard tableaux will be crucial. For instance this should lead to a proof of a "monomial" versions of Conjectures 1.1 and 2.9. More precisely, recall that the expansion of any Schur function in the basis of monomial symmetric functions involves only positive integers. It would thus follow from the Conjectures that the expansion of the difference of products considered have positive integer coefficients when expanded in term of monomial symmetric function. In particular, using definition (4.2), one should have

$$f_{(\lambda,\rho)} \geq f_{(\mu,\nu)}. \tag{4.3}$$

whenever $(\lambda, \rho) = (\mu, \nu)^*$. An independent proof of these facts would clearly lend support to the Conjectures.

5. Acknowledgment

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