Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Vancouver 2004

# Asymptotics of multivariate generating functions

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Let $F(\mathbf{x}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{x}^r r$ be the multivariate generating function encoding the coefficients $\mathbf{r} := (r_1, \ldots, r_d)$. We would like to find estimates for the coefficients $\{a_{\mathbf{r}}\}$ that are asymptotically valid as $\mathbf{r} \to \infty$. In the univariate case, there is a well known, powerful, elegant apparatus for deriving such asymptotics from the analytic behavior of $F$ near its minimal modulus singularity. In more than one variable, this problem is nearly untouched. Writing $F = \sum g_n(r_1, \ldots, r_{d-1}) z_d^{r_d}$, if $g_n$ is asymptotically $g^n$ for some $g$, then theorems by Bender, Richmond, Canfield and Gao yield Gaussian limit laws for $a_{\mathbf{r}}$. No other general results appear to be known.

The present talk will focus on the case of rational generating functions. In the one variable case this class is trivial to analyze, but in the multivariate case even this class poses many unsovled problems. Furthermore, one finds numerous applications within this class. The approach is to write $a_{\mathbf{r}}$ as a multivariate Cauchy integral, and then to use topological techinques to replace this integral with one that is in stationary phase, meaning that it looks locally like $\int_D A(\mathbf{x}) \exp(-|\mathbf{r}| Q(\mathbf{x}))\, d\mathbf{x}$ for some (one hopes postivie definite) quadratic form on a disk-like domain, $D$. Asymptotics can then be read off in a fairly automated way. It is our extreme good fortune that existing results in Stratified Morse Theory are tailor-made to convert the Cauchy integral to the stationary phase integral. A more complete outline of the steps is as follows. This outline is valid for certain geometries of the pole set of $F$.

(1) Write $a_{\mathbf{r}}$ as a Cauchy integral

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} \tag{1}$$

where the torus $T$ is a product of sufficiently small circles around the origin in each coordinate.

(2) The torus $T$ may be replaced by an equivalent $d$-cycle in the homology of $(\mathbb{C}^*)^d$ minus the poles of $F$. Specifically, we denote by $-\infty$ the set where the integrand in (1) is sufficiently small, and represent $T$ in the homology of $(\mathbb{C}^*)^d$ minus the poles of $F$, relative to $-\infty$.

(3) Stratified Morse theory identifies the other homology classes with saddles of the gradient $\mathbf{r} \log \mathbf{z}$ of the function $\mathbf{z}^{\mathbf{r}}$. Each such saddle lives in a stratum of dimension $j < d$ and yields a contribution which is an integral over a product of a cycle ßcyc$^{\|}$ in the stratum with a cycle ßcyc$^{\perp}$ in a transversal to the stratum.

(4) A nonzero contribution at a saddle $\sigma$ occurs when the vector $\mathbf{r}$ is in a certain positive cone determined by the geometry of the pole set of $F$ near $\sigma$.

(5) The integral over ßcyc$^{\perp}$ is equal to an easily computed spline, and the integral over ßcyc$^{\|}$ is then asymptotically evaluated by the saddle point method.

## References

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