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On a Class of Totally Nonnegative f-immanants

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Abstract. *We define a family of totally nonnegative polynomials of the form* $\sum f(\sigma)x_{1,\sigma(1)}\cdots x_{n,\sigma(n)}$ *and show that this family generalizes all known totally nonnegative polynomials of the form* $\Delta_{J,J'}(x)\Delta_{L,L'}(x)-\Delta_{I,I'}(x)\Delta_{K,K'}(x)$*, where* $\Delta_{J,J'}(x),\ldots,\Delta_{K,K'}(x)$ *are matrix minors. We also give new conditions on the sets* $J,\ldots,K'$ *which guarantee that the corresponding polynomials are totally nonnegative.*

RÉSUMÉ. *Nous donons une famille de polynômes totallement nonnegatifs de la forme* $\sum f(\sigma)x_{1,\sigma(1)}\cdots x_{n,\sigma(n)}$ *et montrons que cette famille generalise tous les polynômes totalement nonnegatifs de la forme* $\Delta_{J,J'}(x)\Delta_{L,L'}(x)-\Delta_{I,I'}(x)\Delta_{K,K'}(x)$*, ou* $\Delta_{J,J'}(x),\ldots,\Delta_{K,K'}(x)$ *sont des mineurs des maitrices. Nous donons aussi des conditions nouvelles sur les ensembles* $J,\ldots,K'$ *qui guarantisent que les polynômes correspondents sont totalement nonnegatifs.*

1. Introduction

A real matrix is called *totally nonnegative* (TNN) if the determinant of each of its square submatrices is nonnegative. Such matrices appear in many areas of mathematics and the concept of total nonnegativity has been generalized to apply not only to matrices, but also to other mathematical objects (See e.g. [**10**] and references there.) In particular, a polynomial $p(x)$ in n^2 variables $x=(x_{1,1},\ldots,x_{n,n})$ is called *totally nonnegative* if it satisfies

$$p(A) \underset{\text{def}}{=} p(a_{1,1},\ldots,a_{n,n}) \geq 0$$

for every $n\times n$ TNN matrix $A=[a_{i,j}]$. Obvious examples are the $n\times n$ determinant and the $k\times k$ *minors*, i.e. the determinants of $k\times k$ submatrices. Given subsets $I=\{i_1,\ldots,i_k\}$ and $I'=\{i'_1,\ldots,i'_k\}$ of $[n]=\{1,\ldots,n\}$ we define the (I,I') *minor* to be the polynomial

$$\Delta_{I,I'}(x)=\sum_{\sigma\in S_k}(-1)^{\text{INV}(\sigma)}x_{i_1,i'_{\sigma(1)}}\cdots x_{i_k,i'_{\sigma(k)}}.$$

Thus $\Delta_{I,I'}(A)$ is the determinant of the submatrix of A corresponding to rows I and columns I'.

Some recent interest in TNN polynomials concerns a collection of polynomials arising in the study of canonical bases of quantum groups [**3**]. While this collection, known as the *dual canonical basis* of type A_{n-1}, currently has no simple description, Lusztig [**18**] has proved that it consists entirely of TNN polynomials. Berenstein, Gelfand, and Zelevinsky [**4, 11**] have developed machinery to enumerate the dual canonical basis elements for small n, and further investigation suggests that these polynomials are expressable as subtraction-free Laurent expressions in matrix minors [**9**].

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Progress on the problem of describing the dual canonical basis is obstructed somewhat by the scarcity of nontrivial families of polynomials which are known to be TNN. Providing examples of such families, several authors have conjectured and proved the total nonnegativity of polynomials called *f-immanants*, constructed from functions $f : S_n \to \mathbb{R}$ by

$$\mathrm{Imm}_f(x) = \sum_{\sigma \in S_n} f(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}. \tag{1.1}$$

Stembridge proved the total nonnegativity of the immanants $\mathrm{Imm}_{\chi^\lambda}(x)$ constructed from the irreducible characters $\chi^\lambda : S_n \to \mathbb{R}$ of S_n [**20**, Cor. 3.3]. (See also [**15**].) These immanants are usually abbreviated $\mathrm{Imm}_\lambda(x)$,

$$\mathrm{Imm}_\lambda(x) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}. \tag{1.2}$$

Stembridge also proved the stronger result [**20**, Cor. 3.4] that the immanants

$$\mathrm{Imm}_\lambda(x) - \deg(\chi^\lambda) \det(x) \tag{1.3}$$

are TNN, where $\deg(\chi^\lambda)$ is the dimension of the Specht module $\mathcal{S}^\lambda$, i.e. the number of standard Young tableaux of shape λ.

Discovering another family of TNN immanants, Fallat et. al. [**8**, Thm. 4.6] characterized all TNN immanants of the form

$$\Delta_{J,J}(x)\Delta_{\overline{J},\overline{J}}(x) - \Delta_{I,I}(x)\Delta_{\overline{I},\overline{I}}(x), \tag{1.4}$$

where $\overline{I} = [n] \smallsetminus I$, $\overline{J} = [n] \smallsetminus J$. This result was later strengthened [**19**, Thm. 3.2] to include products of nonprincipal minors

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x). \tag{1.5}$$

(For other work concerning TNN immanants, see [**2, 7**].)

More results of Stembridge [**20**, Sec. 2], [**21**, Sec. 5] suggest that certain quotients of the symmetric group algebra provide important information about TNN polynomials in general. In this paper, we use such a quotient which is isomorphic to the Temperley-Lieb algebra $\mathfrak{t}^n 2$ to define a family of functions

$$\{f_\tau : S_n \to \mathbb{R} \,|\, \tau \text{ a basis element of } \mathfrak{t}^n 2\}$$

and a family of corresponding TNN immanants $\{\mathrm{Imm}_{f_\tau}\}$ whose cone contains all immanants in the family (1.5). We begin in Section 2 with some of the well-known combinatorics of total nonnegativity. Then in Section 3 we introduce the Temperley-Lieb algebra and derive our main results. Finally in Section 4 we give an improved criterion for deciding whether or not an immanant of the form (1.5) is TNN.

2. Total nonnegativity and planar networks

It is possible to prove that some polynomials $p(x)$ are TNN by providing a combinatorial interpretation for $p(A)$ whenever A is a TNN matrix. Typically such a combinatorial interpretation involves a particular class of digraphs which we will call planar networks.

We define a *planar network of order* n to be an acyclic planar directed multigraph $G = (V, E)$ in which $2n$ boundary vertices are labeled counterclockwise as $q_1, \ldots, q_n, q'_n, \ldots, q'_1$. The vertices $q_1, \ldots, q_n$ are called *sources* and the vertices $q'_1, \ldots, q'_n$ are called *sinks.* Each edge $e \in E$ is weighted by a postive real weight $\omega(e)$, and we will define the weight of a set F of edges to be the product of weights of edges in F,

$$\omega(F) = \prod_{e \in F} \omega(e). \tag{2.1}$$

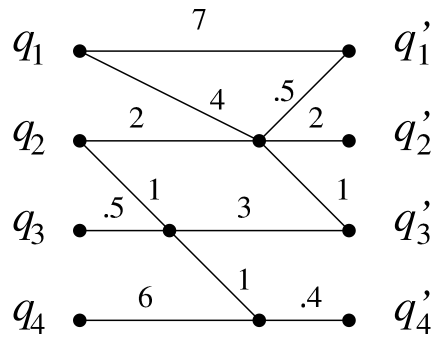


FIGURE 1. A planar network

More generally, we will define the weight a *multiset* of edges to be the analogous product in which weights of edges may appear with multiplicities greater than one. If $m = (m_e)_{e \in F}$ is a vector of multiplicities which defines a multiset of edges in F, we will denote the weight of this multiset by $\omega(F, m)$.

Given a planar network G of order n, we will define a subgraph H of G to be a *planar subnetwork* of G if it is a planar network whose sources and sinks are precisely those of G. We will economize notation by writing $H \subset G$ to denote that H is a planar subnetwork of G.

We define the *path matrix* $A = [a_{i,j}]$ of a planar network G by letting $a_{i,j}$ be the sum

$$a_{i,j} = \sum_{\pi} \omega(\pi),$$

of weights of paths over all paths π from source i (q_i) to sink j (q'_j). The reader may verify that the path matrix of the planar network in Figure 1 is

$$\begin{bmatrix} 9 & 8 & 4 & 0 \\ 1 & 4 & 5 & .4 \\ 0 & 0 & 3 & .2 \\ 0 & 0 & 0 & 2.4 \end{bmatrix}. \tag{2.2}$$

and that this matrix is TNN. (In figures we will assume that all edges are directed from left to right.)

The following famous theorem of Lindström and others [**1**] [**5**] [**6**] [**13**] [**16**] [**17**] explains the connection between planar networks and TNN matrices. (See also [**10**].)

Theorem 2.1. *An $n \times n$ matrix A is totally nonnegative if and only if it is the path matrix of a planar network G of order n. Furthermore, for any k-element subsets $I = \{i_1, \ldots, i_k\}$, $I' = \{i'_1, \ldots, i'_k\}$ of $[n]$, the (I, I') minor of A has the combinatorial interpretation*

$$\Delta_{I,I'}(A) = \sum_{\Pi} \omega(\Pi),$$

where the sum is over all k-tuples $\Pi = (\pi_1, \ldots, \pi_k)$ of paths in G which satisfy

(1) *π_j is a path from q_{i_j} to $q'_{i'_j}$.*
(2) *π_j and π_ℓ do not intersect for $j \neq \ell$.*

The reader may verify that the graph in Figure 1 has three nonintersecting path families from $\{q_1, q_2\}$ to $\{q'_1, q'_3\}$, and that these families have weights 14, 21, and 6. Correspondingly, the $(\{1, 2\}, \{1, 3\})$-minor of the path matrix (2.2) is $41 = 14 + 21 + 6$.

Immediate consequences of Theorem 2.1 are combinatorial interpretations for certain TNN immanants. Fix a planar network G and its path matrix A. The application of the monomial $x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ to A has

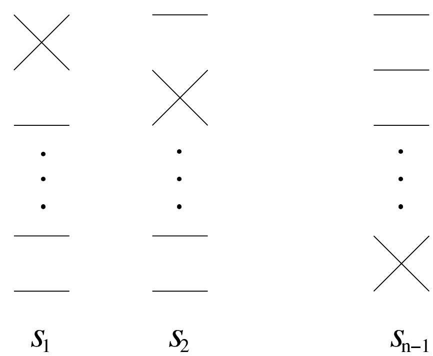


FIGURE 2. Planar networks for the generators of S_n.

the interpretation

$$a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} = \sum_{\Pi} \omega(\Pi),$$

where the sum is over path families $\Pi = (\pi_1, \ldots, \pi_n)$ in G in which π_i is a path from q_i to $q'_{\sigma(i)}$. We will say that such a path family has *type* σ. Also, by choosing $I = I' = [n]$ in Theorem 2.1, we have that

$$\det(A) = \sum_{H \subset G} \omega(H),$$

where the sum is over all planar subnetworks H of G which are unions of n nonintersecting paths. With a bit more work, one can derive a similar combinatorial interpretation for the TNN immanants (1.5),

$$\Delta_{J,J'}(A)\Delta_{\overline{J},\overline{J'}}(A) - \Delta_{I,I'}(A)\Delta_{\overline{I},\overline{I'}}(A) = \sum_{H \in \mathcal{H}} c_H \omega(H),$$

for appropriate collections $\mathcal{H}$ of planar subnetworks which depend on the index sets I, J, etc., and for appropriate constants c_H. (See [**19**, Cor. 3.3].) The problem of finding an analogous combinatorial interpretation for the TNN immanants (1.2) and (1.3) remains open.

To construct more TNN polynomials, we shall examine the planar networks of order n which are unions of n paths. We will say that a path family Π *covers* a planar network $H = (V, E)$ if every edge in E belongs to a path in Π. Since two different path families may cover the edges of a planar network with different multiplicities, we introduce the following notation. Given a planar network $H = (V, E)$ of order n, a sequence $m = (m_e)_{e \in E}$ of positive multiplicities, and a permutation σ in S_n, we define the number $\gamma(G, \sigma, m)$ to be the number of path families Π of type σ which cover H in such a way that each edge e belongs to exactly m_e paths. Note that we may assume that the components of m belong to $[n]$, since each edge of G will be covered at least once and at most n times by n paths. To enumerate the path families which cover H, we will associate to H an element $\beta(H)$ in $\mathbb{Z}[S_n]$ which will serve as an unweighted path generating function,

$$\beta(H) = \sum_{m} \sum_{\sigma \in S_n} \gamma(H, \sigma, m)\sigma,$$

where the first sum is over sequences m.

Certain planar networks which appear often in conjunction with the symmetric group are called *wiring diagrams*. Specifically, to the generators $s_1, \ldots, s_{n-1}$ of S_n we associate the planar networks in Figure 2. Then to an expression

$$\sigma = s_{i_1} \cdots s_{i_k}$$

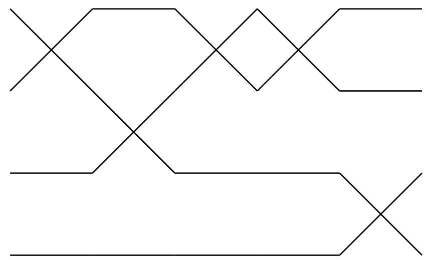
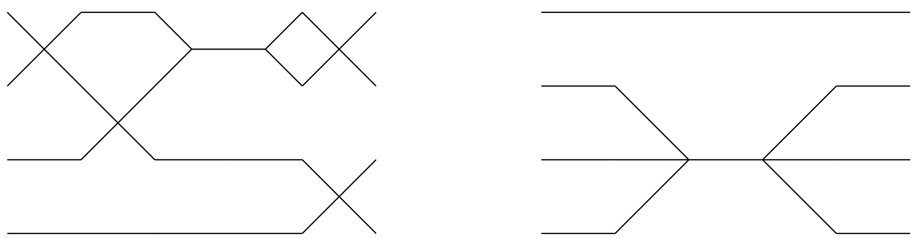


FIGURE 3. A wiring diagram.



(a) (b)

FIGURE 4. A generalized wiring diagram and another planar network.

(not necessarily reduced), we associate the planar network formed by concatenation of the generator networks. It is easy to see that there is at least one path family of type σ which covers a wiring diagram corresponding to any expression for σ. (This family crosses paths at every opportunity.) Furthermore, the path generating function for this planar network

$$(1 + s_{i_1}) \cdots (1 + s_{i_k}).$$

It is easy to see that any family of n paths which covers a wiring diagram of order n covers each edge exactly once. Figure 3 shows the wiring diagram associated to the expression $s_1s_2s_1s_1s_3$ (in S_4). The reader can verify that the corresponding path generating function is

$$2(s_3 + s_1s_3 + s_2s_3 + s_1s_2s_3 + s_2s_1s_3 + s_1s_2s_1s_3).$$

Three necessary conditions for a planar network to be a wiring diagram are the following.

(1) No vertex is contained in three paths.
(2) No edge is contained in two paths.
(3) Path intersections occur in an unambiguous left-to-right order.

Relaxing the first two conditions, we have planar networks such as that in Figure 4 (a).

We will define a planar network of order n to be a *generalized wiring diagram* (of order n) if it is a union of n paths, no three of which intersect in a single vertex.

It is easy to see that the form of a given wiring diagram determines a unique sequence m of multiplicities with which edges are covered.

Lemma 2.2. *Let H be a generalized wiring diagram. If a path family Π and a path family Π' cover the edges of H with multiplicity sequences m and m', respectively, then $m = m'$.*

PROOF. Omitted. □

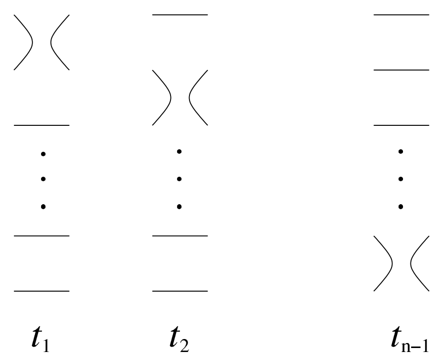


FIGURE 5. Generators of $\mathfrak{t}^n\lambda$.

The path generating functions of generalized wiring diagrams factor just as those of wiring diagrams. On the other hand, the path generating functions of arbitrary unions of n paths do not factor this way. Figure 4 (b) shows a planar network whose path generating function is $1 + s_2 + s_3 + s_2s_3 + s_3s_2 + s_2s_3s_2$. We will denote by $u_{[i,j]}$ the element of $\mathbb{Z}[S_n]$ which is a sum of permutations in the subgroup generated by $s_i, \dots, s_{j-1}$.

Lemma 2.3. *Let H be a planar network which is a union of n paths. If H is a generalized wiring diagram then $\beta(H)$ factors as*

$$\beta(H) = (1 + s_{i_1}) \cdots (1 + s_{i_k})$$

for some generators $s_{i_1}, \dots, s_{i_k}$ of S_n. If H is not a generalized wiring diagram, then $\beta(H)$ can be expressed as a sum of terms of the form

$$u_{[i_1,j_1]} \cdots u_{[i_k,j_k]},$$

where in each such term we have $i_\ell \leq j_\ell - 2$ for at least one index ℓ.

PROOF. Omitted. □

3. Main results

Given an integer λ, we define the *Temperley-Lieb algebra* $\mathfrak{t}^n\lambda$ to be the $\mathbb{Z}$-algebra generated by elements $t_1, \dots, t_{n-1}$ subject to the relations

$$\begin{aligned} t_i^2 &= \lambda t_i, && \text{for } i = 1, \dots, n-1, \\ t_i t_j t_i &= t_i, && \text{if } |i-j| = 1, \\ t_i t_j &= t_j t_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

The rank of $\mathfrak{t}^n\lambda$ as a $\mathbb{Z}$-module is well known to be the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

$\mathfrak{t}^n 2$ is isomorphic to the quotient $\mathbb{Z}[S_n]/I$, where I is the ideal generated by $u_{[1,3]}, u_{[2,4]}, \dots, u_{[n-2,n]}$. (See [**12**, Sec. 2.1].) The isomorphism is given by

$$\begin{aligned} \theta : \mathbb{Z}[S_n] &\to \mathfrak{t}^n 2, \\ s_i &\mapsto t_i - 1. \end{aligned}$$

We will call the elements of the multiplicative monoid generated by $t_1, \dots, t_{n-1}$ the *basis elements* of $\mathfrak{t}^n\lambda$.

Figure 5 shows pictorial representations of the basis elements of $\mathfrak{t}^n\lambda$ which were made popular by Kauffman [**14**, Sec. 4]. Multiplication of generators corresonds to concatenation of diagrams, with cycles contributing λ. Figure 6 shows the multiplication $t_1t_2t_1t_1t_3 = \lambda t_1t_3$ in $T_4(\lambda)$. (We "tighten" long curves to simplify the picture.)

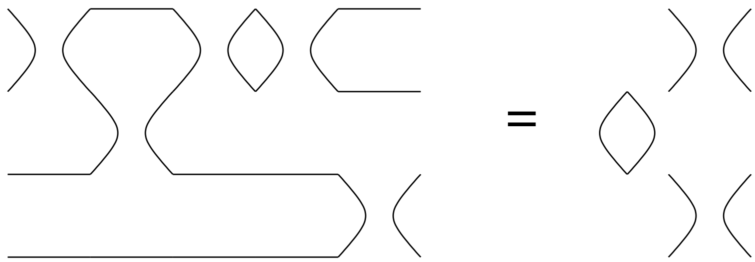


FIGURE 6. Multiplication in $\mathfrak{t}^n\lambda$.

For any basis element τ of $\mathfrak{t}^n 2$, define $f_\tau : S_n \to \mathbb{R}$ to be the function which maps σ to the coefficient of τ in $\theta(\sigma)$.

Given a planar network H which is a union of n paths, define the element $\phi_\lambda(H)$ of $\mathfrak{t}^n\lambda$ by

$$\phi_\lambda(H) = \theta(\beta(H)).$$

If H is a generalized wiring diagram, then by Lemma 2.3, we have that

$$\phi_\lambda(H) = \theta(1 + s_{i_1}) \cdots \theta(1 + s_{i_k}) = t_{i_1} \cdots t_{i_k}$$

for some indices $i_1, \ldots, i_k \in [n]$ and therefore that

$$\phi_\lambda(H) = \lambda^j \tau$$

for some nonnegative integer j and some basis element $\tau = \phi_1(H)$ of $\mathfrak{t}^n\lambda$. We will denote the exponent by $\alpha(H)$,

$$\phi_\lambda(H) = \lambda^{\alpha(H)} \phi_1(H).$$

If, on the other hand, H is not a generalized wiring diagram, then by Lemma 2.3 we have that $\beta(H)$ is equal to a sum of $\mathbb{Z}[S_n]$ elements which belong to the kernel of θ. It follows in this case that $\phi_\lambda(H) = 0$. If H is a generalized wiring diagram, then $\phi_\lambda(H)$ can be computed pictorially as follows.

(1) Contract any doubly covered subpath to a single vertex.
(2) For each vertex v of indegree two and outdegree two, create vertex v' with indegree two and vertex v'' with outdegree two.
(3) Interpret the resulting graph as an element of $\mathfrak{t}^n\lambda$. (See Figures 3 and 6.)

Lemma 3.1. *Let H be a planar network which is a union of n paths. For any basis element τ of $\mathfrak{t}^n 2$ we have*

$$\sum_\Pi f_\tau(\text{type}(\Pi)) = \begin{cases} 2^{\alpha(H)} & \text{if } \phi_1(H) = \tau, \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over path families Π which cover H.

PROOF. Note that we have

$$\sum_\Pi f_\tau(\text{type}(\Pi)) = \sum_m \sum_{\sigma \in S_n} \gamma(H, \sigma, m),$$

which is equal to the coefficient of τ in

$$\theta\left(\sum_m \sum_{\sigma \in S_n} \gamma(H, \sigma, m)\sigma \right) = \theta(\beta(H)) = \phi_2(H). \tag{3.1}$$

This coefficient is $2^{\alpha(H)}$ if $\phi_1(H) = \tau$ and is zero otherwise. □

We may now state and prove our main result.

Theorem 3.2. *For any basis element τ of $\mathfrak{t}^n 2$, the f_τ-immanant $\mathrm{Imm}_{f_\tau}(x)$ is totally nonnegative. In particular, let G be a planar network of order n and let A be its path matrix. Then we have*

$$\mathrm{Imm}_{f_\tau}(A) = \sum_{H \subset G} 2^{\alpha(H)} \omega(H, m),$$

where the sum is over all planar subnetworks H of G which are generalized wiring diagrams and which satisfy $\phi_1(H) = \tau$, and m is the vector of edge multiplicities which is uniquely determined by H.

PROOF. We have

$$\begin{aligned}\mathrm{Imm}_{f_\tau}(A) &= \sum_{\sigma \in S_n} f_\tau(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \\ &= \sum_{\sigma \in S_n} f_\tau(\sigma) \sum_{H \subset G} \sum_m \omega(H, m) \gamma(H, \sigma, m),\end{aligned}$$

where the second sum is over all planar subnetworks H of G which are unions of n paths. Changing the order of summation, we have

$$\begin{aligned}\mathrm{Imm}_{f_\tau}(A) &= \sum_{H \subset G} \sum_m \omega(H, m) \sum_{\sigma \in S_n} f_\tau(\sigma) \gamma(H, \sigma, m) \\ &= \sum_{H \subset G} \sum_m \omega(H, m) \sum_{\Pi} f_\tau(\mathrm{type}(\Pi)),\end{aligned}$$

where the inner sum is over all path families Π which cover H with edge multiplicities m. By Lemma 3.1, this inner sum is $2^{\alpha(H)}$ if H is a generalized wiring diagram, and zero otherwise. In the case that H is a generalized wiring diagram, then Lemma 2.2 implies that the sequence m is completely determined by H, and we have our desired result.

□

4. Improved criterion

Now let us associate to each pair of k-subsets (I, I') of $[n]$ a subset of the basis elements of $\mathfrak{t}^n \lambda$. Labeling the vertices of a basis element generator τ by $q_1, \ldots, q_n, q'_n, \ldots, q'_1$ (counterclockwise), let us say that τ is *compatible with* the pair (I, I') if each edge is incident upon exactly one of the vertices $\{q_i \mid i \in I\} \cup \{q'_j \mid j \in \overline{I'}\}$.

Theorem 4.1. *Let I, I', J, J' be subsets of $[n]$ satisfying $|I| = |I'|$ and $|J| = |J'|$, and let $R(I, I')$, $R(J, J')$ be the subsets of basis elements of $\mathfrak{t}^n \lambda$ which are compatible with (I, I') and (J, J'), respectively. The immananant $\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ is totally nonnegative if and only if $R(I, I')$ is contained in $R(J, J')$. In particular, we have*

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) = \sum_{\tau \in R(J,J') \smallsetminus R(I,I')} \mathrm{Imm}_{f_\tau}(x).$$

PROOF. Omitted. □

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