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Virtual Crystals and the $X = M$ Conjecture

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Abstract. *This is an expository talk on virtual crystals and the $X = M$ conjecture.*
Résumé. *C'est un entretien expositoire sur les cristaux virtuels et la conjecture $X = M$.*

1. Extended Abstract

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$ was introduced independently by Drinfeld [**D**] and Jimbo [**J**] in their study of two dimensional solvable lattice models in statistical mechanics. The parameter q corresponds to the temperature of the underlying model. Kashiwara [**K**] showed that at zero temperature or $q = 0$ the representations of $U_q(\mathfrak{g})$ have bases, which he coined crystal bases, with a beautiful combinatorial structure and favorable properties such as uniqueness and stability under tensor products.

The irreducible finite-dimensional $U'_q(\mathfrak{g})$-modules were classified by Chari and Pressley [**CP1, CP2**] in terms of Drinfeld polynomials. The Kirillov–Reshetikhin modules $W^{r,s}$, labeled by a Dynkin node r and a positive integer s, form a special class of these finite-dimensional modules. They naturally correspond to the weight $s\Lambda_r$, where Λ_r is the r-th fundamental weight of $\mathfrak{g}$. Recently, Hatayama et al. [**HKOTY, HKOTT**] conjectured that the Kirillov–Reshetikhin modules $W^{r,s}$ have a crystal basis denoted by $B^{r,s}$. The existence of such crystals allows the definition of one dimensional configuration sums X, which play an important role in the study of phase transitions of two dimensional exactly solvable lattice models. For $\mathfrak{g}$ of type $A_n^{(1)}$, the existence of the crystal $B^{r,s}$ was settled in [**KKMMNN**], and the one dimensional configuration sums contain the Kostka polynomials, which arise in the theory of symmetric functions, combinatorics, the study of subgroups of finite abelian groups, and Kazhdan–Lusztig theory. In certain limits they are branching functions of integrable highest weight modules.

In [**HKOTY, HKOTT**] fermionic formulas M for the one dimensional configuration sums were conjectured. Fermionic formulas originate in the Bethe Ansatz of the underlying exactly solvable lattice model. The term fermionic formula was coined by the Stony Brook group [**KKMM1, KKMM2**], who interpreted fermionic-type formulas for characters and branching functions of conformal field theory models as partition functions of quasiparticle systems with "fractional" statistics obeying Pauli's exclusion principle. For type $A_n^{(1)}$ the fermionic formulas were proven in [**KSS**] using a generalization of a bijection between crystals and rigged configurations of Kirillov and Reshetikhin [**KR**]. In [**OSS2**] similar bijections were used to prove the fermionic formula for nonexceptional types for crystals $B^{1,1}$. Rigged configurations are combinatorial objects which label the solutions to the Bethe equations. The bijection between crystals and rigged configurations

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reflects two different methods to solve lattice models in statistical mechanics: the corner-transfer-matrix method and the Bethe Ansatz.

The theory of virtual crystals [**OSS1, OSS3**] provides a realization of crystals of type X as crystals of type Y, based on well-known natural embeddings $X \hookrightarrow Y$ of affine algebras:

$$\begin{array}{lcl} C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} & \hookrightarrow & A_{2n-1}^{(1)} \\ A_{2n-1}^{(2)}, B_n^{(1)} & \hookrightarrow & D_{n+1}^{(1)} \\ E_6^{(2)}, F_4^{(1)} & \hookrightarrow & E_6^{(1)} \\ D_4^{(3)}, G_2^{(1)} & \hookrightarrow & D_4^{(1)}. \end{array}$$

Note that under these embeddings every affine Kac–Moody algebra is embedded into one of simply-laced type $A_n^{(1)}$, $D_n^{(1)}$ or $E_6^{(1)}$. Hence, by the virtual crystal method the combinatorial structure of any finite-dimensional affine crystal can be expressed in terms of the combinatorial crystal structure of the simply-laced types. Whereas the affine crystals $B^{r,s}$ of type $A_n^{(1)}$ are already well-understood [**Sh**], this is not the case for $B^{r,s}$ of types $D_n^{(1)}$ and $E_6^{(1)}$.

In this talk we highlight the main results regarding the $X = M$ conjecture of [**HKOTY, HKOTT**] and virtual crystals [**OSS1, OSS3**], which can be summarized as follows:

- Refs. [**HKOTY, HKOTT**] conjecture the existence of $B^{r,s}$ and the identity $X = M$ for general affine Kac-Moody algebras.
- Refs. [**OSS1, OSS3**] introduce the virtual crystal method which yields a description of the combinatorial structure of the crystals $B^{r,s}$ in terms of the combinatorics of $B^{r,s}$ for types $A_n^{(1)}, D_n^{(1)}$ and $E_6^{(1)}$. Similarly, the fermionic formulas and rigged configurations also exhibit this virtual embedding structure. In [**OSS3**] this was used in particular to extend the Kleber algorithm, which provides an efficient algorithm for calculating fermionic formulas, to nonsimply-laced algebras.
- In Ref. [**KSS**] the $X = M$ conjecture was proven for type $A_n^{(1)}$ using a bijection between crystals/tableaux and rigged configurations. This was extended to other nonexceptional types in [**OSS2**] for tensor products of $B^{1,1}$ and in [**SSh**] for tensor products of $B^{1,s}$. Type $D_n^{(1)}$ for tensor products of $B^{r,1}$ was treated in [**S**].
- The combinatorial structure of the crystals $B^{2,s}$ of type $D_n^{(1)}$ is studied in [**SS**]. This work is presented by Philip Sternberg in form of a poster at this conference.

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