



Tamari Lattices and Non-crossing Partitions in Types B and D

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Abstract. *The usual, or type A_n , Tamari lattice is a partial order on T_n^A , the triangulations of an $(n+3)$ -gon. We define a partial order on T_n^B , the set of centrally symmetric triangulations of a $(2n+2)$ -gon. We show that it is a lattice, and that it shares certain other nice properties of the A_n Tamari lattice; it can therefore be considered the B_n Tamari lattice.*

We define a bijection between T_n^B and the non-crossing partitions of type B_n defined by Reiner. Reiner has also defined the noncrossing partitions of type D_n as a subset of those of type B_n . We show that the elements of T_n^B which correspond to the noncrossing partitions of type D_n form a lattice under the order induced from their inclusion in T_n^B , which therefore can be considered the D_n Tamari lattice.

This is a somewhat abridged version of a longer paper with the same title, available at www.arxiv.org/math.CO/0311334.

Résumé. *Le treillis de Tamari standard (de type A_n) est un ordre partiel sur T_n^A , les triangulations d'un $(n+3)$ -gone. Nous définissons un ordre partiel sur T_n^B , l'ensemble des triangulations centralement symétriques d'un $(2n+2)$ -gone. Nous montrons que c'est un treillis et qu'il possède aussi d'autres propriétés intéressantes similaires au treillis de Tamari de type A_n . Ce treillis peut donc être considéré comme le treillis de Tamari de type B_n .*

Nous définissons une bijection entre T_n^B et les partages non-croisés de type B_n définis par Reiner. Reiner a aussi défini les partages non-croisés de type D_n comme un sous-ensemble de ceux de type B_n . Nous montrons que les éléments de T_n^B qui correspondent aux partages non-croisés de type D_n forment un treillis sous l'ordre induit par leur inclusion dans T_n^B , qui peut donc être considéré comme le treillis de Tamari de type D_n .

Cet exposé est une version plus courte d'un exposé du même titre qui est disponible sur www.arxiv.org/math.CO/0311334.

1. Introduction

Let T_n^A denote the set of triangulations of an $(n+3)$ -gon. By a triangulation of a polygon, we mean a division of it into triangles by connecting pairs of its vertices with straight lines which do not cross in the interior of the polygon. Conventionally, we will number the vertices of our $(n+3)$ -gon clockwise from 0 to $n+2$, with a long top edge connecting vertices 0 and $n+2$. An example triangulation is shown in Figure 1 below.

Let $S \in T_n^A$. As in [Lee], we colour the chords of S red and green, as follows. A chord C of S is the diagonal of a quadrilateral $Q(C)$ in S . If C is the diagonal of $Q(C)$ which is connected to the vertex with

1991 *Mathematics Subject Classification.* Primary 05E15.

Key words and phrases. Tamari lattice, non-crossing partitions, classical types, left modularity, EL-labelling.

the largest label, we colour it green; otherwise we colour it red. In Figure 1, the red chords are indicated by thick lines.

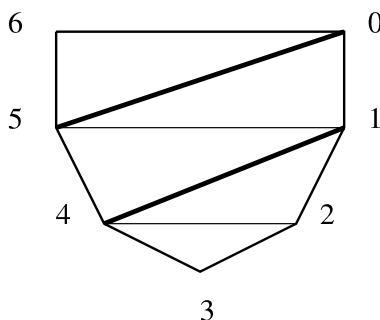


FIGURE 1

We partially order T_n^A by giving covering relations: T covers S if they coincide except that some green chord in S has been replaced by the other diagonal of $Q(C)$ (which is red). This is one way to construct the Tamari lattice, which was introduced in [Tam] and which has since been studied by several authors (see [HT, Pal, BW2]).

Although this is not clear from the elementary description given here, the Tamari lattice should be thought of as being associated to type A . One indication of why can be found in [BW2], where it is shown that T_n^A is a quotient of the weak order on the symmetric group S_{n+1} (the type A_n reflection group). Another reason is that the elements of T_n^A index clusters in the A_n root system (see [FZ]). Once one has the idea that the Tamari lattice is type A , it is natural to ask whether there exist Tamari lattices in other types.

For reasons which we shall go into further below, the B_n triangulations, denoted T_n^B , are the triangulations of a $(2n+2)$ -gon which are fixed under a half-turn rotation. These triangulations have already appeared in the work of Simion [Sim], and in that of Fomin and Zelevinsky [FZ] where they index the clusters in the B_n root system. One goal of our paper is to define a partial order on T_n^B and to prove that it is a lattice. The definition is analogous to that already given for the A_n Tamari lattice: it is given in terms of covering relations, and S covers T in T_n^B if S is obtained from T by replacing a symmetric pair of chords C, \bar{C} by the other diagonals of $Q(C), Q(\bar{C})$. The details of the definition are a trifle complicated, so we defer them for the main body of the paper. This definition was arrived at independently and more or less simultaneously by Reading [Rea]. He has also proved that T_n^B is a lattice, using a rather different approach. Two alternative partial orders on T_n^B with similar (but somewhat easier to describe) covering relations were suggested by Simion [Sim]; one is studied further in [San]. But since neither of these is a lattice, neither is completely satisfying as a type B analogue of the usual Tamari lattice.

What objects should be considered the D_n triangulations is not as settled as in type B_n , although certain information is known, such as the desired cardinality. One candidate is provided in [FZ], and used there to index the clusters in the D_n root system. We follow a different approach. First, we find a bijection between B_n triangulations and B_n non-crossing partitions, which were introduced by Reiner [Rei]. Motivated by Reiner's definition of non-crossing partitions for type D_n as a subset of those for type B_n (which has the desired cardinality), we take our D_n triangulations T_n^D to be the corresponding subset of T_n^B . (It is not clear whether there is any natural bijection between our T_n^D and the D_n triangulations of [FZ].) Our second chief result is to show that the order induced on T_n^D from its inclusion in T_n^B gives it a lattice structure also. In fact, our approach to type D_n works for any of the interpolating pseudo-types indexing hyperplane arrangements between B_n and D_n . (We shall recall the definition of these pseudo-types below.)

We show that T_n^B and T_n^D have an unrefinable chain of left modular elements, a property also shared by the usual Tamari lattice [BS]. One consequence of this, due to Liu [Liu], is that these lattices have

EL-labellings. Using these labellings, we show that, as for the usual Tamari lattice (see [BW2]), the order complex of any interval is either homotopic to a sphere or contractible.

From the results in this paper one could proceed in two directions. One direction is to consider the existence of Tamari lattices in all Coxeter types. The other direction is to investigate further the lattices defined here, to see how many more of the properties of the usual Tamari lattice carry over.

2. Type B Triangulations

Recall that the B_n Weyl group consists of signed permutations of n . We can think of these as permutations of $\{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ fixed under interchanging i and \bar{i} for all $1 \leq i \leq n$. By analogy, B_n triangulations, T_n^B , are defined to be type A triangulations of a $(2n + 2)$ -gon fixed under a half-turn. There is general consensus that this is the correct definition of B_n triangulation: see [Sim, FZ].

We number the vertices of our standard $(2n + 2)$ -gon counterclockwise from $n + 1$ to 1 and then from $\bar{n} + \bar{1}$ to $\bar{1}$. A typical triangulation is shown in Figure 2.

We will frequently distinguish two types of chords: *pure* and *mixed*. A chord is pure if it connects two barred vertices or two unbarred vertices; otherwise it is mixed. For $S \in T_n^B$, consider a chord C of S . The chord C is the diagonal of a quadrilateral, which we denote $Q(C)$. If C is pure, then we colour it red if $Q(C)$ contains another vertex of the same type as those of C whose label is higher, and green otherwise. If it is mixed, we colour it red if $Q(C)$ contains an unbarred vertex whose label is higher than the label of the unbarred vertex of C , or a barred vertex whose label is higher than the label of the barred vertex of C . Otherwise we colour it green. In Figure 2, the red chords are indicated by thick lines.

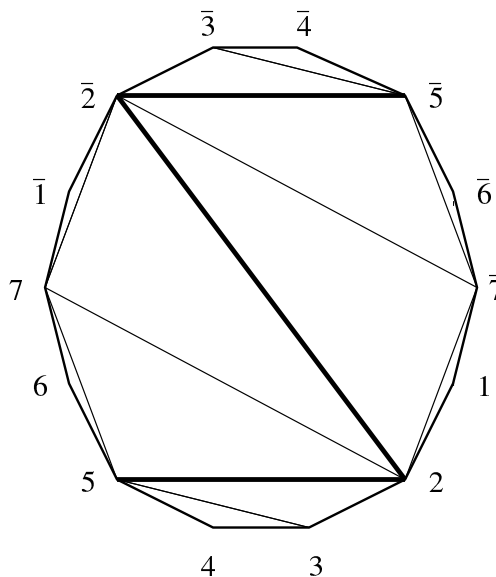


FIGURE 2

For C a chord, we write \bar{C} for its symmetric partner (that is to say, the image of C under a half turn). Observe that C and \bar{C} are assigned the same colour.

Lemma 2.1. *Consider a chord C in a triangulation S . Let S' be the triangulation obtained by replacing C by C' , the other diagonal of $Q(C)$, and also replacing \bar{C} by \bar{C}' . Then the colours of C in S and C' in S' are opposite.*

We can now state the first main theorem of this paper (which, as was already remarked, was arrived at and proved independently and more or less simultaneously by Reading [Rea]).

Theorem 2.2. *There is a lattice structure on T_n^B whose covering relations are given by $S \leq T$ iff S and T differ in that green chords C, \bar{C} in S are replaced in T by the other diagonals of $Q(C)$ and $Q(\bar{C})$ (which will be red). Note that we allow $C = \bar{C}$ (i.e. C being a diameter). We call this lattice the B_n Tamari lattice.*

The first ingredient in our proof of Theorem 1 is some further analysis of the red and green chords of triangulations.

Fix a triangulation S . For $1 \leq i \leq n$, consider those chords of S which are attached to i and let $C_i(S)$ be the first of these encountered in searching clockwise starting at $\bar{1}$. If none is encountered before reaching $i - 1$, then $C_i(S)$ is not defined. Let $R(S)$ be the set of these chords, together with their symmetric partners.

Lemma 2.3. *For any triangulation S , the chords in $R(S)$ are red, the other chords of S are green, and S is the unique triangulation whose red chords are exactly $R(S)$.*

3. Bracket Vectors in types A and B

We briefly recall some well-known facts about the type A Tamari lattice, which serve as motivation for our work in type B.

Any triangulation $S \in T_n^A$ has a bracket vector $r(S) = (r_1(S), \dots, r_{n+1}(S))$. Let $c_i(S)$ be the least vertex attached to i . Then $r_i(S) = i - 1 - c_i(S)$. For example, the bracket vector of the triangulation shown in Figure 1 is $(0, 0, 0, 2, 4)$. This approach to representing elements of the Tamari lattice goes back to [HT], though we make some different choices of convention here.

Proposition 3.1. *An $(n + 1)$ -tuple of positive integers is a bracket vector for some triangulation in T_n^A iff it satisfies the following two properties:*

- (i) For $1 \leq i < j \leq n + 1$, $r_i \leq r_j - (j - i)$ provided $r_j - (j - i)$ is non-negative.
- (ii) $0 \leq r_i \leq i - 1$.

The order relation on triangulations has a simple interpretation in terms of bracket vectors, which we summarize in the following proposition:

Proposition 3.2. *The lattice structure on T_n^A can be described as follows:*

- (i) $S \leq T$ iff $r_i(S) \leq r_i(T)$ for all i .
- (ii) $r(S \wedge T)_i = \min(r_i(S), r_i(T))$.
- (iii) For x any $n + 1$ -tuple of numbers satisfying only the second condition of Proposition 1, there is a unique triangulation $\uparrow(x)$ such that for $S \in T_n^A$,

$$r_i(S) \geq x_i \text{ for all } i \text{ iff } S \geq \uparrow(x).$$

- (iv) $r(S \vee T) = \uparrow(\max(r(S), r(T)))$, where \max is taken coordinatewise.

We now proceed to describe a similar construction in type B. To a triangulation $S \in T_n^B$ we associate a bracket vector $r(S) = (r_1(S), \dots, r_n(S))$, as follows. For $1 \leq i \leq n$, let $c_i(S)$ denote the first vertex adjacent to i encountered proceeding clockwise starting at $\bar{1}$. If the counter-clockwise distance from $i - 1$ to $c_i(S)$ is less than or equal to $n - 1$, set $r_i(S)$ to be that distance. Otherwise, set $r_i(S) = *$. Thus, the triangulation shown in Figure 2 has bracket vector $(0, *, 0, 0, 2, 0)$.

Conventions regarding $*$. $*$ is considered to be greater than any integer. $*$ plus an integer (or $*$) equals $*$.

Lemma 3.3. *The map from T_n^B to bracket vectors is injective.*

Proposition 3.4. *B_n bracket vectors are n -tuples of symbols from $[0, n - 1] \cup \{*\}$ characterized by the following two properties:*

- i) For $1 \leq i < j \leq n$, $r_i \leq r_j - (j - i)$ if $r_j - (j - i)$ is non-negative.

ii) If $* > r_i \geq i$, then $r_{n+i-r_i} = *$.

We will now define an order on T_n^B . For $S, T \in T_n^B$, let $S \leq T$ iff for all i , $r_i(S) \leq r_i(T)$.

Proposition 3.5. *The covering relations in this order on T_n^B are exactly those described by Theorem 1.*

Our next goal is to show that the B_n Tamari order is really a lattice. Before we can do that, we need some preliminary results.

Let M_n denote the n -tuples with entries in $[0, n - 1] \cup \{*\}$, with the Cartesian product order. Let $M_n^{(i)}$ denote the elements of M_n which satisfy condition (i) of Proposition 3. Let $M_n^{(ii)}$ denote the elements of M_n which satisfy condition (ii) of Proposition 3.

Proposition 3.6. *There exist maps $\uparrow: M_n^{(ii)} \rightarrow T_n^B$, $\downarrow: M_n^{(i)} \rightarrow T_n^B$, which satisfy the following conditions:*

$$f \leq r(S) \text{ iff } \uparrow(f) \leq S$$

$$r(S) \leq f \text{ iff } S \leq \downarrow(f).$$

Using these maps, we can prove that meet and join exist in T_n^B by giving simple descriptions of them.

Proposition 3.7. *The Tamari order on T_n^B is a lattice. The lattice operations are as follows: For $S, T \in T_n^B$, $S \vee T = \uparrow(\max(r(S), r(T)))$ and $S \wedge T = \downarrow(\min(r(S), r(T)))$.*

This completes the proof of Theorem 1. The Hasse diagram of T_B^3 is shown in Figure 5, at the end of the paper.

4. Non-crossing partitions

The A_n non-crossing partitions, NC_n^A , are partitions of $n + 1$ into sets such that if v_1, \dots, v_{n+1} are $n + 1$ points on a circle, labelled in cyclic order, and if B_1, \dots, B_r are the convex hulls of the sets of vertices corresponding to the blocks of the partition, then the B_i are non-intersecting.

There is a bijection from T_n^A to NC_n^A as follows. For $S \in T_n^A$, erase all the green chords of S and the vertices 0 and $n + 2$. Then move the endpoints of each red chord ij a little bit, the lower-numbered end point a little clockwise, the higher-numbered endpoint a little counterclockwise (so i and j are both on the upper side of the chord). These chords now divide the vertices in $[n + 1]$ into subsets, which form a non-crossing partition by construction. Figure 3 shows the triangulation from Figure 1, together with the non-crossing partition which it induces: $\{14, 23, 5\}$

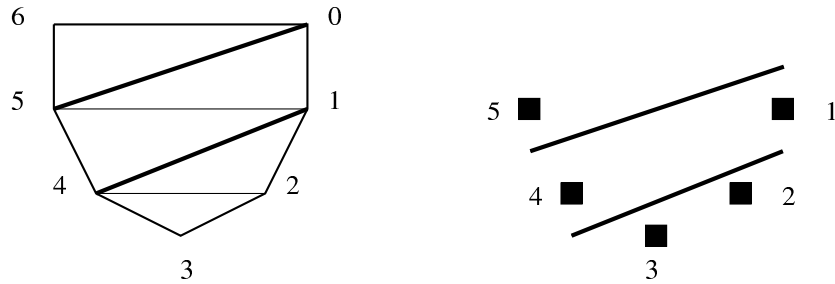


FIGURE 3

Note that the non-crossing partitions are often considered as being ordered by refinement; this order is quite different from the Tamari order.

As defined by Reiner [Rei], the B_n non-crossing partitions, NC_n^B , are partitions of the set $1, \dots, n, \bar{1}, \dots, \bar{n}$, which have the properties that the partition remains fixed under interchanging barred and unbarred elements, and that if $2n$ points are chosen around a circle and labelled cyclically $v_1, \dots, v_n, v_{\bar{1}}, \dots, v_{\bar{n}}$, then the convex hulls of the vertices corresponding to the blocks of the partition do not intersect.

We now define a map ψ from T_n^B to NC_n^B , analogous to that in type A. Erase all green chords. Move both endpoints of mixed red chords slightly counterclockwise. Move the endpoints of pure red chords slightly together (so that the vertices both lie on the side of the chord which includes the larger part of the polygon). Erase the vertices $n + 1$ and $\overline{n + 1}$. The remaining vertices are now partitioned by the red chords, in what is clearly a B_n non-crossing partition. Figure 4 shows the triangulation from Figure 2, together with the B_n non-crossing partition which it induces: $\{\overline{12\overline{56}}, 34, \overline{1256}, \overline{34}\}$.

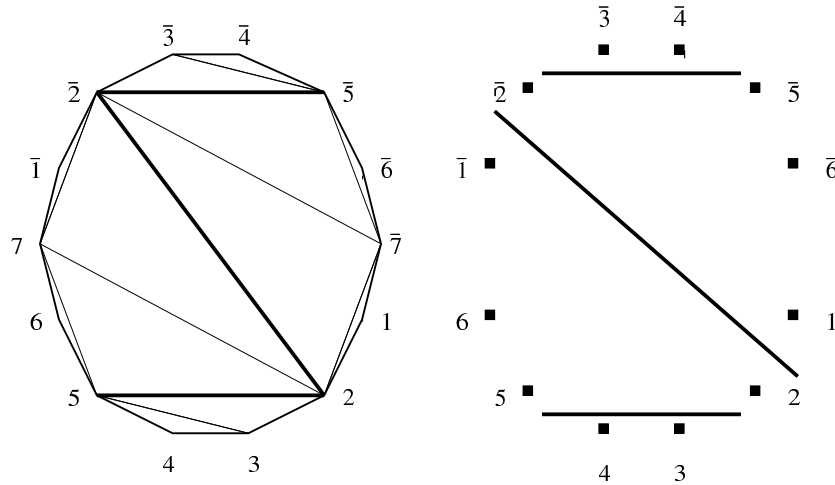


FIGURE 4

Proposition 4.1. *The map ψ is a bijection from T_n^B to NC_n^B .*

5. EL-Shellability

Recall that an element x of a lattice L is said to be left modular if, for all $y < z \in L$,

$$(y \vee x) \wedge z = y \vee (x \wedge z).$$

For $1 \leq i \leq n$ and $t \in [1, n - 1] \cup \{*\}$, let $S_{i,t}$ denote the triangulation with bracket vector as follows:

$$r_j(S_{i,t}) = \begin{cases} * & \text{for } j > i \\ t & \text{for } j = i \\ 0 & \text{for } j < i. \end{cases}$$

Lemma 5.1. *$S_{i,t} \in T_n^B$ is left modular.*

Since the $S_{i,t}$ together with $\hat{0}$ form an unrefinable chain, we have the following theorem:

Theorem 5.2. *T_n^B has an unrefinable chain of left modular elements.*

The analogous fact that T_n^A possesses a maximal chain of left modular elements was first proved by Blass and Sagan [BS].

It was shown in [Liu] that a lattice having an unrefinable chain of left-modular elements has an EL-labelling. In particular, this shows that the order complex of any interval in such a lattice is shellable and hence contractible or homotopy equivalent to a wedge of spheres. For more on EL-labelling, and EL-shellability, see [Bj, BW1, BW2].

Thus, Theorem 2 implies the following corollary:

Corollary 5.3. *T_n^B is EL-shellable.*

6. Homotopy types of intervals

As we have already remarked, the fact that T_n^B is EL-shellable implies that the order complex of any interval is either contractible or has the homotopy type of a wedge of spheres. In this section, we shall show that it is in fact either contractible or homotopic to a single sphere. One reason that such a result is of interest is that it implies that the Möbius function of any interval in T_n^B is 0, -1 , or 1 .

Theorem 6.1. *The order complex of an interval in T_n^B is either contractible or homotopy equivalent to a single sphere.*

To sketch the proof of Theorem 3, we begin by recalling the EL-labelling of [Liu]. Let L be a lattice, and let $\hat{0} = x_0 < x_1 < \dots < x_r = \hat{1}$ be an unrefinable chain of left modular elements. Let \mathcal{W}_i be the set of join irreducibles below x_i but not below x_{i-1} . For $y < z$ in L , let $\mathcal{W}(y, z)$ be the set of irreducibles below z but not below y . For any $S < T$ in L , label the corresponding edge of the Hasse diagram by:

$$\gamma(S, T) = \min\{i \mid \mathcal{W}_i \cap \mathcal{W}(S, T) \neq \emptyset\}.$$

Proposition 6.2 ([Liu]). *For L a lattice with an unrefinable left modular chain, the labelling γ defined above is an EL-labelling.*

In order to interpret this labelling in our case, we need some results about the join-irreducibles of T_n^B .

For $1 \leq t \leq i - 1$, let $W_{i,t}$ denote the triangulation whose bracket vector consists of t in the i -th place, all the other entries being zero.

For $i \leq t < n$, let $W_{i,t}$ denote the triangulation defined by:

$$r_j(W_{i,t}) = \begin{cases} t & \text{for } j = i \\ * & \text{for } j = n + i - t \\ 0 & \text{otherwise} \end{cases}$$

Let $W_{i,*}$ denote the triangulation whose bracket vector consists of a single $*$ in the i -th place, all the other entries being zero.

Write \mathcal{W} for the set of all the $W_{i,t}$.

Proposition 6.3. *The join-irreducibles of T_n^B are exactly \mathcal{W} . The unique join-irreducible below $S_{i,t}$ and not below any smaller $S_{i',t'}$ is $W_{i,t}$.*

Recall from [BW2] that given a poset with an EL-labelling, the order complex of an interval $[y, z]$ is homotopic to a wedge of spheres, one for each unrefinable chain from y to z such that the labels strictly decrease as one reads up the chain. Such chains are called *decreasing chains*.

Thus, Theorem 3 follows from the following lemma:

Lemma 6.4. *For $Y < Z \in T_n^B$, there is at most one decreasing chain from Y to Z .*

7. Generalizing to Type BD_n^S

Here we fix n and a subset S of $[n]$. We will be operating in type BD_n^S , a concept introduced in [Rei] which we now explain. This is not a type in the usual sense. Rather, it refers to a certain hyperplane arrangement between those associated to B_n and D_n .

Recall that a root system gives rise to a hyperplane arrangement by taking all the hyperplanes through the origin perpendicular to roots. The B_n arrangement therefore consists of all those hyperplanes defined by $x_i \pm x_j = 0$, together with those defined by $x_i = 0$, for $1 \leq i, j \leq n$, while the D_n arrangement consists only of those hyperplanes defined by $x_i \pm x_j = 0$ for $1 \leq i, j \leq n$. Now, for $S \subset [n]$, the BD_n^S hyperplane arrangement consists of those hyperplanes defined by $x_i \pm x_j = 0$ together with $x_i = 0$ for $i \notin S$. When $S = \emptyset$ we recover B_n , while if $S = [n]$ we recover D_n .

The B_n partitions, Π_n^B , are by definition those partitions of the set $\{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ which are fixed under the map interchanging i and \bar{i} , and such that there is at most one block which contains any i and \bar{i} simultaneously. This is a suitable definition of Π_n^B because its elements are naturally in bijection with the elements of the intersection lattice of the B_n arrangement. NC_n^B is a subset of Π_n^B .

The intersection lattice of the BD_n^S hyperplane arrangement is a subset of that of type B_n . This allows a natural definition of BD_n^S partitions, Π_n^S , as a subset of Π_n^B . By this approach, one obtains that Π_n^S consists of those partitions of Π_n^B which do not contain any block $\{i, \bar{i}\}$ for $i \in S$. In [Rei], Reiner defined NC_n^S , the non-crossing partitions of type BD_n^S , by $\text{NC}_n^S = \text{NC}_n^B \cap \Pi_n^S$. (For more details on the material sketched in the preceding paragraphs, see [Rei].)

We now define T_n^S to be those triangulations which correspond under ψ to partitions in NC_n^S . We can describe them more directly as follows:

Lemma 7.1. T_n^S consists of those triangulations which do not contain the triangles $i, \bar{i}, i+1$ and $i, \bar{i}, \overline{i+1}$ for any $i \in S$. T_n^S can also be characterized as the set of triangulations T such that $r_i(T) \neq n-1$ for any $i \in S$.

The remainder of the paper is devoted to sketching the proof of the following theorem, which generalizes Theorems 1, 2, and 3 to the broader context of type BD_n^S .

Theorem 7.2. T_n^S admits a lattice structure which is a quotient of that on T_n^B . T_n^S possesses an unrefinable chain of left modular elements, which implies that it is EL-shellable. Further, the order complex of any interval is either contractible or homotopic to a single sphere.

We define an equivalence relation \sim_S on T_n^B as follows: two non-identical triangulations are equivalent iff they differ in that one of them, say T , is not in T_n^S , and the other is the triangulation obtained by removing the diameter of T and replacing it with the other possible diameter.

An equivalence relation \sim on a lattice L is said to be a congruence relation if the lattice operations pass to equivalence classes. In this case, there is an induced lattice structure on the equivalence classes (see [Gr]).

Lemma 7.3. The relation \sim_S on T_n^B is a congruence relation.

Since the equivalence classes of \sim_S each contain a single element of T_n^S , the induced lattice structure on T_n^B / \sim_S gives rise to a lattice structure on T_n^S .

(One could also define a partial order on T_n^S by considering the order induced by its inclusion in T_n^B . It turns out that the order defined in this way coincides with the order we have already defined.)

It is immediate that the property of being left modular passes to equivalence classes, so T_n^S has a maximal chain of left modular elements, and is therefore EL-shellable. This maximal chain is shorter than that of T_n^B , because $S_{i,n-1} \sim_S S_{i,*}$ for $i \in S$.

It is easy to see that the join irreducibles of T_n^S are those $W_{i,t}$ such that either $i \notin S$ or $t \neq n-1$; again, they are in bijection with the elements of the left modular chain.

As in the type B case, the result on homotopy types of intervals follows by showing that there is at most one decreasing chain in any interval in T_n^S .

Acknowledgements

I would like to thank Nathan Reading for suggesting the possibility of a type B Tamari lattice to me, and for his amicable approach to the overlap in our investigations. I would also like to thank Vic Reiner and Christos Athanasiadis for discussions in which many of the ideas that appear in this paper were formed. I would like to thank Vic Reiner, Marcelo Aguiar, Nathan Reading, Nirit Sandman, and two anonymous referees for their comments on previous versions of this manuscript.

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