Counting non-equivalent coverings and non-isomorphic maps for Riemann surfaces *

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Abstract

The main result of the paper is a new formula for the number of conjugacy classes of subgroups of a given index in a finitely generated group. As application of this result a simple proof of the formula for the number of non-equivalent coverings over surface (orientable or not, bordered or not) is given. Another application is a formula for the number of non-isomorphic unrooted maps on an orientable closed surface with a given number of edges.

Keywords: *number of subgroups, conjugacy class of subgroups, surface coverings, unrooted maps*

1 Introduction

Let $M_\Gamma(n)$ denote the number of subgroups of index n in a group Γ, and $N_\Gamma(n)$ be the number of conjugacy classes of such groups. The last function counts the isomorphism classes of transitive permutation representations of degree n of Γ and hence, also the equivalence classes of n-fold unbranched connected coverings of a topological space with fundamental group Γ.

M. Hall [4] determined the numbers of subgroups $M_\Gamma(n)$ for a free group $\Gamma = \mathrm{F}_r$ of rank r. Later V. Liskovets [9] developed a new method for calculation of $N_\Gamma(n)$ for the same group. Both functions $M_\Gamma(n)$ and $N_\Gamma(n)$ for the fundamental group Γ of a closed surface were obtained in [15] and [16] for orientable and non-orientable surfaces, respectively. See also [17] and [3] for the case of the fundamental group of the Klein bottle and a survey [8] for related problems. In all these cases the problem of calculating of $M_\Gamma(n)$ was solved essentially by using representation theory of symmetric groups, contributed by Hurwitz and Frobenius, as the main tool ([6], [7]). The solution for the problem to finding $N_\Gamma(n)$ was based on the further development of the Liskovets

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method ([9], [10]). In [11] and [12], these ideas were applied to determine $M_\Gamma(n)$ for the fundamental groups of some Seifert spaces. Asymptotic formulas for $M_\Gamma(n)$ in many important cases were obtained in series of papers by T. W. Müller and his collaborators ([19],[20],[21]). An excellent exposition of the above results and related subjects is given in the book [14].

In the present paper, a new formula for the number of conjugacy classes of subgroups of given index in an arbitrary finitely generated group is obtained.

The main counting principle suggested in Section 2 of the paper is rather universal. It can be applied to Fuchsian groups to calculate the number of non-equivalent surface coverings (Section 3) as well as the number of unrooted maps on the surface (Section 4). Remark that the results of Section 3 were obtained in [9, 15, 16] by making use of cumbersome combinatorial technique. In the present paper they are rederived as simple consequences of Theorem 1 in Section 2. Another application of Theorem 1 is given in Section 4 where a new approach to determination of the number of unrooted maps on the closed oriented surface with given number of edges is suggested. Earlier, in more complicated way this result was obtained in [18].

2 Counting conjugacy classes of subgroup

Denote by $\mathrm{Epi}(K, \mathbb{Z}_\ell)$ the set of epimorphisms of a group K onto the cyclic group $\mathbb{Z}_\ell$ of order ℓ and by $|E|$ the cardinality of a set E.

The main result of this paper is the following counting principle.

Theorem 1 *Let Γ be a finitely generated group. Then the number of conjugacy classes of subgroups of index n in the group Γ is given by the formula*

$$N_\Gamma(n) = \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell m = n}} \sum_{K \underset{m}{<} \Gamma} |\mathrm{Epi}(K, \mathbb{Z}_\ell)|,$$

where the sum $\sum_{K \underset{m}{<} \Gamma}$ is taken over all subgroups K of index m in the group Γ.

Proof: Let P be a subgroup in Γ and $N(P,\Gamma)$ is the normalizer of P in the group Γ. We need the following two elementary lemmas.

Lemma 1 *The number of conjugacy classes of subgroups of index n in the group Γ is given by the formula*

$$N_\Gamma(n) = \frac{1}{n} \sum_{P \underset{n}{<} \Gamma} |N(P,\Gamma)/P|.$$

Proof: Let E be a conjugacy class of subgroups of index n in the group Γ. We claim that

$$\sum_{P\in E}|N(P,\Gamma)/P| = n.$$

Indeed, let $P'\in E$. Then $|E| = |\Gamma : N(P',\Gamma)|$ and for any $P\in E$ the groups $N(P,\Gamma)/P$ and $N(P',\Gamma)/P'$ are isomorphic. We have

$$\sum_{P\in E}|N(P,\Gamma)/P| = |E||N(P',\Gamma)/P'| = |\Gamma : N(P',\Gamma)||N(P',\Gamma) : P'| = |\Gamma : P'| = n.$$

Hence,

$$n\,N(n) = \sum_{E} n = \sum_{E}\sum_{P\in E}|N(P,\Gamma)/P| = \sum_{P\underset{n}{<}\Gamma}|N(P,\Gamma)/P|,$$

where the sum $\sum_{E}$ is taken over all conjugacy classes E of subgroups of index n in the group Γ. □

Lemma 2 *Let P be a subgroup of index n in the group Γ. Then*

$$|N(P,\Gamma)/P| = \sum_{\substack{\ell\mid n\\ \ell m=n}}\ \sum_{P\underset{Z_\ell}{\triangleleft}K\underset{m}{<}\Gamma}\phi(\ell),$$

where $\phi(\ell)$ is the Euler function and the second sum is taken over all subgroups K of index m in Γ containing P as a normal subgroup with $K/P\cong Z_\ell$. The sum vanishes if there are no such subgroups.

Proof: Set $G = N(P,\Gamma)/P$. Since $P\triangleleft N(P,\Gamma)<\Gamma$ and $P\underset{n}{<}\Gamma$, the order of any cyclic subgroup of G is a divisor of n.

Note that there is a one-to-one correspondence between cyclic subgroups Z_ℓ in the group G and subgroups K satisfying $P\underset{Z_\ell}{\triangleleft}K\underset{m}{<}\Gamma$, where $\ell m = n$.

Given a cyclic subgroup $Z_\ell<G$ there are exactly $\phi(\ell)$ elements of G which generate Z_ℓ.

Hence,

$$|G| = \sum_{\ell\mid n}\phi(\ell)\sum_{Z_\ell<G}1 = \sum_{\ell\mid n}\phi(\ell)\sum_{P\underset{Z_\ell}{\triangleleft}K\underset{m}{<}\Gamma}1 = \sum_{\ell\mid n}\ \sum_{P\underset{Z_\ell}{\triangleleft}K\underset{m}{<}\Gamma}\phi(\ell).$$

□

We finish the proof of the theorem by applying Lemma 1 and Lemma 2 for $\ell m = n$:

$$n\,N(n) = \sum_{P \underset{n}{<} \Gamma} |N(P,\Gamma)/P| =$$

$$\sum_{P \underset{n}{<} \Gamma} \sum_{\ell | n} \sum_{P \underset{Z_\ell}{\triangleleft} K \underset{m}{<} \Gamma} \phi(\ell) = \sum_{\ell | n} \sum_{P \underset{n}{<} \Gamma} \sum_{P \underset{Z_\ell}{\triangleleft} K \underset{m}{<} \Gamma} \phi(\ell) =$$

$$\sum_{\ell | n} \sum_{K \underset{m}{<} \Gamma} \sum_{P \underset{Z_\ell}{\triangleleft} K} \phi(\ell) = \sum_{\ell | n} \sum_{K \underset{m}{<} \Gamma} |\mathrm{Epi}(K, \mathsf{Z}_\ell)|.$$

The last equality is a consequence of the following observation. Given subgroup P, $P \underset{Z_\ell}{\triangleleft} K$ there are exactly $\phi(\ell)$ epimorphisms $\psi : K \to \mathsf{Z}_\ell$, with $\mathrm{Ker}\,(\psi) = \mathrm{P}$. Indeed, any two of them differ one from other by an element of the group $\mathrm{Aut}(\mathsf{Z}_\ell)$ consisting of $\phi(\ell)$ elements. □

Denote by $\mathrm{Hom}(\Gamma,\, \mathsf{Z}_\ell)$ the set of homomorphisms of a group Γ into the cyclic group Z_ℓ of order ℓ. Following G. Jones [1] we note that $|\mathrm{Hom}(\Gamma,\, \mathsf{Z}_\ell)| = \sum_{d|\ell} |\mathrm{Epi}(\Gamma,\, \mathsf{Z}_d)|$. Hence, by the Möbius inversion formula [2, §8.3, p. 148] we have the following result

Lemma 3

$$|\mathrm{Epi}(\Gamma,\, Z_\ell)| = \sum_{d|\ell} \mu(\frac{\ell}{d}) |\mathrm{Hom}(\Gamma,\, Z_d)|,$$

where $\mu(n)$ is the Möbius function.

This lemma essentially simplifies the calculation of $|\mathrm{Epi}(\Gamma,\, \mathsf{Z}_\ell)|$ for a finitely generated group Γ. Indeed, let $\mathrm{H}_1(\Gamma) = \Gamma/[\Gamma,\, \Gamma]$ be the first homology group of Γ. Suppose that $\mathrm{H}_1(\Gamma) = \mathsf{Z}_{\mathrm{m}_1} \oplus \mathsf{Z}_{\mathrm{m}_2} \oplus \ldots \oplus \mathsf{Z}_{\mathrm{m}_\mathrm{s}} \oplus \mathsf{Z}^{\mathrm{r}}$. Then we have

Lemma 4

$$|\mathrm{Epi}(\Gamma,\, Z_\ell)| = \sum_{d|\ell} \mu(\frac{\ell}{d})\,(m_1, d)\,(m_2, d)\ldots(m_s, d)\,d^r,$$

where (m, d) is the greatest common divisor of m and d.

Proof: Note that $|\mathrm{Hom}(\mathsf{Z}_m,\, \mathsf{Z}_d)| = (m, d)$ and $|\mathrm{Hom}(\mathsf{Z},\, \mathsf{Z}_d)| = d$. Since the group Z_d is Abelian, we obtain

$$|\mathrm{Hom}(\Gamma,\, \mathsf{Z}_d)| = |\mathrm{Hom}(\mathrm{H}_1(\Gamma),\, \mathsf{Z}_d)| = (m_1, d)\,(m_2, d)\ldots(m_s, d)\,d^r.$$

Then the result follows from Lemma 3. □

In particular, we have

Corollary 1

(i) *Let* $\mathrm{F_r}$ *be a free group of rank* r. *Then* $\mathrm{H_1(F_r)} = Z^{\mathrm{r}}$ *and* $|\mathrm{Epi}(\mathrm{F}_r,\, Z_\ell)| = \sum\limits_{d|\,\ell} \mu(\frac{\ell}{d})\, d^r$.

(ii) *Let* $\Phi_g = \langle a_1, b_1, \ldots, a_g, b_g : \prod\limits_{i=1}^{g} [a_i,\, b_i] = 1\rangle$ *be the fundamental group of a closed orientable surface of genus* g. *Then* $\mathrm{H_1(\Phi_g)} = Z^{2\mathrm{g}}$ *and* $|\mathrm{Epi}(\Phi_r,\, Z_\ell)| = \sum\limits_{d|\,\ell} \mu(\frac{\ell}{d})\, d^{2g}$.

(iii) *Let* $\Lambda_p = \langle a_1, a_2, \ldots, a_p : \prod\limits_{i=1}^{p} a_i^2 = 1\rangle$ *be the fundamental group of a closed non-orientable surface of genus* p. *Then* $\mathrm{H_1(\Lambda_p)} = Z_2 \oplus Z^{\mathrm{p}-1}$ *and* $|\mathrm{Epi}(\Lambda_p,\, Z_\ell)| = \sum\limits_{d|\,\ell} \mu(\frac{\ell}{d})\, (2, d)\, d^{p-1}$.

Note that the fundamental group of any compact surface (orientable or not, possibly, with non-empty boundary) is one of the three groups F_r, Φ_g and Λ_p listed in Corollary 1. In the next two sections we identify the number of conjugacy classes of subgroups of index n in the group Γ and the number of equivalence classes of n-fold unbranched connected coverings of a manifold with fundamental group Γ.

3 Counting surface coverings

Recall that the fundamental group $\pi_1(\mathcal{B})$ of a bordered surface $\mathcal{B}$ of Euler characteristic $\chi = 1 - r$, $r \geq 0$, is a free group F_r of rank r. An example of such a surface is the disc $\mathcal{D}_r$ with r holes. As the first application of the counting principle (Theorem 1) we have the following result obtained earlier by V. Liskovets [9]

Theorem 2 *Let* $\mathcal{B}$ *be a bordered surface with the fundamental group* $\pi_1(\mathcal{B}) = \mathrm{F}_r$. *Then the number of non-equivalent* $n-$*fold coverings of* $\mathcal{B}$ *is given by the formula*

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|\,n \\ \ell\, m=n}} \sum_{d|\,\ell} \mu(\frac{\ell}{d})\, d^{(r-1)m+1} M(m),$$

where $M(m)$ *is the number of subgroups of index* m *in the group* F_r.

Proof: Note that all subgroups of index m in F_r are isomorphic to $\Gamma_m = \mathrm{F}_{(r-1)m+1}$. By Theorem 1 we have

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|\,n \\ \ell\, m=n}} |\mathrm{Epi}(\Gamma_m,\, \mathrm{Z}_\ell)| \cdot M(m).$$

By Corollary 1(i) we get

$$|\mathrm{Epi}(\Gamma_m,\, \mathrm{Z}_\ell)| = \sum_{d|\,\ell} \mu(\frac{\ell}{d})\, d^{(r-1)m+1}$$

and the result follows. $\square$

By the M. Hall recursive formula [4] the number of subgroups of index m in the group F_r is equal to $M(m) = \dfrac{t_{m,r}}{(m-1)!}$, where

$$t_{m,r} = m!^r - \sum_{j=1}^{m-1} \binom{m-1}{j-1} (m-j)!^r\, t_{j,r}, \;\; t_{1,r} = 1.$$

The next result was obtained in [15] in a rather complicated way.

Theorem 3 *Let $\mathcal{S}$ be a closed orientable surface with the fundamental group $\pi_1(\mathcal{S}) = \Phi_g$. Then the number of non-equivalent $n-$fold coverings of $\mathcal{S}$ is given by the formula*

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|\,n \\ \ell\, m=n}} \sum_{d|\,\ell} \mu(\frac{\ell}{d})\, d^{2(g-1)m+2} M(m),$$

where $M(m)$ is the number of subgroups of index m in the group Φ_g.

Proof: Recall that any subgroup K_m of index m in the group Φ_g is isomorphic to $\Phi_{g'}$, where g' and g are related by the Riemann-Hurwitz formula [6] $2g' - 2 = m(2g-2)$. Hence, $K_m = \Phi_{(g-1)m+1}$. By the main counting principle (Theorem 1) we have

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|\,n \\ \ell\, m=n}} |\mathrm{Epi}(K_m,\, \mathrm{Z}_\ell)| \cdot M(m),$$

where

$$|\mathrm{Epi}(K_m,\, \mathrm{Z}_\ell)| = \sum_{d|\,\ell} \mu(\frac{\ell}{d})\, d^{2(g-1)m+2}$$

is given by Corollary 1(ii) . $\square$

Let $\mathcal{N}$ be a closed non-orientable surface of genus p with the fundamental group $\pi_1(\mathcal{N}) = \Lambda_p$. Denote by $\mathcal{N}_m^+$ and $\mathcal{N}_m^-$ an orientable and non-orientable $m-$fold coverings of $\mathcal{N}$, respectively and set $\Gamma_m^+ = \pi_1(\mathcal{N}_m^+)$ and $\Gamma_m^- = \pi_1(\mathcal{N}_m^-)$. For simplicity, we will refer to Γ_m^+ and Γ_m^- as orientable and non-orientable subgroups of index m in Λ_p, respectively. By the Riemann-Hurwitz formula we get

$$2\,\mathrm{genus}\,(\mathcal{N}_m^+) - 2 = m(p-2) \;\text{ and }\; \mathrm{genus}\,(\mathcal{N}_m^-) - 2 = m(p-2),$$

where $p = \text{genus}\,(\mathcal{N})$. Hence $\Gamma_m^+ = \Phi_{\frac{m}{2}(p-2)+1}$ and $\Gamma_m^- = \Lambda_{m(p-2)+2}$.

By the main counting principle, the number of non-equivalent $n-$fold coverings of N is given by the formula

$$N(n) = \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell\, m = n}} (|\mathrm{Epi}(\Gamma_m^+,\, \mathsf{Z}_\ell)| \cdot M^+(m) + |\mathrm{Epi}(\Gamma_m^-,\, \mathsf{Z}_\ell)| \cdot M^-(m)),$$

where $M^+(m)$ and $M^-(m)$ are the numbers of orientable and non-orientable subgroups of index m in the group Λ_p, respectively.

By Corollary 1(ii) and Corollary 1(iii), we have

$$|\mathrm{Epi}(\Gamma_m^+,\, \mathsf{Z}_\ell)| = \sum_{d \mid \ell} \mu(\frac{\ell}{d})\, d^{m(p-2)+2} \;\text{ and }\; |\mathrm{Epi}(\Gamma_m^-,\, \mathsf{Z}_\ell)| = \sum_{d \mid \ell} \mu(\frac{\ell}{d})\, (2, d)\, d^{m(p-2)+1}.$$

As a result, we have proved the following theorem obtained earlier in [16] by making use of a cumbersome combinatorial technique.

Theorem 4 *Let n be a closed orientable surface with the fundamental group $\pi_1(\mathcal{N}) = \Lambda_p$. Then the number of non-equivalent $n-$fold coverings of $\mathcal{N}$ is given by the formula*

$$N(n) = \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell\, m = n}} \sum_{d \mid \ell} \mu(\frac{\ell}{d})\, (d^{m(p-2)+2} M^+(m) + (2, d)\, d^{m(p-2)+1} M^-(m)),$$

where $M^+(m)$ and $M^-(m)$ are the numbers of orientable and non-orientable subgroups of index m in the group Λ_p, respectively.

For completeness note that ([15], [16]) if $\Gamma = \Phi_g$ or Λ_p then the number $M(m)$ of subgroups of index m in the group Γ is equal to

$$R_\nu(m) = m \sum_{s=1}^{m} \frac{(-1)^{s+1}}{s} \sum_{\substack{i_1 + i_2 + \ldots + i_s = m \\ i_1, i_2, \ldots, i_s \geq 1}} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_s},$$

where $\beta_k = \sum\limits_{\chi \in D_k} \big(\frac{k!}{f^\chi}\big)^\nu$, D_k is the set of irreducible representations of a symmetric group S_k, f^χ is the degree of the representation χ, $\nu = 2g - 2$ for $\Gamma = \Phi_g$ and $\nu = p - 2$ for $\Gamma = \Lambda_p$. Moreover, in the latter case, $M^+(m) = 0$ if m is odd, $M^+(m) = R_{2\nu}(\frac{m}{2})$ if m is even, and $M^-(m) = M(m) - M^+(m)$. Also, the number of subgroups can be found by the following recursive formula

$$M(m) = m\, \beta_m - \sum_{j=1}^{m-1} \beta_{m-j} M(j), \quad M(1) = 1.$$

4 Non-isomorphic maps on surface

In this section we deal with the problem of enumeration of oriented unrooted maps of given genus g. From now on *a surface* is a connected, orientable surface without a border. A *map* is a 2-cell decomposition of a surface. Standardly, maps on surfaces are described as 2-cell embeddings of graphs. An embedded graph is a 4-tuple (D, V, I, L), where D and V are disjoint sets of *darts* and *vertices*, respectively, I is an incidence function $I : D \to V$ assigning to each dart an initial vertex, and L is the dart-reversing involution. Edges of a graph are orbits of L. Note that some edges may be incident just with one vertex, such edges will be called *semiedges*. In what follows we shall deal with the category of *oriented maps*, that means one of the two global orientations of the underlying surface is fixed. Recall, that a map is called *rooted* if it has one distinguished dart x called a *root*. An *isomorphism* between rooted maps takes root onto root. Recall that if (M, x) and (M, y) are two rooted maps based on the same map with a dart set D then the number of isomorphism classes for (M, x) and (M, y) is the same. There is a 1-1 correspondence between isomorphism classes of rooted maps defined in the category of oriented maps, and isomorphism classes of rooted maps in the category of maps on orientable surfaces as they are defined, for instance, in monograph [13, page 7].

We fix the set of darts D and consider different maps based on D. We will determine the number of isomorphism classes of (unrooted) maps with n darts and of given genus g. This number will be denoted by $NUM_g(n)$.

Denote by $Orb(S_g/Z_\ell)$ the set of all orbifolds arising as cyclic quotients under some action of Z_ℓ on maps on a surface of genus g and by $NRM_O(m)$ the number of rooted quotient maps for a given orbifold type O which lift onto maps on a surface of genus g, having $n = \ell m$ darts. We note that if the map contains no semi-edges then the number of darts $n = 2\,e$, where e is the number of edges of the map.

Let S_g be an orientable surface of genus g and Z_ℓ be a cyclic group of automorphisms of S_g. Denote by $(\gamma; m_1, m_2, \ldots, m_r)$, $2 \le m_1 \le m_2 \le \ldots \le m_r \le \ell$, the signature of orbifold $O = S_g/Z_\ell$. That is, the underlying space of O is an oriented surface of genus γ and the regular cyclic covering $S_g \to O = S_g/Z_\ell$ is branched over r points of O with branch indices $m_1, m_2, \ldots, m_r$, respectively. W.J. Harvey [5] obtained necessary and sufficient conditions for an existence of a cyclic orbifold S_g/Z_ℓ with signature $(\gamma; m_1, m_2, \ldots, m_r)$.

Given orbifold O of the signature $(\gamma; m_1, m_2, \ldots, m_r)$ define an orbifold fundamental group $\pi_1(O)$ to be an F-group generated by 2γ generators $a_1, b_1, a_2, b_2, \ldots, a_\gamma, b_\gamma$ and by r generators e_j, $j = 1, \ldots, r$ satisfying the relations

$$\prod_{i=1}^{\gamma}[a_i, b_i]\prod_{j=1}^{r} e_j = 1,\ e_j^{m_j} = 1 \text{ for every } \ j = 1, \ldots, r,$$

where $[a, b] = aba^{-1}b^{-1}$.

An epimorphism $\pi_1(O) \to Z_\ell$ onto a cyclic group of order ℓ is called *order preserving* if it preserves the orders of generators e_j, $j = 1, \ldots, r$. Equivalently, an order preserving epimorphism $\pi_1(O) \to Z_\ell$ has a torsion-free kernel. We denote by $Epi_0(\pi_1(O), Z_\ell)$ the number of order preserving epimorphisms $\pi_1(O) \to Z_\ell$.

An elementary consideration based on Theorem 1 enables us to prove the following theorem [18].

Theorem 5 *With the above notation the following enumeration formula holds:*

$$NUM_g(n) = \frac{1}{n} \sum_{\ell|n, n=\ell m} \sum_{O \in Orb(S_g/Z_\ell)} Epi_0(\pi_1(O), Z_\ell) NRM_O(m).$$

The number of rooted maps on the orbifold as well as the number of order preserving epimorphisms are explicitly determined in joint paper with R. Nedela [18]. Numerical tables for the number of non-isomorphic (unrooted) maps on closed surface of genus 1, 2 and 3 with up to 30 edges are also given in the paper.

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