HALL–LITTLEWOOD FUNCTIONS AND THE A_2 ROGERS–RAMANUJAN IDENTITIES

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Abstract. We prove an identity for Hall–Littlewood symmetric functions labelled by the Lie algebra A_2. Through specialization this yields a simple proof of the A_2 Rogers–Ramanujan identities of Andrews, Schilling and the author.

Nous démontrons une identité pour les functions symétriques de Hall–Littlewood associée à l'algèbre de Lie A_2. En spécialisant cette identité, nous obtenons une démonstration simple des identités du type Rogers–Ramanujan associées á A_2 d'Andrews, Schilling et l'auteur.

1. Introduction

The Rogers–Ramanujan identities, given by [10]

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \tag{1.1a}$$

and

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-2})(1-q^{5n-3})}, \tag{1.1b}$$

are two of the most famous q-series identities, with deep connections with number theory, representation theory, statistical mechanics and various other branches of mathematics.

Many different proofs of the Rogers–Ramanujan identities have been given in the literature, some bijective, some representation theoretic, but the vast majority basic hypergeometric. In 1990, J. Stembridge, building on work of I. Macdonald, found a proof of the Rogers–Ramanujan identities quite unlike any of the previously known proofs. In particular he discovered that Rogers–Ramanujan-type identities may be obtained by appropriately specializing identities for Hall–Littlewood polynomials. The Hall–Littlewood polynomials and, more generally, Hall–Littlewood functions are an important class of symmetric functions, generalizing the well-known Schur functions. Stembridge's Hall–Littlewood approach to Rogers–Ramanujan identities has been further generalized in recent work by Fulman [2], Ishikawa *et al.* [5] and Jouhet and Zeng [7].

Several years ago Andrews, Schilling and the present author generalized the two Rogers–Ramanujan identities to three identities labelled by the Lie algebra A_2 [1]. The simplest of these, which takes the place of (1.1a) when A_1 is replaced by A_2

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reads

$$(1.2)\qquad \sum_{n_1,n_2=0}^{\infty} \frac{q^{n_1^2-n_1n_2+n_2^2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_1+n_2}} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{7n-1})^2(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-6})^2},$$

where $(q;q)_0 = 1$ and $(q;q)_n = \prod_{i=1}^{n}(1-q^i)$ is a q-shifted factorial.

An important question is whether (1.2) and its companions can again be understood in terms of Hall–Littlewood functions. This question is especially relevant since the A_n analogues of the Rogers–Ramanujan identities have so far remained elusive, and an understanding of (1.2) in the context of symmetric functions might provide further insight into the structure of the full A_n generalization of (1.1).

In this paper we will show that the theory of Hall–Littlewood functions may indeed be applied to yield a proof of (1.2). In particular we will prove the following A_2-type identity for Hall–Littlewood functions.

Theorem 1.1. *Let $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$ and let $P_\lambda(x;q)$ and $P_\mu(y;q)$ be Hall–Littlewood functions indexed by the partitions λ and μ. Then*

$$(1.3)\qquad \sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x;q)P_\mu(y;q) = \prod_{i\geq 1} \frac{1}{(1-x_i)(1-y_i)} \prod_{i,j\geq 1} \frac{1-x_iy_j}{1-q^{-1}x_iy_j}.$$

In the above λ' and μ' are the conjugates of λ and μ, $(\lambda|\mu) = \sum_{i\geq 1}\lambda_i\mu_i$, and $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$.

An appropriate specialization of Theorem 1.1 leads to a q-series identity of [1] which is the key-ingredient in proving (1.2).

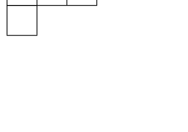
In the next section we give the necessary background material on Hall–Littlewood functions. Section 3 contains a proof of Theorem 1.1 and in Section 4 we present a proof of the A_2 Rogers–Ramanujan identities (1.2) based on Theorem 1.1.

2. Hall-Littlewood functions

We review some basic facts from the theory of Hall-Littlewood functions. For more details the reader may wish to consult Chapter III of Macdonald's book on symmetric functions [9].

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition, i.e., $\lambda_1 \geq \lambda_2 \geq \dots$ with finitely many λ_i unequal to zero. The length and weight of λ, denoted by $\ell(\lambda)$ and $|\lambda|$, are the number and sum of the non-zero λ_i (called parts), respectively. The unique partition of weight zero is denoted by 0, and the multiplicity of the part i in the partition λ is denoted by $m_i(\lambda)$.

We identify a partition with its diagram or Ferrers graph in the usual way, and, for example, the diagram of $\lambda = (6,3,3,1)$ is given by



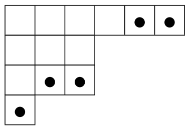
The conjugate λ' of λ is the partition obtained by reflecting the diagram of λ in the main diagonal. Hence $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$.

A standard statistic on partitions needed repeatedly is

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda'_i}{2}.$$

We also need the usual scalar product $(\lambda|\mu) = \sum_{i\geq 1}\lambda_i\mu_i$ (which in the notation of [9] would be $|\lambda\mu|$). We will occasionally use this for more general sequences of integers, not necessarily partitions.

If λ and μ are two partions then $\mu \subset \lambda$ iff $\lambda_i \geq \mu_i$ for all $i \geq 1$, i.e., the diagram of λ contains the diagram of μ. If $\mu \subset \lambda$ then the skew-diagram $\lambda - \mu$ denotes the set-theoretic difference between λ and μ, and $|\lambda - \mu| = |\lambda| - |\mu|$. For example, if $\lambda = (6,3,3,1)$ and $\mu = (4,3,1)$ then the skew diagram $\lambda - \mu$ is given by the marked squares in



and $|\lambda - \mu| = 5$.

For $\theta = \lambda - \mu$ a skew diagram, its conjugate $\theta' = \lambda' - \mu'$ is the (skew) diagram obtained by reflecting θ in the main diagonal. Following [9] we define the components of θ and θ' by $\theta_i = \lambda_i - \mu_i$ and $\theta'_i = \lambda'_i - \mu'_i$. Quite often we only require knowledge of the sequence of components of a skew diagram θ, and by abuse of notation we will occasionally write $\theta = (\theta_1, \theta_2, \dots)$, even though the components θ_i alone do not fix θ.

A skew diagram θ is a horizontal strip if $\theta'_i \in \{0,1\}$, i.e., if at most one square occurs in each column of θ. The skew diagram in the above example is a horizontal strip since $\theta' = (1,1,1,0,1,1,0,0,\dots)$.

Let S_n be the symmetric group, $\Lambda_n = \mathbb{Z}[x_1,\dots,x_n]^{S_n}$ be the ring of symmetric polynomials in n independent variables and Λ the ring of symmetric functions in countably many independent variables.

For $x = (x_1,\dots,x_n)$ and λ a partition such that $\ell(\lambda) \leq n$ the Hall–Littlewood polynomials $P_\lambda(x;q)$ are defined by

$$P_\lambda(x;q) = \sum_{w\in S_n/S_n^\lambda} w\Big(x^\lambda \prod_{\lambda_i>\lambda_j}\frac{x_i - qx_j}{x_i - x_j}\Big). \tag{2.1}$$

Here S_n^λ is the subgroup of S_n consisting of the permutations that leave λ invariant, and $w(f(x)) = f(w(x))$. When $\ell(\lambda) > n$,

$$P_\lambda(x;q) = 0. \tag{2.2}$$

The Hall–Littlewood polynomials are symmetric polynomials in x, homogeneous of degree $|\lambda|$, with coefficients in $\mathbb{Z}[q]$, and form a $\mathbb{Z}[q]$ basis of $\Lambda_n[q]$. Thanks to the stability property $P_\lambda(x_1,\dots,x_n,0;q) = P_\lambda(x_1,\dots,x_n;q)$ the Hall–Littlewood polynomials may be extended to the Hall–Littlewood functions in an infinite number of variables $x_1, x_2, \dots$ in the usual way, to form a $\mathbb{Z}[q]$ basis of $\Lambda[q]$. The indeterminate q in the Hall–Littlewood symmetric functions serves as a parameter interpolating between the Schur functions and monomial symmetric functions; $P_\lambda(x;0) = s_\lambda(x)$ and $P_\lambda(x;1) = m_\lambda(x)$.

We will also need the symmetric functions $Q_\lambda(x;q)$ (also referred to as Hall-Littlewood functions) defined by

$$Q_\lambda(x;q) = b_\lambda(q) P_\lambda(x;q), \tag{2.3}$$

where

$$b_\lambda(q) = \prod_{i=1}^{\lambda_1} (q;q)_{m_i(\lambda)}.$$

We already mentioned the homogeneity of the Hall–Littlewood functions;

$$P_\lambda(ax;q) = a^{|\lambda|} P_\lambda(x;q), \tag{2.4}$$

where $ax = (ax_1, ax_2, \dots)$. Another useful result is the specialization

$$P_\lambda(1, q, \dots, q^{n-1}; q) = \frac{q^{n(\lambda)} (q;q)_n}{(q;q)_{n-\ell(\lambda)} b_\lambda(q)}, \tag{2.5}$$

where $1/(q;q)_{-m} = 0$ for m a positive integer, so that $P_\lambda(1, q, \dots, q^{n-1}; q) = 0$ if $\ell(\lambda) > n$ in accordance with (2.2). By (2.3) this also implies the particularly simple

$$Q_\lambda(1, q, q^2, \dots; q) = q^{n(\lambda)}. \tag{2.6}$$

The skew Hall–Littlewood functions $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ are defined by

$$P_\lambda(x, y; q) = \sum_\mu P_{\lambda/\mu}(x;q) P_\mu(y;q) \tag{2.7}$$

and

$$Q_\lambda(x, y; q) = \sum_\mu Q_{\lambda/\mu}(x;q) Q_\mu(y;q),$$

so that

$$Q_{\lambda/\mu}(x;q) = \frac{b_\lambda(q)}{b_\mu(q)} P_{\lambda/\mu}(x;q). \tag{2.8}$$

An important property is that $P_{\lambda/\mu}$ is zero if $\mu \not\subset \lambda$. Some trivial instances of the skew functions are given by $P_{\lambda/0} = P_\lambda$ and $P_{\lambda/\lambda} = 1$. By (2.8) similar statements apply to $Q_{\lambda/\mu}$.

The Cauchy identity for (skew) Hall–Littlewood functions is given by [11, Lemma 3.1]

$$\sum_\lambda P_{\lambda/\mu}(x;q) Q_{\lambda/\nu}(y;q) = \sum_\lambda P_{\nu/\lambda}(x;q) Q_{\mu/\lambda}(y;q) \prod_{i,j\ge 1} \frac{1 - q x_i y_j}{1 - x_i y_j}. \tag{2.9}$$

We conclude our introduction of the Hall–Littlewood functions with the following two important definitions. Let $\lambda \supset \mu$ be partitions such that $\theta = \lambda - \mu$ is a horizontal strip, i.e., $\theta'_i \in \{0, 1\}$. Let I be the set of integers $i \ge 1$ such that $\theta'_i = 1$ and $\theta'_{i+1} = 0$. Then

$$\phi_{\lambda/\mu}(q) = \prod_{i \in I} (1 - q^{m_i(\lambda)}).$$

Similarly, let J be the set of integers $j \ge 1$ such that $\theta'_j = 0$ and $\theta'_{j+1} = 1$. Then

$$\psi_{\lambda/\mu}(q) = \prod_{j \in J} (1 - q^{m_j(\mu)}).$$

For example, if $\lambda = (5,3,2,2)$ and $\mu = (3,3,2)$ then θ is a horizontal strip and $\theta' = (1,1,0,1,1,0,0,\dots)$. Hence $I = \{2,5\}$ and $J = \{3\}$, leading to

$$\phi_{\lambda/\mu}(q) = (1 - q^{m_2(\lambda)})(1 - q^{m_5(\lambda)}) = (1 - q^2)(1 - q)$$

and

$$\psi_{\lambda/\mu}(q) = (1 - q^{m_3(\mu)}) = (1 - q^2).$$

The skew Hall–Littlewood functions $Q_{\lambda/\mu}(x;q)$ and $P_{\lambda/\mu}(x;q)$ can be expressed in terms of $\phi_{\lambda/\mu}(q)$ and $\psi_{\lambda/\mu}(q)$ [9, p. 229]. For our purposes we only require a special instance of this result corresponding to the case that x represents a single variable. Then

$$Q_{\lambda/\mu}(x;q) = \begin{cases} \phi_{\lambda/\mu}(q) x^{|\lambda-\mu|} & \text{if } \lambda - \mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise} \end{cases} \tag{2.10a}$$

and

$$P_{\lambda/\mu}(x;q) = \begin{cases} \psi_{\lambda/\mu}(q) x^{|\lambda-\mu|} & \text{if } \lambda - \mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.10b}$$

3. Proof of Theorem 1.1

Throughout this section z represents a single variable.

To establish (1.3) it is enough to show its truth for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, and by induction on m it then easily follows that we only need to prove

$$\sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x;q) P_\mu(y,z;q) = \frac{1}{1-z} \prod_{i=1}^{n} \frac{1 - zx_i}{1 - q^{-1}zx_i} \sum_{\lambda,\mu} q^{n(\lambda)+n(\mu)-(\lambda'|\mu')} P_\lambda(x;q) P_\mu(y;q), \tag{3.1}$$

where we have replaced y_{m+1} by z.

If on the left we replace μ by ν and use (2.7) (with $\lambda \to \nu$ and $x \to z$) we get

$$\mathrm{LHS}(3.1) = \sum_{\lambda,\mu,\nu} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} P_\lambda(x;q) P_\mu(y;q) P_{\nu/\mu}(z;q).$$

From (2.9) with $\mu = 0$, $x = (x_1, \dots, x_n)$ and $y \to z/q$ it follows that

$$P_\nu(x;q) \prod_{i=1}^{n} \frac{1 - zx_i}{1 - q^{-1}zx_i} = \sum_\lambda Q_{\lambda/\nu}(z/q;q) P_\lambda(x;q).$$

Using this on the right of (3.1) with λ replaced by ν yields

$$\mathrm{RHS}(3.1) = \frac{1}{1-z} \sum_{\lambda,\mu,\nu} q^{n(\mu)+n(\nu)-(\mu'|\nu')} P_\lambda(x;q) P_\mu(y;q) Q_{\lambda/\nu}(z/q;q).$$

Therefore, by equating coefficients of $P_\lambda(x;q)P_\mu(y;q)$ we find that the problem of proving (1.3) boils down to showing that

$$\sum_\nu q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} P_{\nu/\mu}(z;q) = \frac{1}{1-z} \sum_\nu q^{n(\mu)+n(\nu)-(\mu'|\nu')} Q_{\lambda/\nu}(z/q;q).$$

Next we use (2.10) to arrive at the equivalent but more combinatorial statement that

$$\sum_{\substack{\nu\supset\mu\\ \nu-\mu \text{ hor. strip}}} q^{n(\lambda)+n(\nu)-(\lambda'|\nu')} z^{|\nu-\mu|}\psi_{\nu/\mu}(q) = \frac{1}{1-z}\sum_{\substack{\nu\subset\lambda\\ \lambda-\nu \text{ hor. strip}}} q^{n(\mu)+n(\nu)-(\mu'|\nu')}(z/q)^{|\lambda-\nu|}\phi_{\lambda/\nu}(q). \tag{3.2}$$

To make further progress we need a lemma [12].

Lemma 3.1. *For k a positive integer let $\omega=(\omega_1,\dots,\omega_k)\in\{0,1\}^k$, and let $J=J(\omega)$ be the set of integers j such that $\omega_j=0$ and $\omega_{j+1}=1$. For $\lambda\supset\mu$ partitions let $\theta'=\lambda'-\mu'$ be a skew diagram. Then*

$$\sum_{\substack{\lambda\supset\mu\\ \lambda-\mu \text{ hor. strip}\\ \theta'_i=\omega_i,\ i\in\{1,\dots,k\}}} q^{n(\lambda)} z^{|\lambda-\mu|}\psi_{\lambda/\mu}(q) = \frac{q^{n(\mu)+(\mu'|\omega)}z^{|\omega|}}{1-z}(1-z(1-\omega_k)q^{\mu'_k})\prod_{j\in J}(1-q^{m_j(\mu)}).$$

The restriction $\theta'_i=\omega_i$ for $i\in\{1,\dots,k\}$ in the sum over λ on the left means that the first k parts of λ' are fixed. The remaining parts are free subject only to the condition that $\lambda-\mu$ is a horizontal strip, i.e., that $\lambda'_i-\mu'_i\in\{0,1\}$.

In view of Lemma 3.1 it is natural to rewrite the left side of (3.2) as

$$\mathrm{LHS}(3.2)=\sum_{\omega\in\{0,1\}^{\lambda_1}}\ \sum_{\substack{\nu\supset\mu\\ \nu-\mu \text{ hor. strip}\\ \theta'_i=\omega_i,\ i\in\{1,\dots,\lambda_1\}}} q^{n(\lambda)+n(\nu)-(\lambda'|\mu')-(\lambda'|\omega)} z^{|\nu-\mu|}\psi_{\nu/\mu}(q),$$

where $\theta=\nu-\mu$, and where we have used that $\theta'_i\in\{0,1\}$ as follows from the fact that $\nu-\mu$ is a horizontal strip.

Now the sum over ν can be performed by application of Lemma 3.1 with $\lambda\to\nu$ and $k\to\lambda_1$, resulting in

$$\mathrm{LHS}(3.2)=\frac{q^{n(\lambda)+n(\mu)-(\lambda'|\mu')}}{1-z}\sum_{\omega\in\{0,1\}^{\lambda_1}} q^{(\mu'|\omega)-(\lambda'|\omega)}z^{|\omega|} \times(1-z(1-\omega_{\lambda_1})q^{\mu'_{\lambda_1}})\prod_{j\in J}(1-q^{m_j(\mu)})$$

with $J=J(\omega)\subset\{1,\dots,\lambda_1-1\}$ the set of integers j such that $\omega_j<\omega_{j+1}$.

For the right-hand side of (3.2) we introduce the notation $\tau_i=\lambda'_i-\nu'_i$, so that the sum over ν can be rewritten as a sum over $\tau\in\{0,1\}^{\lambda_1}$. Using that

$$n(\nu)=\sum_{i=1}^{\lambda_1}\binom{\nu'_i}{2}=\sum_{i=1}^{\lambda_1}\binom{\lambda'_i-\tau_i}{2}=n(\lambda)-(\lambda'|\tau)+|\tau|$$

this yields

$$\mathrm{RHS}(3.2)=\frac{q^{n(\lambda)+n(\mu)-(\lambda'|\mu')}}{1-z}\sum_{\tau\in\{0,1\}^{\lambda_1}} q^{(\mu'|\tau)-(\lambda'|\tau)}z^{|\tau|}\prod_{i\in I}(1-q^{m_i(\lambda)}),$$

with $I = I(\tau) \subset \{1, \ldots, \lambda_1\}$ the set of integers i such that $\tau_i > \tau_{i+1}$ (with the convention that $\lambda_1 \in I$ if $\tau_{\lambda_1} = 1$).

Equating the above two results for the respective sides of (3.2) gives

$$\sum_{\omega \in \{0,1\}^{\lambda_1}} q^{(\mu'|\omega)-(\lambda'|\omega)} z^{|\omega|} (1 - z(1-\omega_{\lambda_1}) q^{\mu'_{\lambda_1}}) \prod_{j \in J} (1 - q^{m_j(\mu)}) \\ = \sum_{\tau \in \{0,1\}^{\lambda_1}} q^{(\mu'|\tau)-(\lambda'|\tau)} z^{|\tau|} \prod_{i \in I} (1 - q^{m_i(\lambda)}).$$

Using that $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ it is not hard to see that this is the

$$k \to \lambda_1, \quad b_{k+1} \to 1, \quad a_i \to z q^{\mu'_i}, \quad b_i \to q^{\lambda'_i}, \quad i \in \{1, \ldots, \lambda_1\}$$

specialization of the more general

$$\sum_{\omega \in \{0,1\}^k} (a/b)^\omega (1 - (1-\omega_k) a_k/b_{k+1}) \prod_{j \in J} (1 - a_j/a_{j+1}) \\ = \sum_{\tau \in \{0,1\}^k} (a/b)^\tau \prod_{i \in I} (1 - b_i/b_{i+1}),$$

where $(a/b)^\omega = \prod_{i=1}^k (a_i/b_i)^{\omega_i}$ and $(a/b)^\tau = \prod_{i=1}^k (a_i/b_i)^{\tau_i}$. Obviously, the set $J \subset \{1, \ldots, k-1\}$ should now be defined as the set of integers j such that $\omega_j < \omega_{j+1}$ and the the set $I \subset \{1, \ldots, k\}$ as the set of integers i such that $\tau_i > \tau_{i+1}$ (with the convention that $k \in I$ if $\tau_k = 1$).

Next we split both sides into the sum of two terms as follows:

$$\Biggl(\sum_{\omega \in \{0,1\}^k} - (a_k/b_{k+1}) \sum_{\substack{\omega \in \{0,1\}^k \\ \omega_k = 0}} \Biggr) (a/b)^\omega \prod_{j \in J} (1 - a_j/a_{j+1}) \\ = \Biggl(\sum_{\tau \in \{0,1\}^k} - (b_k/b_{k+1}) \sum_{\substack{\tau \in \{0,1\}^k \\ \tau_k = 1}} \Biggr) (a/b)^\tau \prod_{\substack{i \in I \\ i \neq k}} (1 - b_i/b_{i+1}).$$

Equating the first sum on the left with the first sum on the right yields

$$\sum_{\omega \in \{0,1\}^k} (a/b)^\omega \prod_{j \in J} (1 - a_j/a_{j+1}) = \sum_{\tau \in \{0,1\}^k} (a/b)^\tau \prod_{\substack{i \in I \\ i \neq k}} (1 - b_i/b_{i+1}). \tag{3.3}$$

If we equate the second sum on the left with the second sum on the right and use that $k-1 \notin J(\omega)$ if $\omega_k = 0$ and $k-1 \notin I(\tau)$ if $\tau_k = 1$, we obtain $(a_k/b_{k+1})((3.3)_{k \to k-1})$.

Slightly changing our earlier convention we thus need to prove that

$$\sum_{\omega \in \{0,1\}^k} (a/b)^\omega \prod_{j \in J} (1 - a_j/a_{j+1}) = \sum_{\tau \in \{0,1\}^k} (a/b)^\tau \prod_{i \in I} (1 - b_i/b_{i+1}), \tag{3.4}$$

where from now on $I \subset \{1, \ldots, k-1\}$ denotes the set of integers i such that $\tau_i > \tau_{i+1}$ (so that no longer $k \in I$ if $\tau_k = 1$). It is not hard to see by multiplying out the respective products that boths sides yield $((1+\sqrt{2})^{k+1} - (1-\sqrt{2})^{k+1})/(2\sqrt{2})$ terms. To see that the terms on the left and right are in one-to-one correspondence we again resort to induction. First, for $k = 1$ it is readily checked that both sides yield $1 + a_1/b_1$. For $k = 2$ we on the left get

$$\underbrace{1}_{\omega=(0,0)} + \underbrace{(a_1/b_1)}_{\omega=(1,0)} + \underbrace{(a_2/b_2)(1-a_1/a_2)}_{\omega=(0,1)} + \underbrace{(a_1a_2/b_1b_2)}_{\omega=(1,1)}$$

and on the right

$$\underbrace{1}_{\tau=(0,0)} + \underbrace{(a_1/b_1)(1-b_1/b_2)}_{\tau=(1,0)} + \underbrace{(a_2/b_2)}_{\tau=(0,1)} + \underbrace{(a_1a_2/b_1b_2)}_{\tau=(1,1)}$$

which both give

$$1 + a_1/b_1 + a_2/b_2 - a_1/b_2 + a_1a_2/b_1b_2.$$

Let us now assume that (3.4) has been shown to be true for $1 \le k \le K-1$ with $K \ge 3$ and prove the case $k = K$.

On the left of (3.4) we split the sum over ω according to

$$\sum_{\omega\in\{0,1\}^k} = \sum_{\substack{\omega\in\{0,1\}^k\\ \omega_1=1}} + \sum_{\substack{\omega\in\{0,1\}^k\\ \omega_1=\omega_2=0}} + \sum_{\substack{\omega\in\{0,1\}^k\\ \omega_1=0,\ \omega_2=1}} .$$

Defining $\bar\omega \in \{0,1\}^{k-1}$ and $\bar{\bar\omega} \in \{0,1\}^{k-2}$ by $\bar\omega = (\omega_2,\dots,\omega_k)$ and $\bar{\bar\omega} = (\omega_3,\dots,\omega_k)$, and also setting and $\bar a_j = a_{j+1}$, $\bar b_j = b_{j+1}$, and $\bar{\bar a}_j = a_{j+2}$, $\bar{\bar b}_j = b_{j+2}$, this leads to

$$\begin{aligned}
\mathrm{LHS}(3.4) = (a_1/b_1) &\sum_{\bar\omega\in\{0,1\}^{k-1}} (\bar a/\bar b)^{\bar\omega} \prod_{j\in J(\bar\omega)} (1-\bar a_j/\bar a_{j+1}) \\
&+ \sum_{\substack{\bar\omega\in\{0,1\}^{k-1}\\ \bar\omega_1=0}} (\bar a/\bar b)^{\bar\omega} \prod_{j\in J(\bar\omega)} (1-\bar a_j/\bar a_{j+1}) \\
&+ (1-a_1/a_2) \sum_{\substack{\bar\omega\in\{0,1\}^{k-1}\\ \bar\omega_1=1}} (\bar a/\bar b)^{\bar\omega} \prod_{j\in J(\bar\omega)} (1-\bar a_j/\bar a_{j+1}) \\
&= (1+a_1/b_1) \sum_{\bar\omega\in\{0,1\}^{k-1}} (\bar a/\bar b)^{\bar\omega} \prod_{j\in J(\bar\omega)} (1-\bar a_j/\bar a_{j+1}) \\
&\quad - (a_1/a_2) \sum_{\substack{\bar\omega\in\{0,1\}^{k-1}\\ \bar\omega_1=1}} (\bar a/\bar b)^{\bar\omega} \prod_{j\in J(\bar\omega)} (1-\bar a_j/\bar a_{j+1}) \\
&= (1+a_1/b_1) \sum_{\bar\omega\in\{0,1\}^{k-1}} (\bar a/\bar b)^{\bar\omega} \prod_{j\in J(\bar\omega)} (1-\bar a_j/\bar a_{j+1}) \\
&\quad - (a_1/b_2) \sum_{\bar{\bar\omega}\in\{0,1\}^{k-2}} (\bar{\bar a}/\bar{\bar b})^{\bar{\bar\omega}} \prod_{j\in J(\bar{\bar\omega})} (1-\bar{\bar a}_j/\bar{\bar a}_{j+1}).
\end{aligned}$$

On the right of (3.4) we split the sum over τ according to

$$\sum_{\tau\in\{0,1\}^k} = \sum_{\substack{\tau\in\{0,1\}^k\\ \tau_1=0}} + \sum_{\substack{\tau\in\{0,1\}^k\\ \tau_1=\tau_2=1}} + \sum_{\substack{\tau\in\{0,1\}^k\\ \tau_1=1,\ \tau_2=0}} .$$

Defining $\bar{\tau} \in \{0,1\}^{k-1}$ and $\bar{\bar{\tau}} \in \{0,1\}^{k-2}$ by $\bar{\tau} = (\tau_2, \dots, \tau_k)$ and $\bar{\bar{\tau}} = (\tau_3, \dots, \tau_k)$, this yields

$$\begin{aligned}
\mathrm{RHS}(3.4) = &\sum_{\bar{\tau}\in\{0,1\}^{k-1}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1-\bar{b}_j/\bar{b}_{j+1}) \\
&+ (a_1/b_1) \sum_{\substack{\bar{\tau}\in\{0,1\}^{k-1}\\ \bar{\tau}_1=1}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1-\bar{b}_j/\bar{b}_{j+1}) \\
&+ (a_1/b_1)(1-b_1/b_2) \sum_{\substack{\bar{\tau}\in\{0,1\}^{k-1}\\ \bar{\tau}_1=0}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1-\bar{b}_j/\bar{b}_{j+1}) \\
= &(1+a_1/b_1) \sum_{\bar{\tau}\in\{0,1\}^{k-1}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1-\bar{b}_j/\bar{b}_{j+1}) \\
&- (a_1/b_2) \sum_{\substack{\bar{\tau}\in\{0,1\}^{k-1}\\ \bar{\tau}_1=0}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1-\bar{b}_j/\bar{b}_{j+1}) \\
= &(1+a_1/b_1) \sum_{\bar{\tau}\in\{0,1\}^{k-1}} (\bar{a}/\bar{b})^{\bar{\tau}} \prod_{j\in J(\bar{\tau})} (1-\bar{b}_j/\bar{b}_{j+1}) \\
&- (a_1/b_2) \sum_{\bar{\bar{\tau}}\in\{0,1\}^{k-2}} (\bar{\bar{a}}/\bar{\bar{b}})^{\bar{\bar{\tau}}} \prod_{j\in J(\bar{\bar{\tau}})} (1-\bar{\bar{b}}_j/\bar{\bar{b}}_{j+1}).
\end{aligned}$$

By our induction hypothesis this equates with the previous expression for the left-hand side of (3.4), completing the proof.

4. The A_2 Rogers–Ramanujan identities

Let $(a;q)_0 = 1$, $(a;q)_n = \prod_{i=1}^n (1-aq^{i-1})$ and $(a_1,\dots,a_k;q)_n = (a_1;q)_n \cdots (a_k;q)_n$.

Proposition 4.1. *There holds*

$$\sum_{\lambda,\mu} \frac{a^{|\lambda|}b^{|\mu|}q^{(\lambda'|\lambda')+(\mu'|\mu')-(\lambda'|\mu')}}{(q;q)_{n-\ell(\lambda)}(q;q)_{m-\ell(\mu)}b_\lambda(q)b_\mu(q)} = \frac{(abq;q)_{n+m}}{(q,aq,abq;q)_n(q,bq,abq;q)_m}. \tag{4.1}$$

Proof. In Theorem 1.1 set $x_i = aq^i$ for $1 \le i \le n$, $x_i = 0$ for $i > n$, $y_j = bq^j$ for $1 \le j \le m$ and $y_j = 0$ for $j > m$. Using the homogeneity (2.4) and specialization (2.5), and noting that $2n(\lambda) + |\lambda| = (\lambda'|\lambda')$, gives (4.1). □

We remark that (4.1) is a bounded version of the A_2 case of the following identity for the A_n root system due to Hua [4] (and corrected in [3]):

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \frac{q^{\frac{1}{2}\sum_{i,j=1}^n C_{ij}(\lambda^{(i)\prime}|\lambda^{(j)\prime})}\prod_{i=1}^n a_i^{|\lambda^{(i)}|}}{\prod_{i=1}^n b_{\lambda^{(i)}}(q)} = \prod_{\alpha\in\Delta_+} \frac{1}{(a^\alpha q;q)_\infty}. \tag{4.2}$$

Here $C_{ij} = 2\delta_{i,i} - \delta_{i,j-1} - \delta_{i,j+1}$ is the (i,j) entry of the A_n Cartan matrix and Δ_+ is the set of positive roots of A_n, i.e., the set (of cardinality $\binom{n+1}{2}$) of roots of the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ with $1 \le i \le j \le n$, where $\alpha_1, \dots, \alpha_n$ are the simple roots of A_n. Furthermore, if $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ then $a^\alpha = a_i a_{i+1} \cdots a_j$.

For $M=(M_1,\dots,M_n)$ with M_i a non-negative integer, we define the following bounded analogue of the sum in (4.2):

$$R_M(a_1,\dots,a_n;q)=\sum_{\lambda^{(1)},\dots,\lambda^{(n)}}\frac{q^{\frac{1}{2}\sum_{i,j=1}^n C_{ij}(\lambda^{(i)\prime}|\lambda^{(j)\prime})}\prod_{i=1}^n a_i^{|\lambda^{(i)}|}}{\prod_{i=1}^n (q;q)_{M_i-\ell(\lambda^{(i)})}b_{\lambda^{(i)}}(q)}.$$

By construction $R_M(a_1,\dots,a_n;q)$ satisfies the following invariance property.

Lemma 4.1. *We have*

$$\sum_{r_1=0}^{M_1}\cdots\sum_{r_n=0}^{M_n}\frac{q^{\frac{1}{2}\sum_{i,j=1}^n C_{ij}r_ir_j}\prod_{i=1}^n a_i^{r_i}}{\prod_{i=1}^n (q;q)_{M_i-r_i}}R_r(a_1,\dots,a_n;q)=R_M(a_1,\dots,a_n;q).$$

Proof. Take the definition of R_M given above and replace each of $\lambda^{(1)},\dots,\lambda^{(n)}$ by its conjugate. Then introduce the non-negative integer r_i and the partition $\mu^{(i)}$ with largest part not exceeding r_i through $\lambda^{(i)}=(r_i,\mu_1^{(i)},\mu_2^{(i)},\dots)$. Since $b_{\lambda'}(q)=(q;q)_{r-\mu_1}b_{\mu'}(q)$ for $\lambda=(r,\mu_1,\mu_2,\dots)$ this implies the identity of the lemma after again replacing each of $\mu^{(1)},\dots,\mu^{(n)}$ by its conjugate. □

Next is the observation that the left-hand side of (4.1) corresponds to $R_{(n,m)}(a,b;q)$. Hence we may reformulate the A_2 instance of Lemma 4.1.

Theorem 4.1. *For M_1 and M_2 non-negative integers*

$$\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2}\frac{a^{r_1}b^{r_2}q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{M_1-r_1}(q;q)_{M_2-r_2}}\frac{(abq;q)_{r_1+r_2}}{(q,aq,abq;q)_{r_1}(q,bq,abq;q)_{r_2}}=\frac{(abq;q)_{M_1+M_2}}{(q,aq,abq;q)_{M_1}(q,bq,abq;q)_{M_2}}. \tag{4.3}$$

To see how this leads to the A_2 Rogers–Ramanujan identity (1.2) and its higher moduli generalizations, let k_1,k_2,k_3 be integers such that $k_1+k_2+k_3=0$. Making the substitutions

$$\begin{aligned} &r_1\to r_1-k_1-k_2, && a\to q^{k_2-k_3}, && M_1\to M_1-k_1-k_2,\\ &r_2\to r_2-k_1, && b\to q^{k_1-k_2}, && M_2\to M_2-k_1,\end{aligned}$$

in (4.3), we obtain

$$\sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2}\frac{q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{M_1-r_1}(q;q)_{M_2-r_2}(q;q)^2_{r_1+r_2}}\begin{bmatrix}r_1+r_2\\r_1+k_1\end{bmatrix}\begin{bmatrix}r_1+r_2\\r_1+k_2\end{bmatrix}\begin{bmatrix}r_1+r_2\\r_1+k_3\end{bmatrix}=\frac{q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2)}}{(q)^2_{M_1+M_2}}\begin{bmatrix}M_1+M_2\\M_1+k_1\end{bmatrix}\begin{bmatrix}M_1+M_2\\M_1+k_2\end{bmatrix}\begin{bmatrix}M_1+M_2\\M_1+k_3\end{bmatrix}, \tag{4.4}$$

where

$$\begin{bmatrix}n\\m\end{bmatrix}=\begin{bmatrix}n\\m\end{bmatrix}_q=\begin{cases}\dfrac{(q^{n-m+1};q)_m}{(q;q)_m} & \text{for } m\geq 0,\\ 0 & \text{otherwise}\end{cases}$$

is a q-binomial coefficient. The identity (4.4) which is equivalent to the type-II A_2 Bailey lemma of [1, Theorem 4.3].

The idea is now to apply (4.4) to the A_2 Euler identity [1, Equation (5.15)]

$$\sum_{k_1+k_2+k_3=0} q^{\frac{3}{2}(k_1^2+k_2^2+k_3^2)} \times \sum_{w\in S_3} \epsilon(w) \prod_{i=1}^{3} q^{\frac{1}{2}(3k_i-w_i+i)^2-w_ik_i} \begin{bmatrix} M_1+M_2 \\ M_1+3k_i-w_i+i \end{bmatrix} = \begin{bmatrix} M_1+M_2 \\ M_1 \end{bmatrix}, \tag{4.5}$$

where $w\in S_3$ is a permutation of $(1,2,3)$ and $\epsilon(w)$ denotes the signature of w.

Replacing M_1, M_2 by r_1, r_2 in (4.5), then multiplying both sides by

$$\frac{q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{M_1-r_1}(q;q)_{M_2-r_2}(q;q)_{r_1+r_2}^2},$$

and finally summing over r_1 and r_2 using (4.4) (with $k_i \to 3k_i - w_i + i$), yields

$$\sum_{k_1+k_2+k_3=0} q^{\frac{3}{2}(k_1^2+k_2^2+k_3^2)} \sum_{w\in S_3} \epsilon(w) \prod_{i=1}^{3} q^{(3k_i-w_i+i)^2-w_ik_i} \begin{bmatrix} M_1+M_2 \\ M_1+3k_i-w_i+i \end{bmatrix} = \sum_{r_1=0}^{M_1}\sum_{r_2=0}^{M_2} \frac{q^{r_1^2-r_1r_2+r_2^2}(q;q)_{M_1+M_2}^2}{(q;q)_{M_1-r_1}(q;q)_{M_2-r_2}(q;q)_{r_1}(q;q)_{r_2}(q;q)_{r_1+r_2}}. \tag{4.6}$$

Letting M_1 and M_2 tend to infinity, and using the Vandermonde determinant

$$\sum_{w\in S_3} \epsilon(w) \prod_{i=1}^{3} x_i^{i-w_i} = \prod_{1\le i<j\le 3} (1-x_jx_i^{-1})$$

with $x_i \to q^{7k_i+2i}$, gives

$$\frac{1}{(q;q)_\infty^3} \sum_{k_1+k_2+k_3=0} q^{\frac{21}{2}(k_1^2+k_2^2+k_3^2)-k_1-2k_2-3k_3} \times (1-q^{7(k_2-k_1)+2})(1-q^{7(k_3-k_2)+2})(1-q^{7(k_3-k_1)+4}) = \sum_{r_1,r_2=0}^{\infty} \frac{q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{r_1}(q;q)_{r_2}(q;q)_{r_1+r_2}}.$$

Finally, by the A_2 Macdonald identity [8]

$$\sum_{k_1+k_2+k_3=0} \prod_{i=1}^{3} x_i^{3k_i} q^{\frac{3}{2}k_i^2-ik_i} \prod_{1\le i<j\le 3} (1-x_jx_i^{-1}q^{k_j-k_i}) = (q;q)_\infty^2 \prod_{1\le i<j\le 3} (x_i^{-1}x_j, qx_ix_j^{-1};q)_\infty$$

with $q\to q^7$ and $x_i \to q^{2i}$ this becomes

$$\sum_{r_1,r_2=0}^{\infty} \frac{q^{r_1^2-r_1r_2+r_2^2}}{(q;q)_{r_1}(q;q)_{r_2}(q;q)_{r_1+r_2}} = \frac{(q^2,q^2,q^3,q^4,q^5,q^5,q^7,q^7;q^7)_\infty}{(q;q)_\infty^3}.$$

This result is easily recognized as the A_2 Rogers–Ramanujan identity (1.2).

The identity (4.6) can be further iterated using (4.4). Doing so and repeating the above calculations (requiring the Vandermonde determinant with $x_i \to q^{(3n+1)k_i+ni}$

and the Macdonald identity with $q \to q^{3n+1}$ and $x_i \to q^{ni}$) yields the following A_2 Rogers–Ramanujan-type identity for modulus $3n+1$ [1, Theorem 5.1; $i = k$]:

$$\sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu)\leq n-1}} \frac{q^{(\lambda|\lambda)+(\mu|\mu)-(\lambda|\mu)}}{b_{\lambda'}(q)b_{\mu'}(q)(q;q)_{\lambda_{n-1}+\mu_{n-1}}} = \frac{(q^n, q^n, q^{n+1}, q^{2n}, q^{2n+1}, q^{2n+1}, q^{3n+1}, q^{3n+1}; q^{3n+1})_\infty}{(q;q)^3_\infty}.$$

In the large n limit ones recovers the A_2 case of Hua's identity (4.2) with $a_1 = a_2 = 1$.

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