The Alternating Sign Matrix Polytope

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- entries $\in \{0, 1, -1\}$
- the sum of the entries in each row and column equals 1
- nonzero entries in each row and column alternate in sign

Examples of ASMs

- $n = 3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Examples of ASMs

- $n = 4$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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Thus a polytope can be specified by a set of points or by a set of linear inequalities.

Examples of polytopes

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Note that the permutohedron is the image of the Birkhoff polytope under the projection $\phi_z : \mathbb{R}^{n^2} \rightarrow \mathbb{R}^n$ defined by $\phi_z(X) = zX$.

Definition of ASM_n

The alternating sign matrix polytope ASM_n is defined as the convex hull of n-by-n alternating sign matrices.

Quick comparison of ASM_n and B_n

	B_n	ASM_n
Dimension	$(n-1)^2$	$(n-1)^2$
Inequality	rows and columns sum to 1	
Description	entries ≥ 0	partial sums ≥ 0
Vertices	$n!$	$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$
Facets	n^2	$4[(n-2)^2+1]$

Inequality description of B_n

Theorem (Birkhoff–von Neumann). *B_n consists of all n-by-n real matrices X satisfying:*

$$x_{ij} \geq 0, \qquad \forall\, 1 \leq i, j \leq n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \qquad \forall\, 1 \leq j \leq n$$

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Such matrices are called (nonnegative) doubly stochastic matrices.

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$$\begin{pmatrix} .5 & .2 & .3 \\ 0 & .4 & .6 \\ .5 & .4 & .1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} .7 & 0 & .3 \\ 0 & .4 & .6 \\ .3 & .6 & .1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} .1 & .6 & .3 \\ 0 & .4 & .6 \\ .9 & 0 & .1 \end{pmatrix}$$

Inequality description of ASM_n

Theorem (S.). *ASM_n consists of all n-by-n real matrices X with:*

$$0 \le \sum_{i=1}^{i'} x_{ij} \le 1, \qquad \forall\, 1 \le i' \le n$$

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Inequality description of ASM_n

$$\begin{pmatrix} 0 & .4 & .5 & .1 & 0 \\ .4 & -.4 & .5 & 0 & .5 \\ .6 & .4 & -.3 & -.1 & .4 \\ 0 & .3 & -.3 & .9 & .1 \\ 0 & .3 & .6 & .1 & 0 \end{pmatrix}$$

Inequality description of ASM_n

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Row partial sums

$$\begin{pmatrix} 0 & .4 & .9 & 1 & 1 \\ .4 & 0 & .5 & .5 & 1 \\ .6 & 1 & .7 & .6 & 1 \\ 0 & .3 & 0 & .9 & 1 \\ 0 & .3 & .9 & 1 & 1 \end{pmatrix}$$

Column partial sums

$$\begin{pmatrix} 0 & .4 & .5 & .1 & 0 \\ .4 & 0 & 1 & .1 & .5 \\ 1 & .4 & .7 & 0 & .9 \\ 1 & .7 & .4 & .9 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

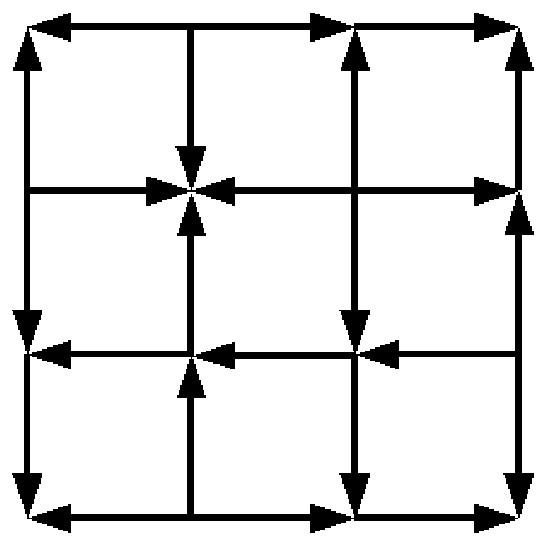
Projection to the permutohedron

Theorem (S.). *Let $z = (z_1, z_2, \ldots, z_n)$ be a strictly increasing (or decreasing) vector and X an n-by-n ASM. Then $\phi_z(X) = zX$ is in the convex hull of the permutations of $\{z_1, z_2, z_3, \ldots, z_n\}$ so that $\phi_z(ASM_n) = P_z$. That is, matrix multiplication by a strictly monotone vector z projects ASM_n onto P_z.*

Thus under this projection, ASM_n and B_n are mapped to the same permutohedron.

Simple flow grids

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



Facets and vertices

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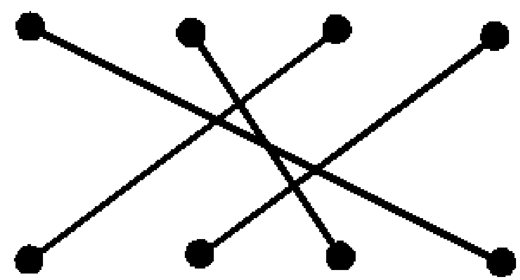
- The vertices of ASM_n are the alternating sign matrices.
- ASM_n has $4[(n-2)^2+1]$ facets.
- The number of facets of ASM_n on which an ASM A lies is given by

$$\begin{cases} 2(n-1)(n-2)+2, & \text{if } A \text{ has two corner 1's} \\ 2(n-1)(n-2)+1, & \text{if } A \text{ has one corner 1} \\ 2(n-1)(n-2), & \text{otherwise} \end{cases}$$

Face lattice of B_n

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



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Corollary (Billera and Sarangarajan, 1994)**.** *The graphs representing edges of* B_n *are the elementary subgraphs of* $K_{n,n}$ *which have exactly one cycle.*

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An *elementary flow grid* G is a directed graph on an n-by-n array of vertices such that the edge set of G is the union of the edge sets of simple flow grids.

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Theorem (S_). *The face lattice of* ASM_n *is isomorphic to the lattice of all* $n \times n$ *elementary flow grids ordered by inclusion.*

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Given an elementary flow grid G, define a *doubly directed region* as a collection of cells in G completely bounded by double directed edges but containing no double directed edges in the interior.

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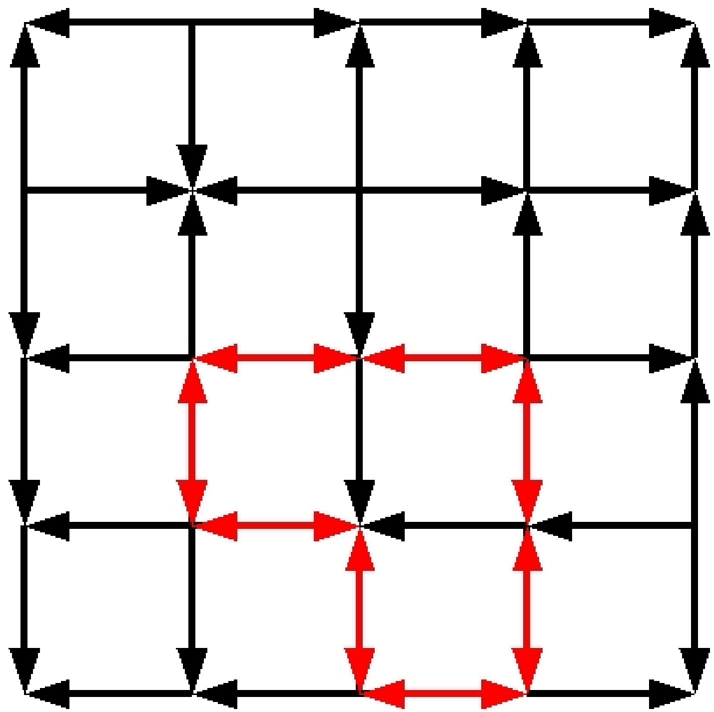
Corollary (S₋). *The m–dimensional faces of ASM_n are represented by the elementary flow grids in which the number of doubly directed regions equals m. In particular, the edges of ASM_n are represented by elementary flow grids containing exactly one cycle (which is traversable in both directions).*

Face lattice of ASM_n

The elementary flow grid corresponding to an edge of ASM_5



Face lattice of ASM_n

The elementary flow grid corresponding to a 3-dimensional face of ASM_5