Unbounded Product-Form Petri Nets

Patricia Bouyer-Decitre[1], Serge Haddad[1] & Vincent Jugé[1,2]

1: LSV (CNRS, ENS Paris-Saclay & Inria) — 2: LIGM (Univ. Paris-Est Marne-la-Vallée)

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In the last decades, interest in quantitative systems rose dramatically. *Efficient* quantitative analysis of *high-level,* expressive systems with *infinite* underlying state space was *a major* difficulty.

From the performance evaluation *community* arose the idea of *studying* "product forms"...

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Discrete-time & continuous-time Markov chains

Discrete-time Markov chain

Stochastic dynamical system evolving at integer time-points: $(X_n)_{n \in \mathbb{Z}_{\geqslant 0}}$

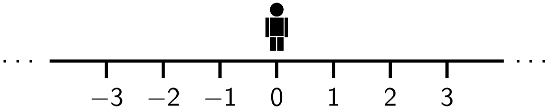
Example: Random walk

Discrete-time & continuous-time Markov chains

Discrete-time Markov chain

Stochastic dynamical system evolving at integer time-points: $(X_n)_{n \in \mathbb{Z}_{\geqslant 0}}$

Example: Random walk — $(X_n)_{n \geqslant 0} = 0$



Discrete-time & continuous-time Markov chains

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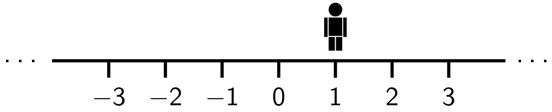
Example: Random walk — $(X_n)_{n \geqslant 0} = 0, 1$



Discrete-time & continuous-time Markov chains

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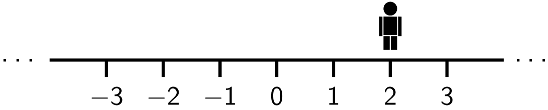
Example: Random walk — $(X_n)_{n \geqslant 0} = 0, 1, 2$



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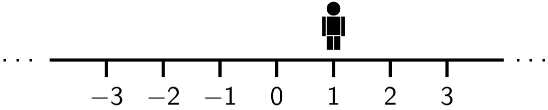

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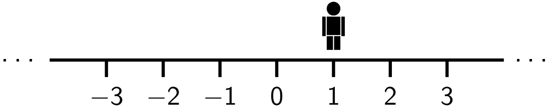
Markov Property: $\{(X_k)_{k \geqslant n} \mid (X_k)_{k \leqslant n}\} = \{(X_k)_{k \geqslant n} \mid X_n\}$



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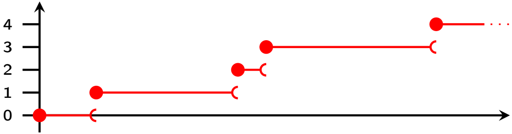

Continuous-time Markov chain

Random dynamical system evolving at **real** time-points: $(X_t)_{t \in \mathbb{R}_{\geqslant 0}}$

Example: Counting bus arrivals

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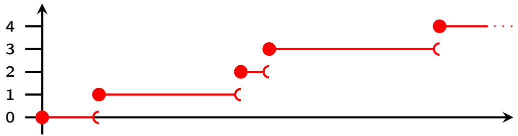

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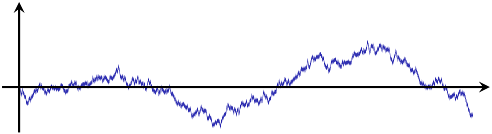

Continuous-time Markov chain ≠ Markov process

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Ergodic Markov chains

What does a Markov chain look like after a long time?

The Markov chain is **ergodic** if it has a **limit** probability measure $\mu = \lim_{t \to +\infty} X_t$.

Examples: Random walk on $\mathbb{Z}$: **not ergodic**

Random walk on $\mathbb{Z}/7\mathbb{Z}$: **ergodic**

Communication networks: ?

Ergodic Markov chains

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The Markov chain is **ergodic** if it has a **limit** probability measure $\mu = \lim_{t \to +\infty} X_t$: μ is the unique **invariant probability measure**.

Examples: Random walk on $\mathbb{Z}$: **not ergodic**

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Communication networks: ?

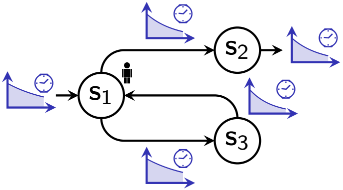
Our goal: Check ergodicity & compute μ **efficiently** in relevant cases.

(undecidable in general)

Product-form Markov chains

Fixed-rate queuing systems (≈ Jackson networks)

Clients follow edges when fixed-rate **exponential clocks** ring.



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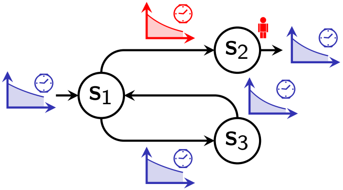

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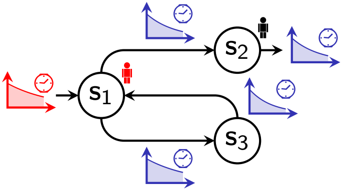

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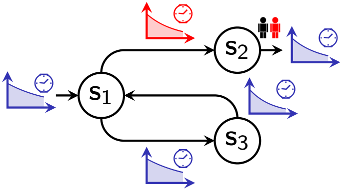

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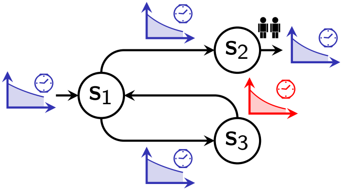

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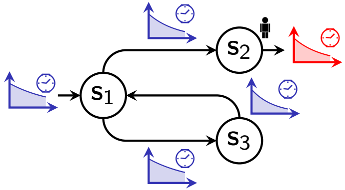

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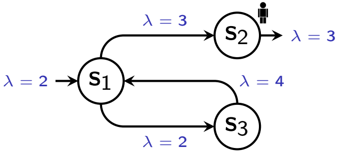

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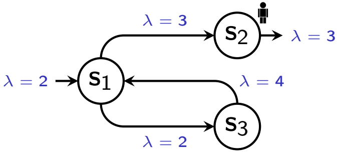
Clients follow edges when fixed-rate **exponential clocks** ring.
Such systems have **product-form** invariant measures.



$\mathbf{n}_i = \#$ 🧍 in $\mathbf{s}_i$

$$\mu(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) = \mathbf{K}\mathbf{p}_1^{\mathbf{n}_1}\mathbf{p}_2^{\mathbf{n}_2}\mathbf{p}_3^{\mathbf{n}_3}$$

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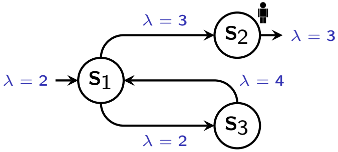


$\mathbf{n}_i = \#$ clients in $\mathbf{s}_i$ **Ergodic** Markov chain!

$$\mu(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) = \mathbf{K}\mathbf{p}_1^{\mathbf{n}_1}\mathbf{p}_2^{\mathbf{n}_2}\mathbf{p}_3^{\mathbf{n}_3}$$

$$\mathbf{p}_1 = \mathbf{p}_2 = 2/3,\ \mathbf{p}_3 = 1/3,\ \mathbf{K} = 27/2$$

Product-form invariant measures

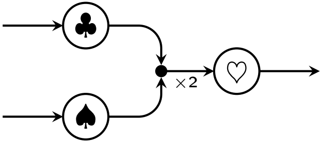
Linear algebra gives us the coefficients $\mathbf{p}_i$ in polynomial time.
It remains to compute the normalising constant $\mathbf{K}$ without enumerating the reachability set. (easy in queuing systems, hard in general)

Beyond queuing systems

What about concurrency?

What if people interact with each other?

Example: Marrying people



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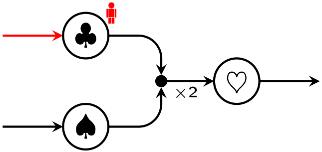

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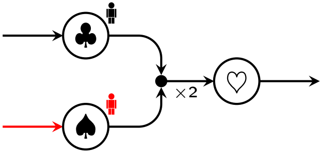

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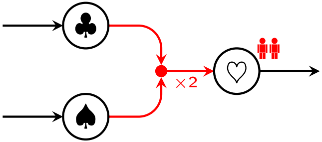

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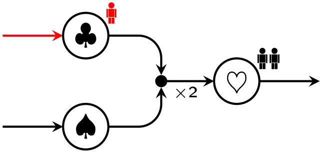

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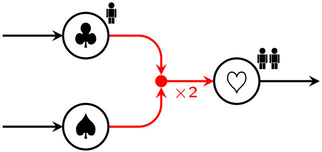

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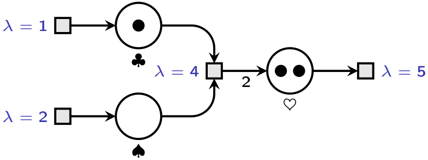
Use **stochastic Petri nets**!



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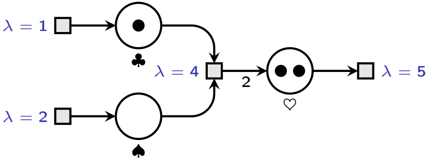

Next step: Investigate qualitative & quantitative properties, e.g.:

- boundedness,
- liveness,
- ergodicity,
- reachability,
- coverability,
- invariant measure. . .

Stochastic Petri nets & Π^2-nets

In spite of their greater complexity...

Some classes of stochastic Petri nets are product form!

- Coleman et al.'s condition (depends on firing rates)
- Π^2-nets (independent of firing rates)

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Complexity results in Π^2-nets

- Checking that a net is a Π^2-net: **PTime** ☺
- Coverability: **ExpSpace-hard** ☹
- Liveness: **PSpace-hard** ☹ (even in safe nets)
- Reachability: **PSpace-hard** ☹ (even in safe nets)

Efficiently computing normalising constants seems out of reach!

Π^2-nets & closed Π^3-nets ($\approx$ nested queuing systems)

Π^2-nets are too general: let us focus on (closed) Π^3-nets!

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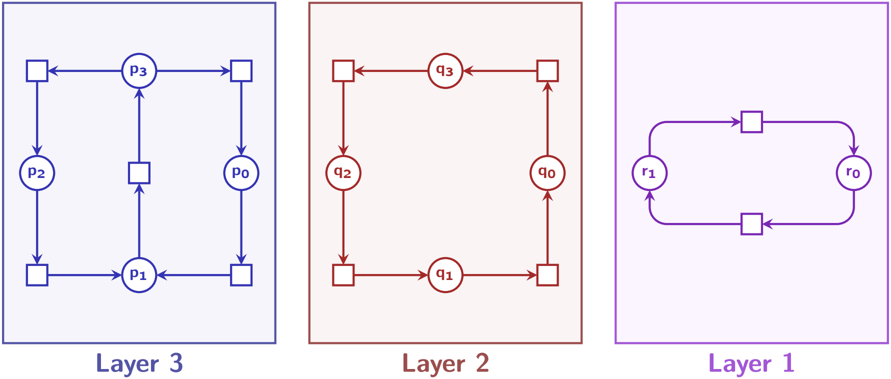

- Layers = Strongly connected finite-state machines

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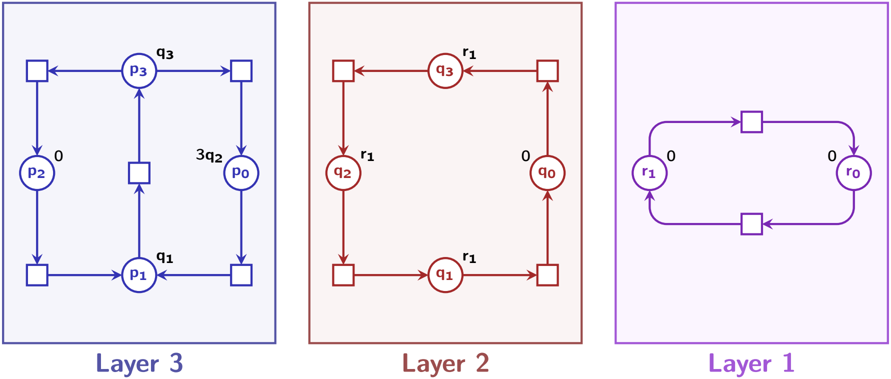

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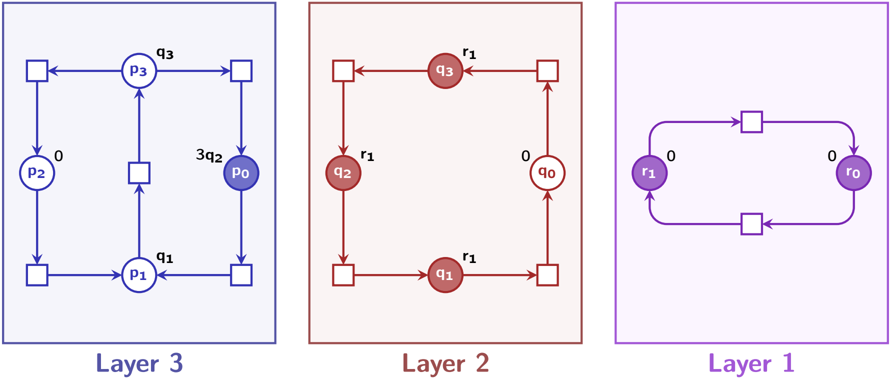

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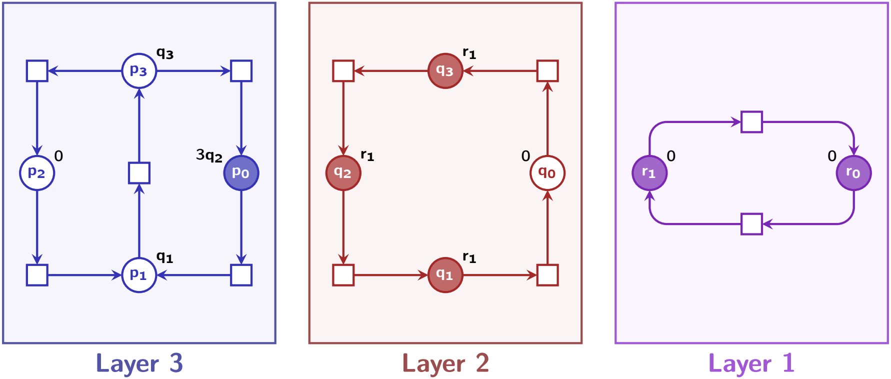

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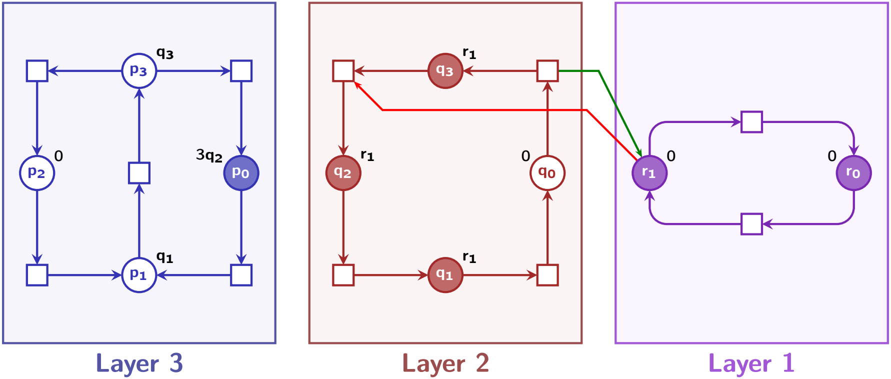

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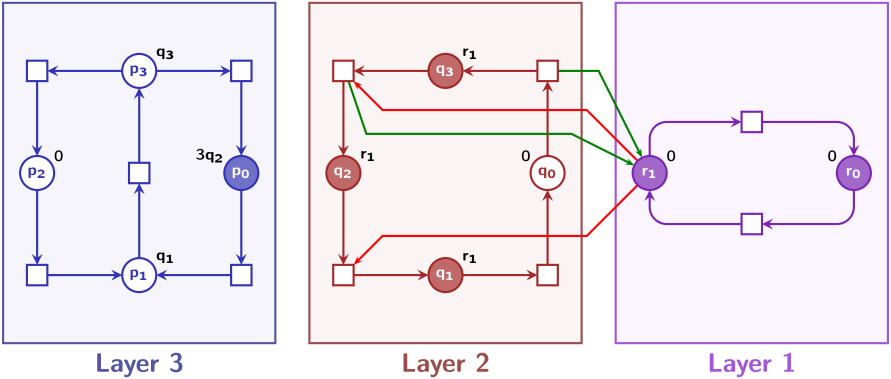

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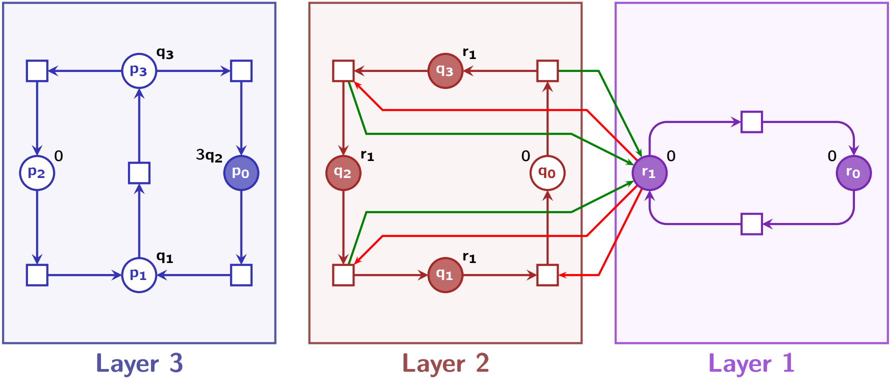

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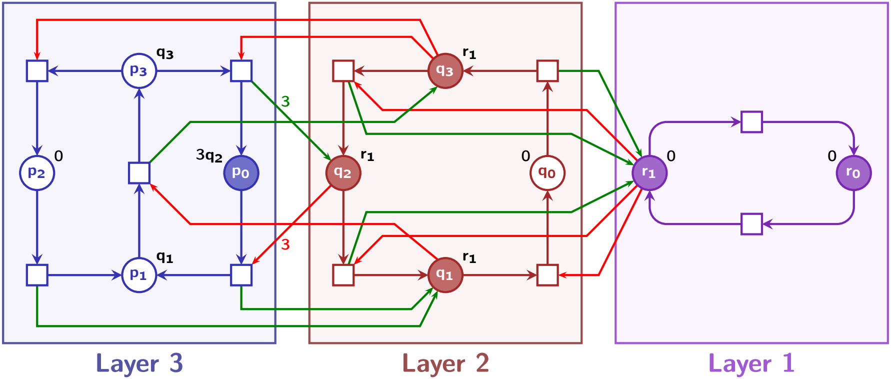

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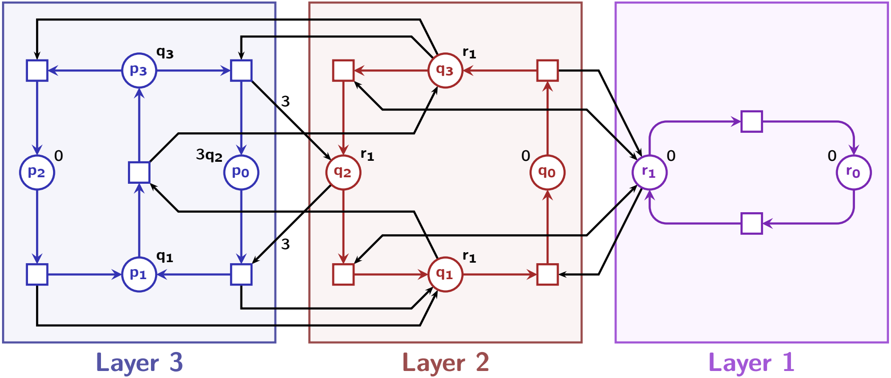
Closed Π^3-nets & open Π^3-nets

Closed Π^3-nets cannot model open systems: can we handle them?



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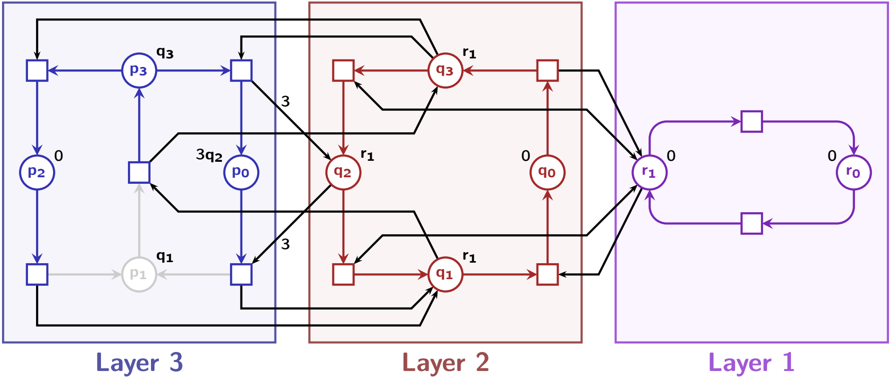

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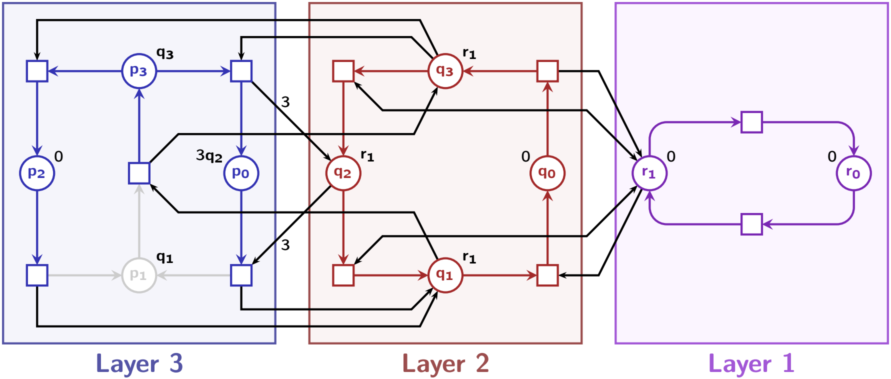

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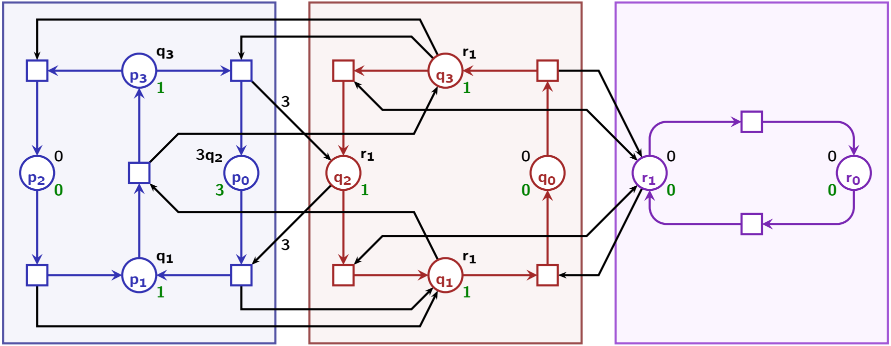
1-layer open/closed Π^3-nets = open/closed (connected) Jackson networks

Qualitative properties in (closed or open) Π^3-Nets

- Liveness: linear constraints — involves **potential**



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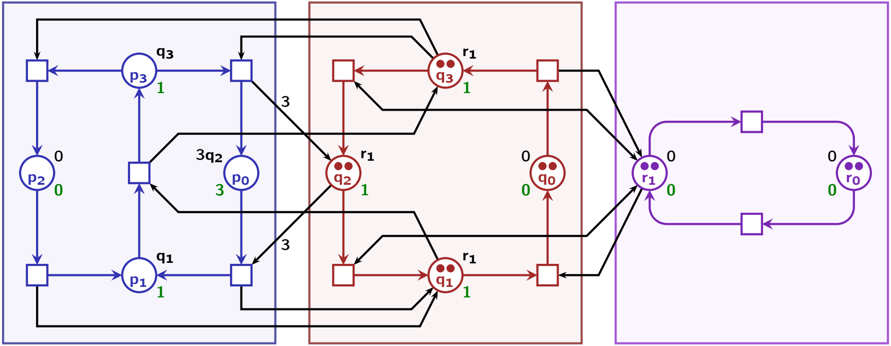

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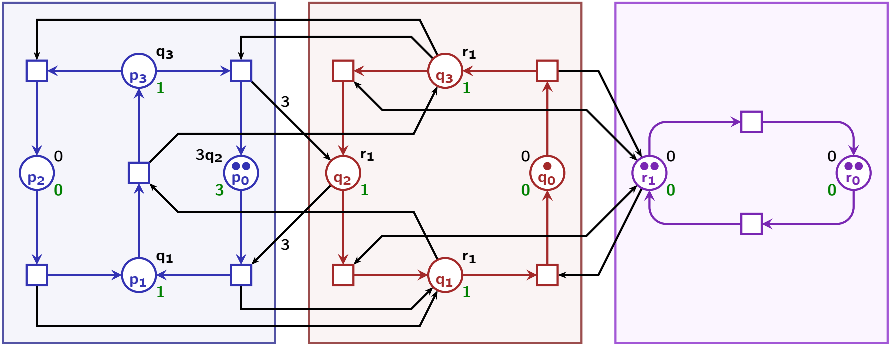

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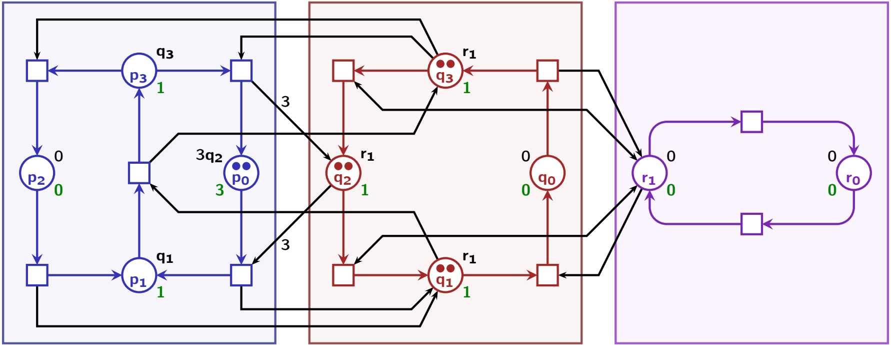

Not live ✗ (Layer 2 is dead)

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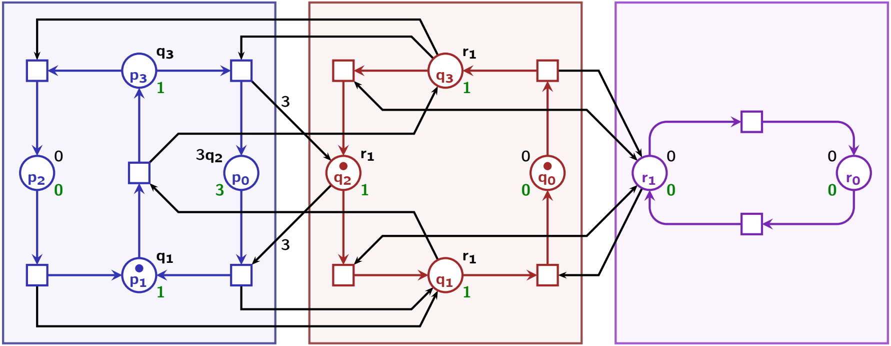

Live ✓

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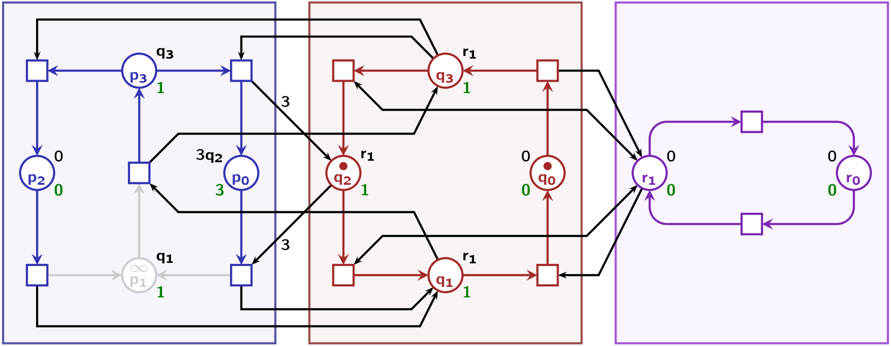

Live ✓

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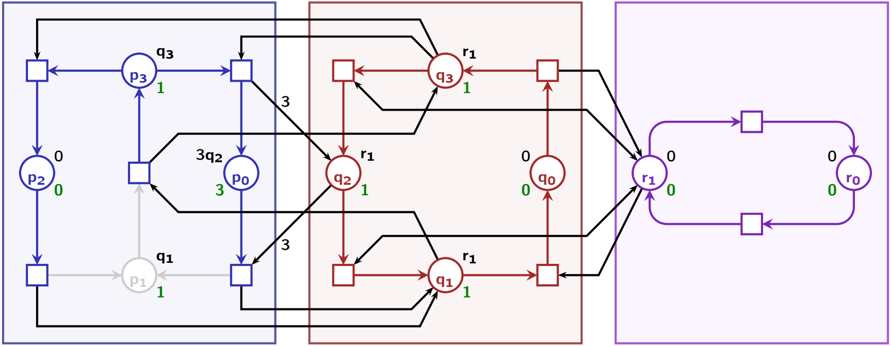

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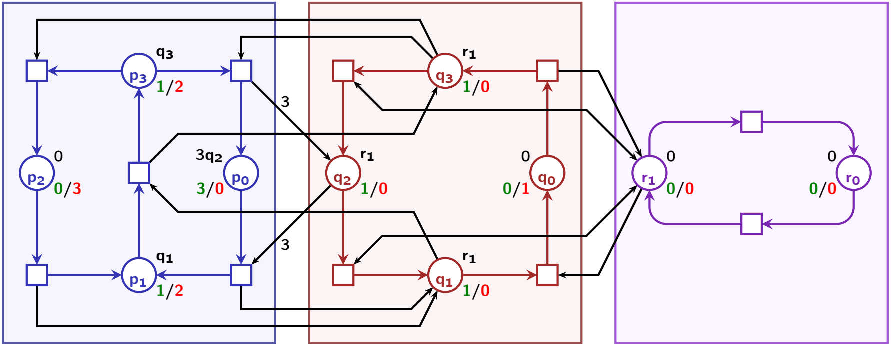

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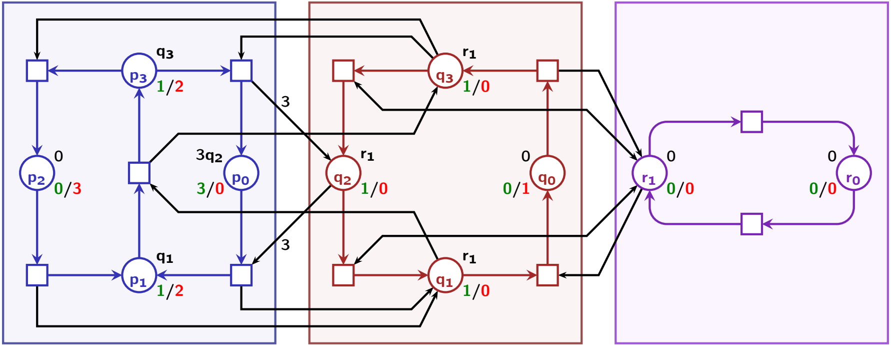

$\sum_i \mathbf{p}_i = \mathbf{C}_3$ $\sum_i \mathbf{q}_i + 2\mathbf{p}_1 + 3\mathbf{p}_2 + 2\mathbf{p}_3 = \mathbf{C}_2$ $\sum_i \mathbf{r}_i + \mathbf{q}_0 = \mathbf{C}_1$

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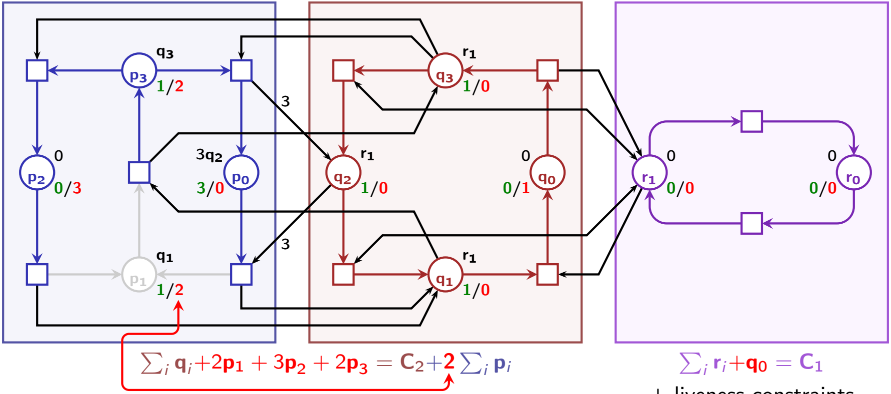

$\sum_i q_i + 2p_1 + 3p_2 + 2p_3 = C_2 + 2\sum_i p_i$

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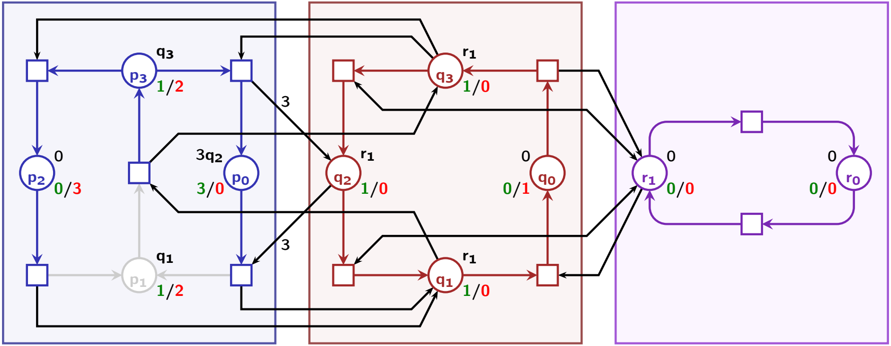

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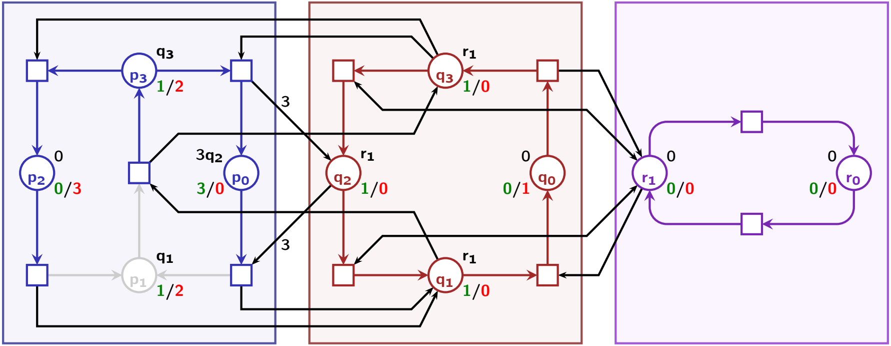
- Coverability in **live** nets: **PTime** ☺ (NP hard in general ☹)
- Boundedness in **live** nets: **PTime** ☺ (co-NP hard in general ☹)



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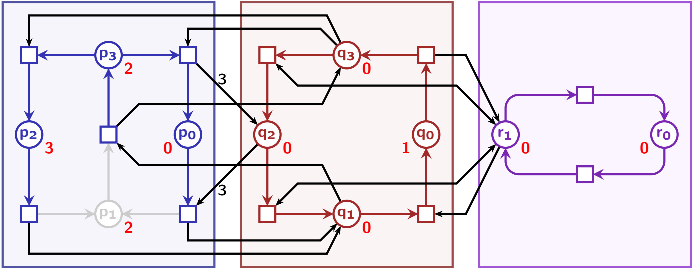

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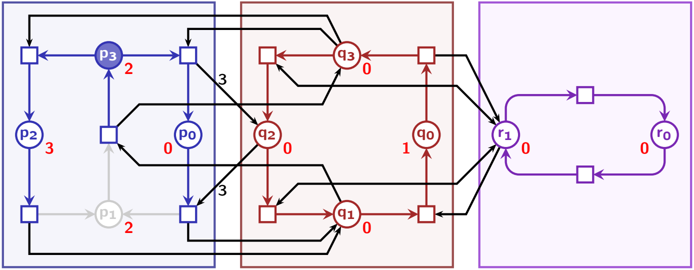

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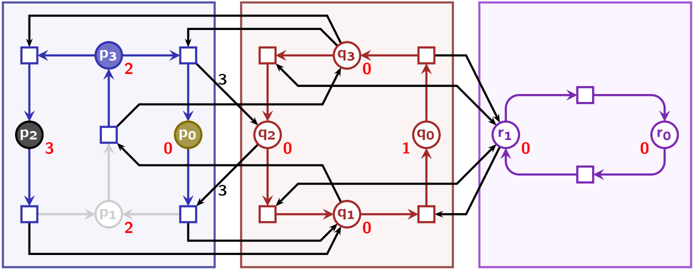

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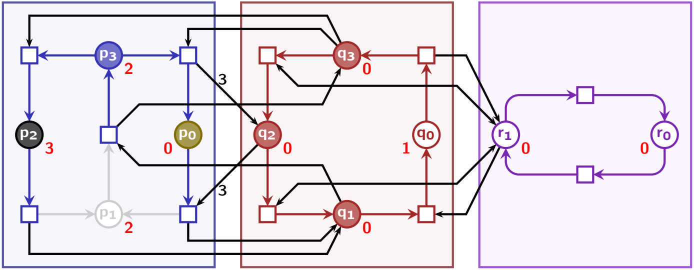

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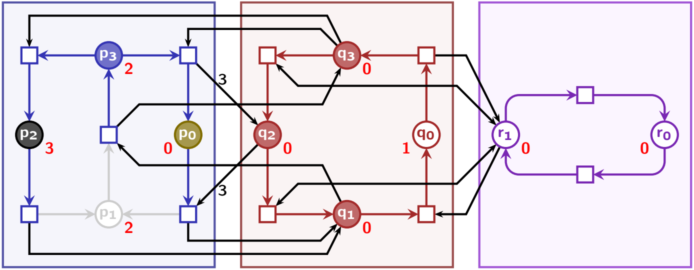

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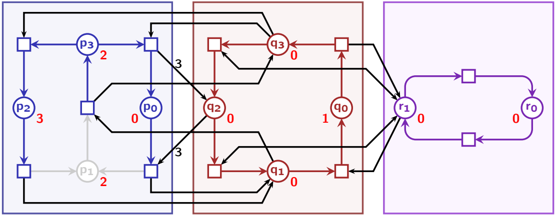
$$\mu = \mathsf{K} \prod_{\text{place } p} \mu_p^{\text{tokens in } p}$$



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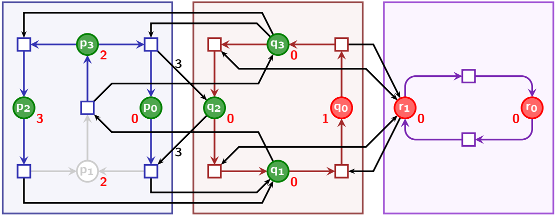

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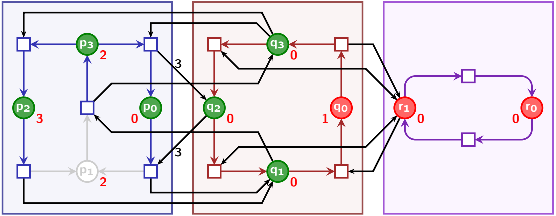

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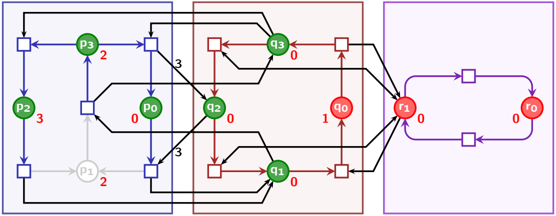

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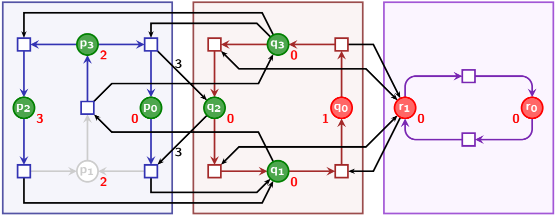

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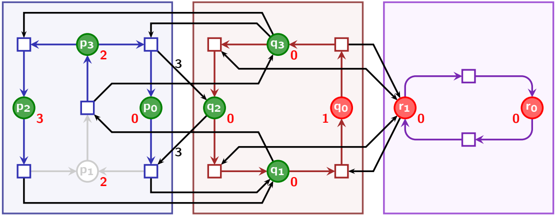

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Theorem (Bouyer, Haddad & Jugé 2017)

One can check if a Π^3-net is ergodic in polynomial time.
If yes, one can compute the invariant measure in pseudo-polynomial time.

Conclusion & Future Work

A **new** class of stochastic Petri nets: (open) Π^3-nets

- allows handling resources in open queuing systems;
- product-form Petri nets;
- efficient characterisation of live Π^3-nets and
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A short bibliography

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