Finite bisimulations for dynamical systems with overlapping trajectories

Béatrice Bérard[1], Patricia Bouyer[2,3] & Vincent Jugé[4]

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07/09/2018

Contents

Hybrid systems

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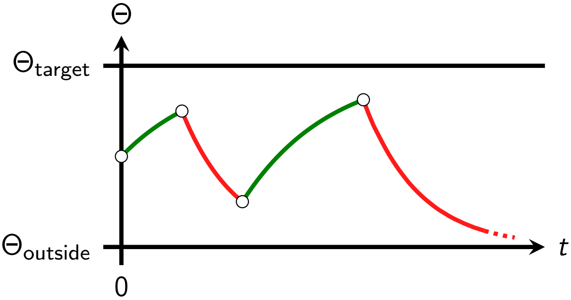

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1. Heater **on**: $\mathrm{d}\Theta/\mathrm{d}t = \alpha(\Theta_{\text{target}} - \Theta)$
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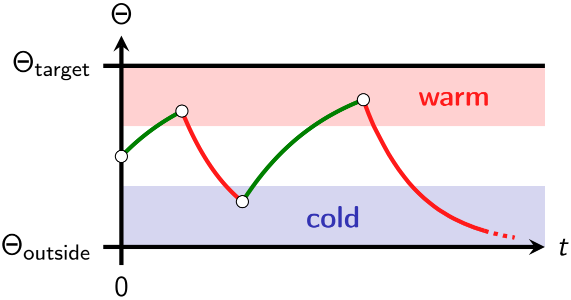

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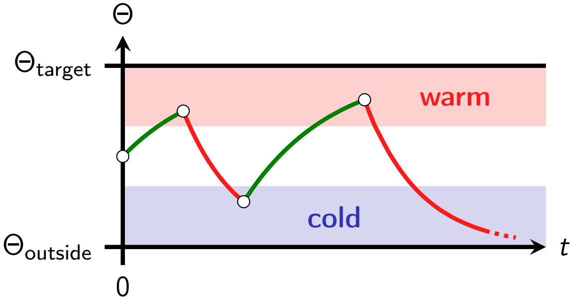

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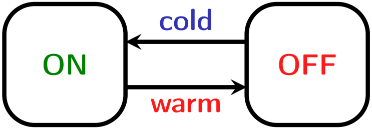
Duality between:

1. **Discrete** set of system modes
2. **Continuous** system evolution



How do hybrid systems behave?

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In this talk: Focus on the special case of dynamical systems

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2. Several possible **trajectories**
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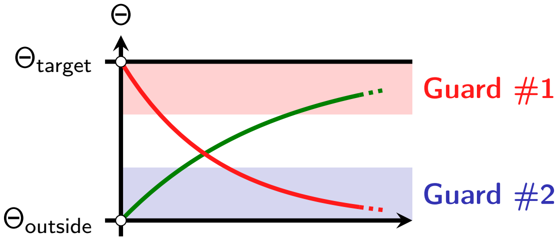

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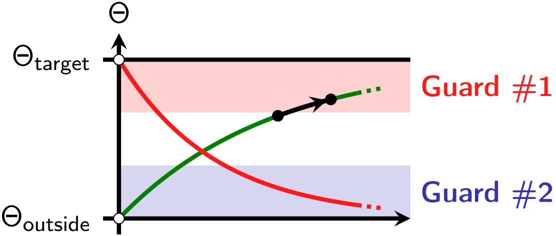

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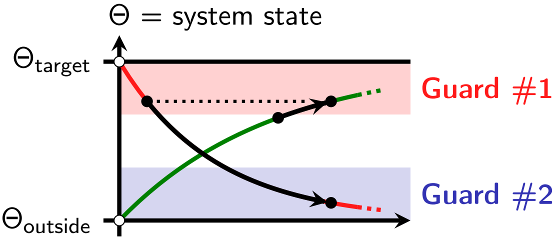

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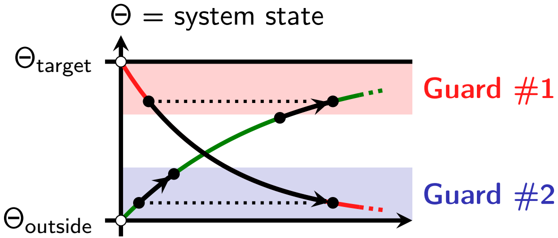

Bisimulation in dynamical systems – 1/2

Dynamical system: Labelled graph induced by

- **Trajectories**: Functions f : Time parameters $\rightarrow$ System states
  - **Underlying graph**: Edges $f(t) \rightarrow f(t')$ for all $t \leqslant t'$

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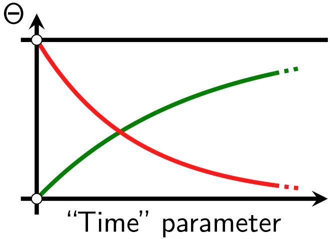

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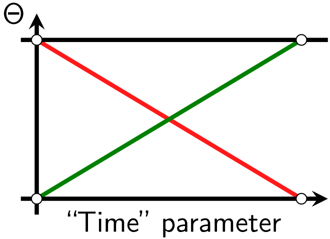

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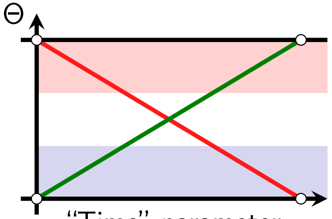
3 labels: **cold**, **normal** and **warm**



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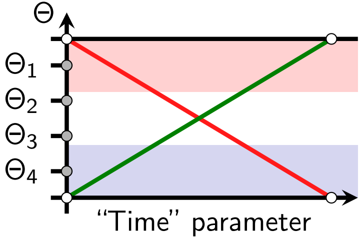

Bisimulation: Splitting states by possible behaviours

- $\Theta_i \approx \Theta_j \Leftrightarrow i = j$ or $\{i, j\} = \{2, 3\}$
- Induced partition: $\{\Theta_1\}, \{\Theta_2, \Theta_3\}, \{\Theta_4\}$

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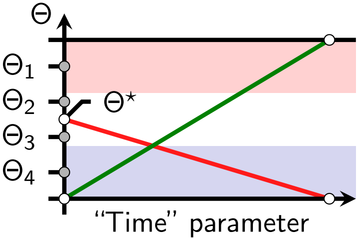

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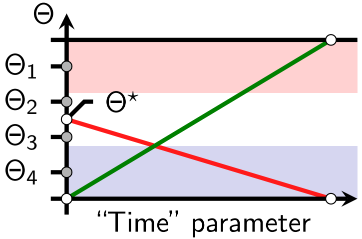

***k*-step Bisimulation**: Splitting states by possible k-step behaviours

- $\Theta_i \overset{0}{\approx} \Theta_j \Leftrightarrow i = j$ or $\{i, j\} = \{2, 3\}$ – $\Theta_i \overset{1}{\approx} \Theta_j \Leftrightarrow i = j$
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Bisimulation in dynamical systems – 2/2

Theorem (Folklore)

1. Bisimulation is **undecidable** in general
2. For all $k \geqslant 0$, k-step bisimulation is **decidable** (under mild assumptions)

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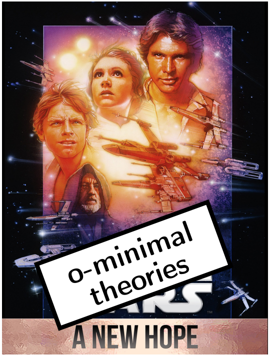

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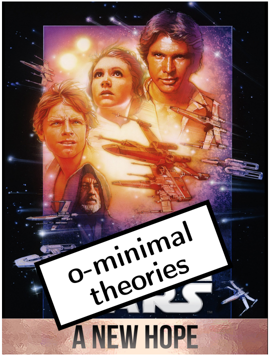
Theorem (Lafferriere, Pappas & Sastry, '00)

Bisimulation is **decidable** and induces a **finite** partition whenever:

1. Parameters $= \mathbb{R}$, System states $= \mathbb{R}^n$
2. Trajectories are
   - solutions of $\mathrm{d}\gamma(x,t)/\mathrm{d}t = F(\gamma(x,t))$
   - definable in an **o-minimal theory** of $\mathbb{R}$



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O-minimal structures: Definitions

Definition #1

A **First-Order** theory is **o-minimal** if:

- it concerns a **linearly ordered set** $(\mathcal{M}, \leqslant)$−with additional predicates.
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. . . and counter-examples: $(\mathbb{Q}, \leqslant, +, \times)$

$$x^2 \leqslant 2 \Leftrightarrow -\sqrt{2} \leqslant x \leqslant \sqrt{2}$$

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$$\exists z, x = z + z \Leftrightarrow x \text{ is even}$$

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$$(\exists t, t = \sin(t) = \sin(x)) \Leftrightarrow x \in \pi\mathbb{Z}$$

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Definition #2

A dynamical system is **o-minimal** if it is definable in an o-minimal theory:
Trajectory $\gamma_{\vec{p}}$ maps **time parameter** t to **system state** $\vec{z}$ iff $(\vec{p}, t, \vec{z}) \models \varphi$

O-minimal structures: Key properties

Key property #1 (Pillay & Steinhorn, '88)

Let $(\mathcal{M}, \leqslant, \dots)$ be o-minimal and $f : \mathcal{M} \to \mathcal{M}$ be definable. There exists a **finite** partition $(\mathcal{I}_1, \dots, \mathcal{I}_k)$ of $\mathcal{M}$ into **intervals** s.t., for all $j \leqslant k$:

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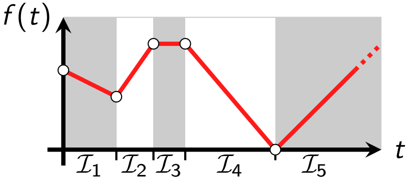

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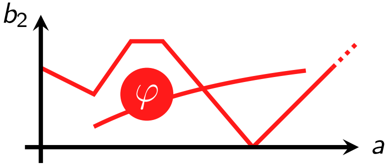
Key property #2 (Pillay & Steinhorn, '88)

Let φ be an ℓ-variable formula. There exists $\mathbf{N}_\varphi \in \mathbb{Z}$ s.t., for all $b_2, \ldots, b_\ell \in \mathcal{M}$, the set $\{a \in \mathcal{M} \mid (a, b_2, \ldots, b_\ell) \models \varphi\}$ is a union of $\mathbf{N}_\varphi$ intervals.



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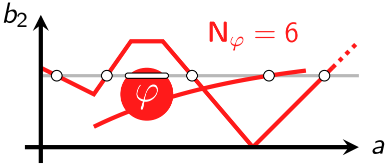

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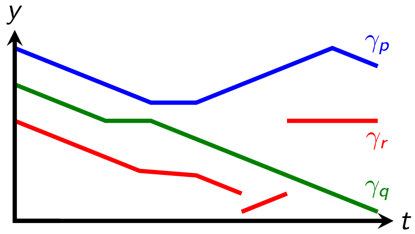

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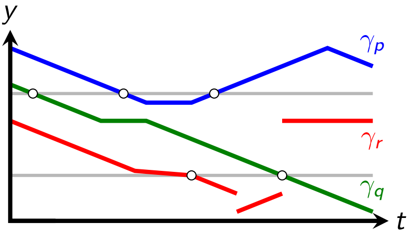

$\gamma_p(\mathcal{M}) \cap \gamma_q(\mathcal{M}) \neq \emptyset$
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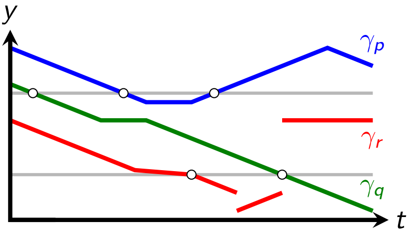

$p \sim q$

$p \not\sim r$

$q \sim r$

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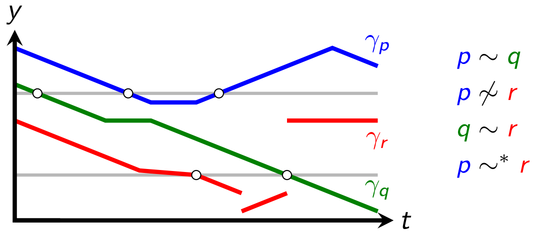
Theorem (Bérard, Bouyer & Jugé, '18)

In an o-minimal dynamical system such that:

- $V_1^*(x) \stackrel{\text{def}}{=\!=} \{x' \mid x \sim^* x'\}$ is **finite** for all x, (FINITE CROSSING)
  the bisimulation relation is **decidable**; (if the theory is decidable)



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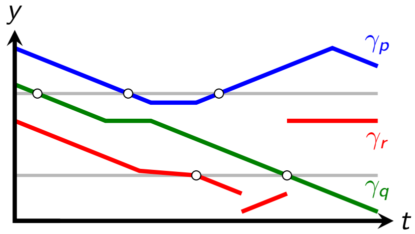
- the sizes $|V_1^*(x)|$ are **uniformly bounded**, (UNIFORM CROSSING)
  the bisimulation relation is **definable** and induces **finite** partition.



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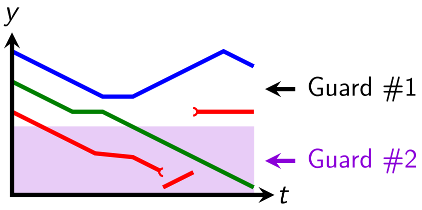

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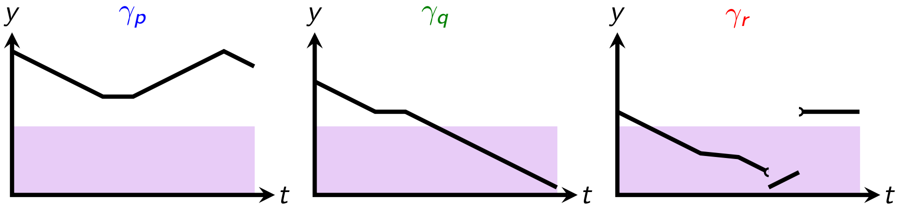

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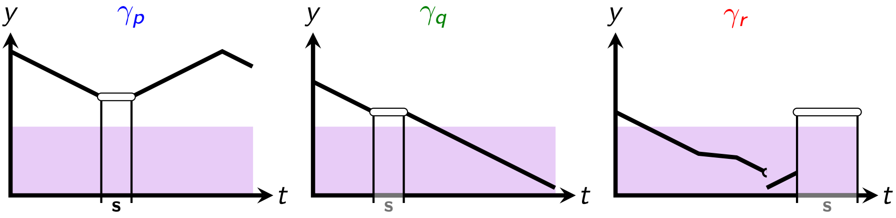

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- $|\mathcal{I}| = \infty$ and $|\gamma_x(\mathcal{I})| = 1$

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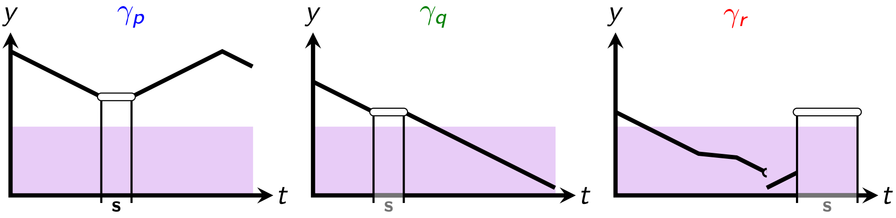
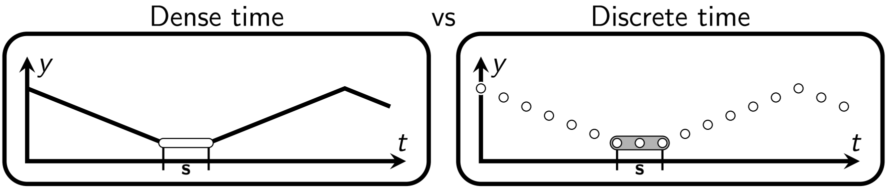
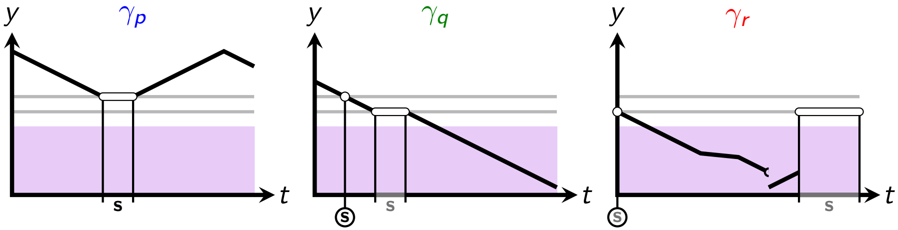


Staticity: $\mathcal{I}$ is ***x*-static** if

- $|\mathcal{I}| \geqslant 2$ and $|\gamma_x(\mathcal{I})| = 1$



Interval partition



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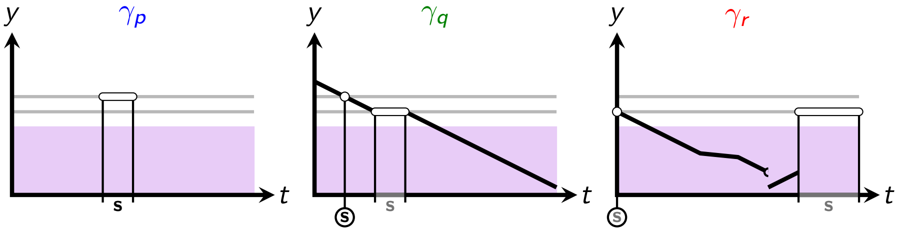

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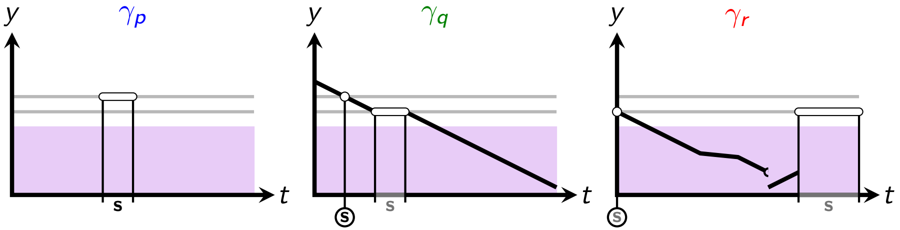
  one in each set $\{x_j\} \times \mathcal{I}_j$

Interval partition



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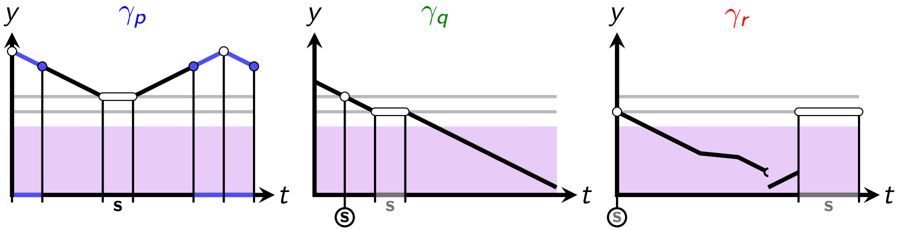

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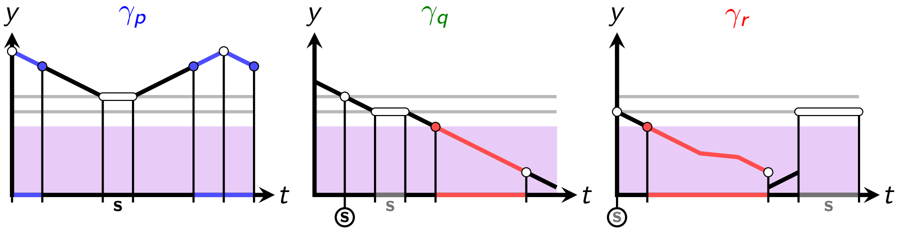

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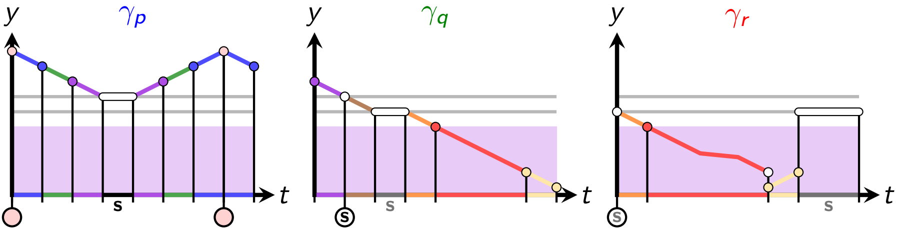

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Interval partition

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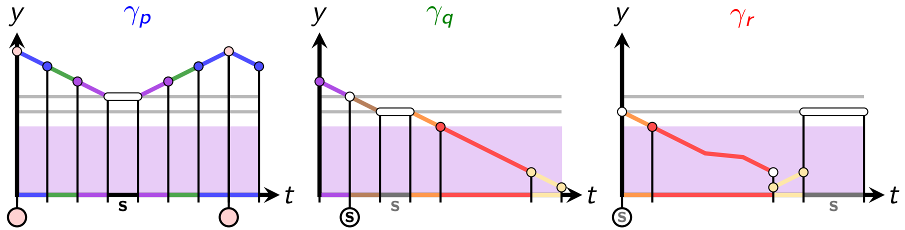

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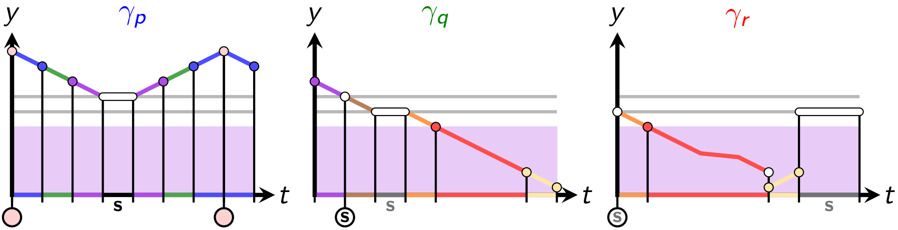

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Decomposition lemma

For all trajectories γ_x:

1. if $V_1(x) \stackrel{\text{def}}{=\!=} \{x' \mid x \sim x'\}$ is **finite**, then the time set is a **finite**, **disjoint**, **definable** union of **maximal** x-static and x-adaptable intervals;

Interval partition

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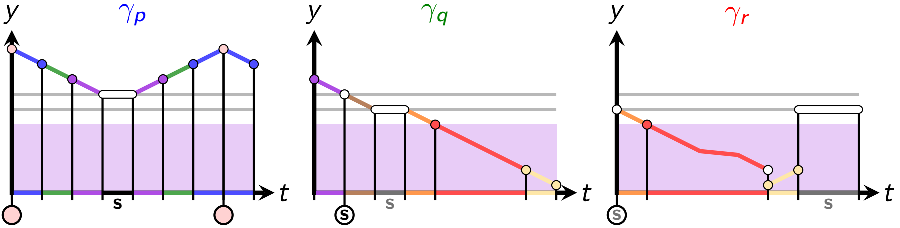

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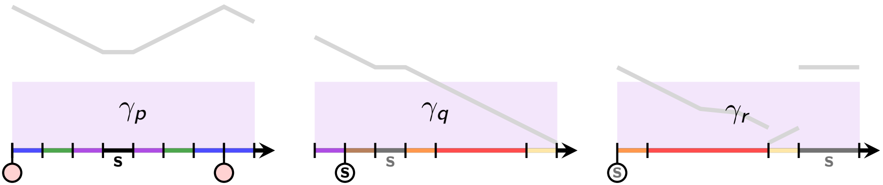
2. if $\mathcal{I}$ is x-static or x-adaptable, all states in $\gamma_x(\mathcal{I})$ are bisimilar.

Bisimulation graph



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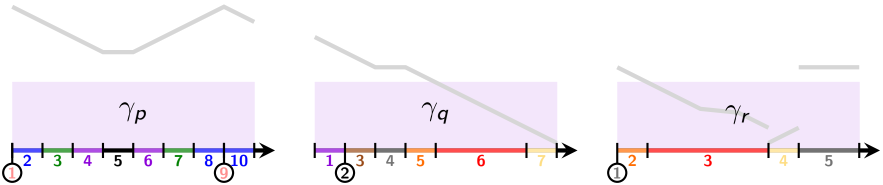

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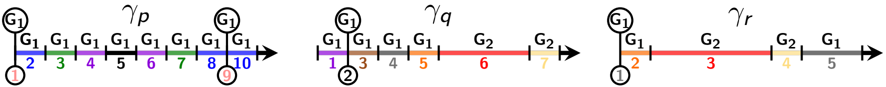

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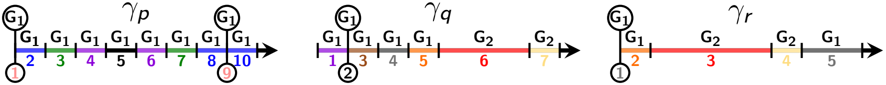
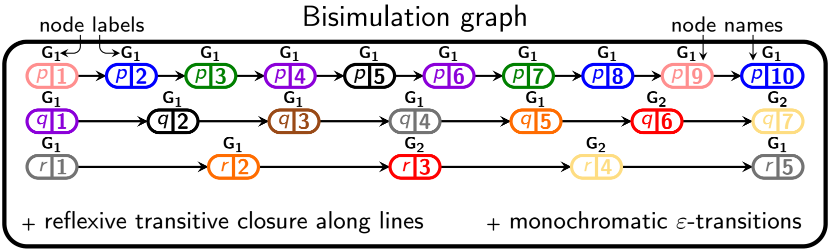
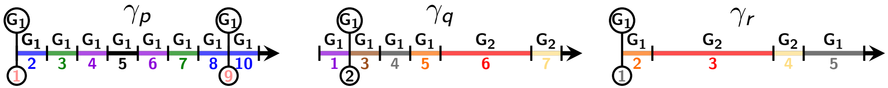




Bisimulation graph

The bismulation graph is sound & complete

Two **states** $\gamma_x(t)$ and $\gamma_{x'}(t')$ are (k-**step**) **bisimilar** iff there exist **intervals** $\mathcal{I} \ni t$ and $\mathcal{I}' \ni t'$ s.t. $(x, \mathcal{I})$ and $(x', \mathcal{I}')$ are (k-**step**) **bisimilar**.



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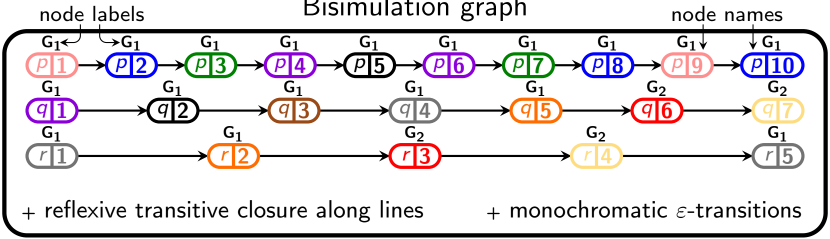

The result

Theorem (Bérard, Bouyer & Jugé, '18)

In an o-minimal dynamical system such that:

- $V_1^*(x) \stackrel{\text{def}}{=\!=} \{x' \mid x \sim^* x'\}$ is **finite** for all x, (FINITE CROSSING)
  the bisimulation relation is **decidable**; (if the theory is decidable)

Proof ideas:

- compute ***k*-step bisimulations** on $\Gamma(x) \cup \Gamma(x')$ for all $k \geqslant 0$, where
  $$\Gamma(x) = \{\gamma_{\hat{x}}(\hat{t}) \mid \hat{x} \sim^* x\};$$
- **finite** bisimulation graph fragment $\Rightarrow$ **convergent** refinement process:
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- the sizes $|V_1^*(x)|$ are **uniformly bounded**, (Uniform Crossing)
  the bisimulation relation is **definable** and induces **finite** partition.

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- some κ works for all x, x': **κ-step bisimulation $=$ bisimulation**.

Conclusion

Going further:

- Refine the definition of $V_1^*(x)$, Crossing conditions, ...;
- Add modes with restricted transitions.

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