On the Worst-Case Complexity of Timsort

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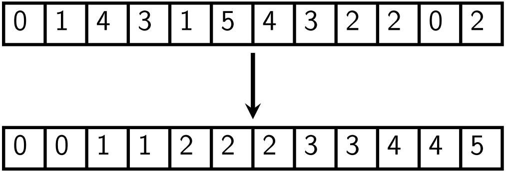
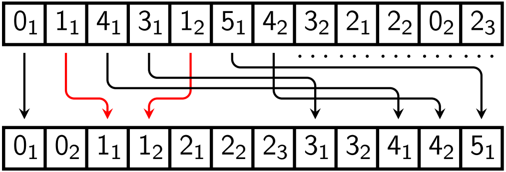
Sorting data



Sorting data – in a stable manner



Sorting data – in a stable manner

0_1	1_1	4_1	3_1	1_2	5_1	4_2	3_2	2_1	2_2	0_2	2_3

$$\downarrow$$

0_1	0_2	1_1	1_2	2_1	2_2	2_3	3_1	3_2	4_1	4_2	5_1

Mergesort has a **worst-case time complexity** of $\mathcal{O}(n \log(n))$

Can we do better?

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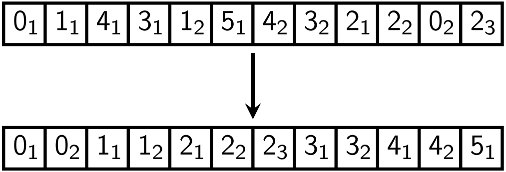


Mergesort has a **worst-case time complexity** of $\mathcal{O}(n \log(n))$

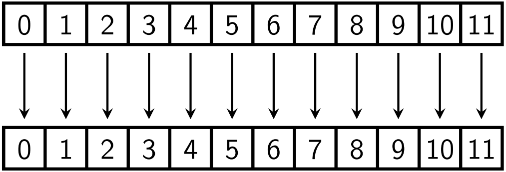
Can we do better? No!

Proof:

- There are $n!$ possible reorderings
- Each element comparison gives a 1-bit information
- Thus $\log_2(n!) \sim n \log_2(n)$ tests are required

Cannot we ever do better?

In some cases, we should. . .



Let us do better!

0	1	4	3	1	5	4	3	2	2	0	2

1. Chunk your data in **monotonic runs**

Let us do better!

4 runs of lengths 3, 2, 6 and 1

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1. Chunk your data in **monotonic runs**
2. New parameters: **Number of runs** (ρ) and their **lengths** ($r_1, \ldots, r_\rho$)

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   $\leqslant \log_2(\rho) \leqslant \log_2(n)$

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Theorem (Auger – Jugé – Nicaud – Pivoteau 2018)

Timsort has a **worst-case time complexity** of $\mathcal{O}(n + n\log(\rho))$

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Timsort has a **worst-case time complexity** of $\mathcal{O}(n + n\,\mathcal{H})$

We cannot do better than $\Omega(n + n\,\mathcal{H})$![2]

- Reading the whole input requires a time $\Omega(n)$
- There are **X** possible reorderings, with $\mathbf{X} \geqslant 2^{1-\rho}\binom{n}{r_1 \ldots r_\rho} \geqslant 2^{n\,\mathcal{H}/2}$

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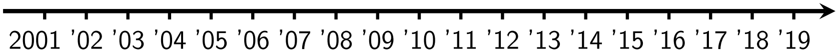
A brief history of Timsort



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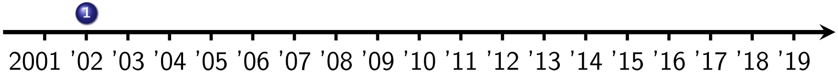

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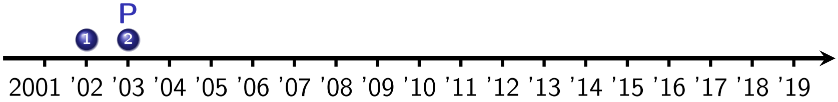

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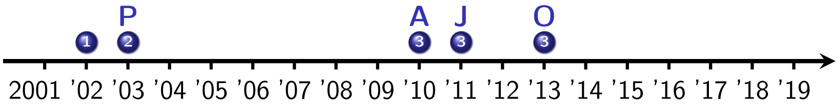

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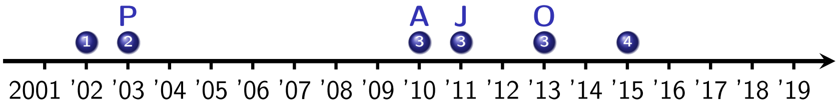

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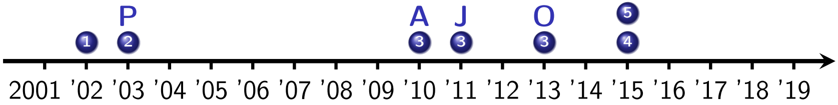

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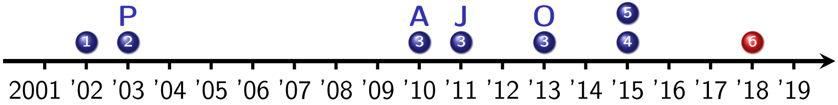

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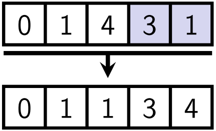
5. 1st worst-case complexity analysis[4] – Timsort works in time $\mathcal{O}(n \log n)$
6. Another stack size bug uncovered (Java version)

   Refined worst-case analysis: both versions work in time $\mathcal{O}(n + n\mathcal{H})$

The principles of Timsort (1/3)

Algorithm based on **merging** adjacent runs



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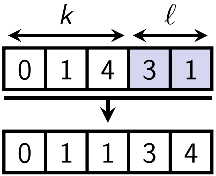

1. **Run merging** algorithm: standard + many optimizations
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   - memory $\mathcal{O}(\min(k, \ell))$

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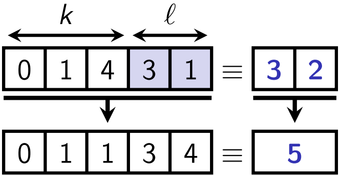

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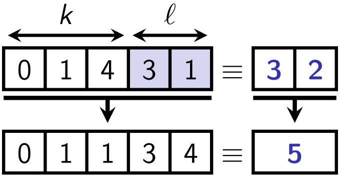

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Let us forget array values – only remember **run lengths**!

The principles of Timsort (2/3)

0	1	4	3	1	5	4	3	2	2	0	2	$\equiv$	**3**	**2**	**6**	**1**

STACK

Run merge policy:

- Maintain a **stack** of runs
- Until the array is sorted, either:
  1. discover & push a new run length onto the stack
  2. merge the top 1[st] and 2[nd] runs
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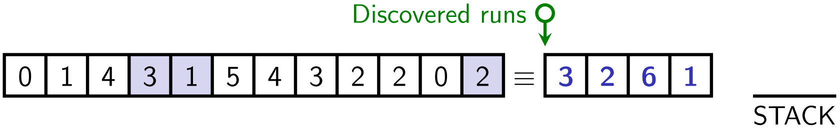

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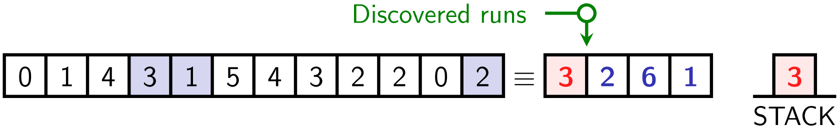

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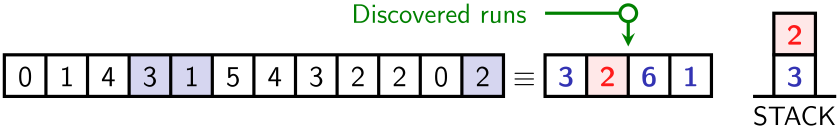

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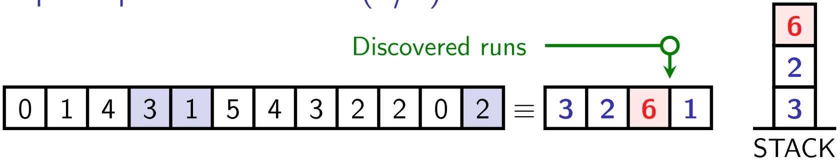

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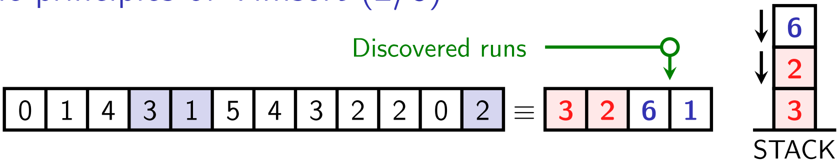

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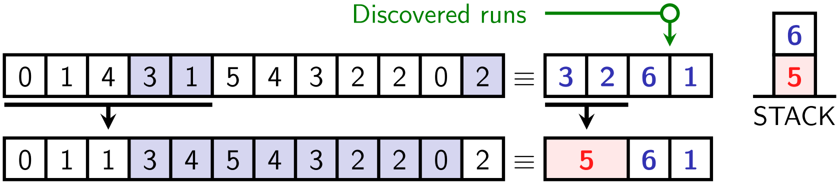

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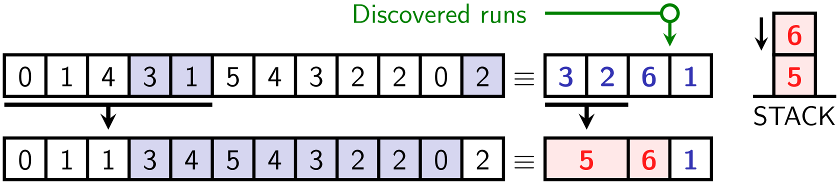

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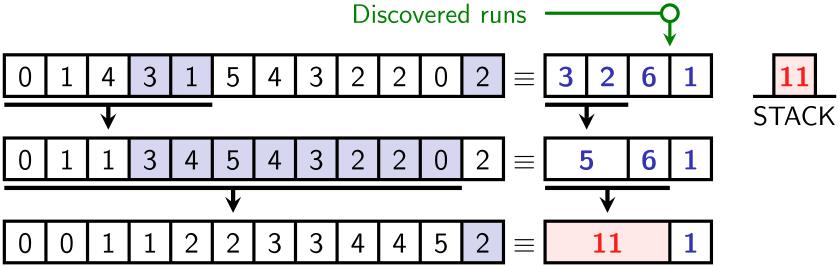

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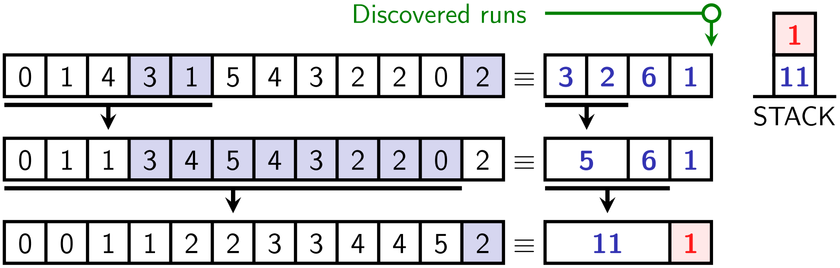

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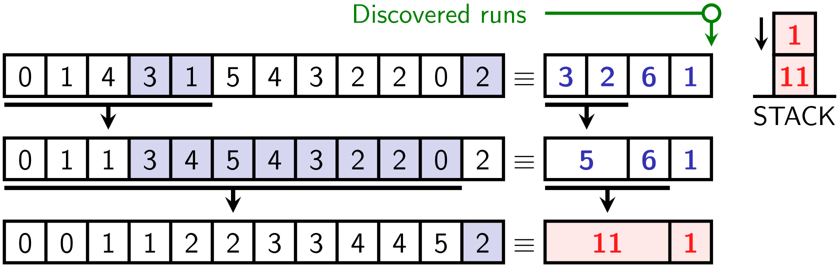

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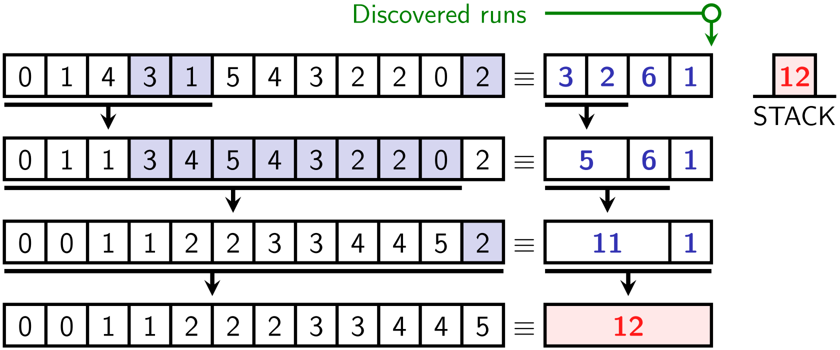

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- Each run r **pays** $\mathcal{O}(r)$ to
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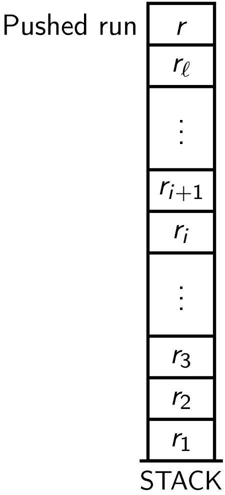

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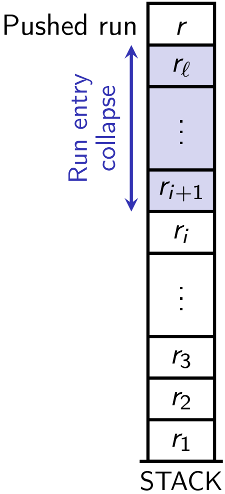

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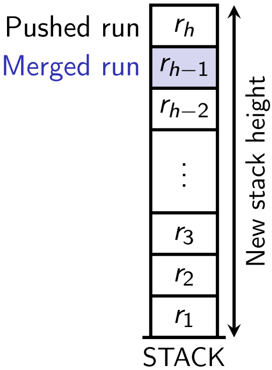

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  - $(r_i)_{i \geqslant 1}$ has **exponential** decay when r is pushed
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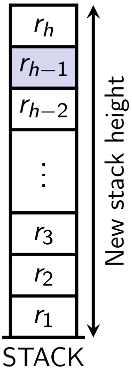
Pushed run

Merged run



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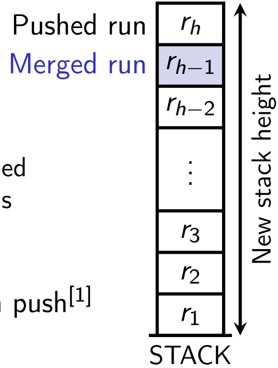

Implementation in **Timsort**:

- **Fibonacci constraints** $r_i > r_{i+1} + r_{i+2}$ on run push[1]
- Merge r_{h-2} and r_{h-1} whenever $r_{h-2} \leqslant r_h$

Intermezzo: Intelligent design & amortized analysis

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- Each run r **pays** $\mathcal{O}(r)$ to
  - **enter** the stack (before its 1st merge) ✓
  - **go down** 1 floor (after its 1st merge) 🔍
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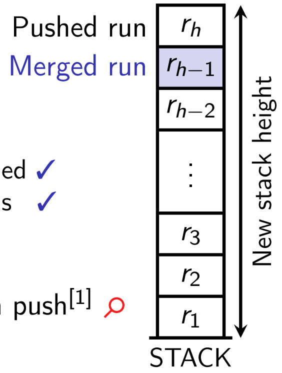

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The principles of Timsort (3/3)

Choice rules for options

1. discover & push a new run length onto the stack
2. merge the top 1$^{\text{st}}$ and 2$^{\text{nd}}$ runs
3. merge the top 2$^{\text{nd}}$ and 3$^{\text{nd}}$ runs

Choice algorithm

if $r_{h-2} \leqslant r_h$: choose ③
else if $r_{h-1} \leqslant r_h$, $r_{h-2} \leqslant r_{h-1} + r_h$ or $r_{h-3} \leqslant r_{h-2} + r_{h-1}$: choose ②
else: choose ① (or ② if ① is unavailable)

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Fibonacci constraints:

- $r_i > r_{i+1} + r_{i+2}$ for all $i \leqslant h - 4$ (induction)
- $r_i > r_{i+1} + r_{i+2}$ for $i \geqslant h - 3$ on run push

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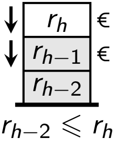
Making runs pay for going down:



The principles of Timsort (3/3)

Choice rules for options

1. discover & push a new run length onto the stack
2. merge the top 1^{st} and 2^{nd} runs
3. merge the top 2^{nd} and 3^{nd} runs

Choice algorithm

if $r_{h-2} \leqslant r_h$: choose ③
else if $\underline{r_{h-1} \leqslant r_h}$, $r_{h-2} \leqslant r_{h-1} + r_h$ or $r_{h-3} \leqslant r_{h-2} + r_{h-1}$: choose ②
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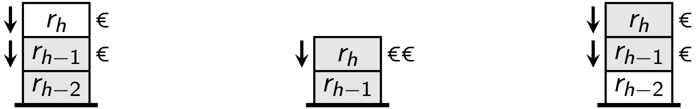

$r_{h-2} \leqslant r_h$ | $r_{h-1} \leqslant r_h$ | $r_{h-2} \leqslant r_{h-1} + r_h$, $r_{h-3} \leqslant r_{h-2} + r_{h-1}$

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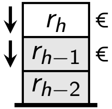
Making runs pay (with 1-step delay) **for going down**:



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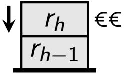
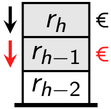


$r_{h-1} \leqslant r_h$



$r_{h-2} \leqslant r_{h-1} + r_h$
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Stack size bugs in Java Timsort

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The increase was not sufficient!

Bug raised by `igm.univ-mlv.fr/~pivoteau/Timsort/TimSort.java`

Java Timsort complexity analysis

Key steps:

- Study of the creation of **consecutive** Fibonacci **constraint failures**

$$r_{i-4} + r_{i-3} < r_{i-2}$$
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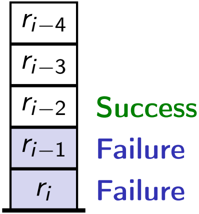

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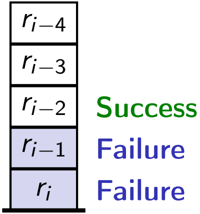

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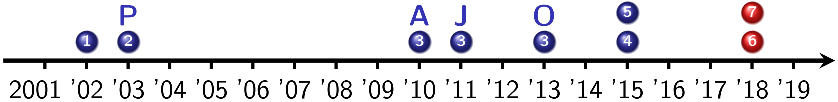
7 Suggested fix[3] now implemented in Java (JDK 11)!



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[1] Tim Peters' description of Timsort,
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[2] *On compressing permutations and adaptive sorting*, Barbay & Navarro (2013)
[3] *OpenJDK's java.utils.Collection.sort() is broken*, de Gouw et al. (2015)
[4] *Merge Strategies: from Merge Sort to Timsort*, Auger et al. (2015)
[5] *Strategies for stable merge sorting*, Buss & Knop (2018)
[6] *Nearly-optimal mergesorts*, Munro & Wild – to be presented **now** (2018)

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