Finding automatic sequences with few correlations

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14/06/2022

What does this title mean?

Our goal today:

Find **simple deterministic** algorithms for computing sequences $(u_n)_{n \geqslant 0}$

that **share similarities** with i.i.d. symbol sequences

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terms $(u_n)_{n \geqslant 0}$ should be equidistributed

pairs $(u_n, u_{n+a})_{n \geqslant 0}$ should be equidistributed

triples $(u_n, u_{n+a}, u_{n+a+b})_{n \geqslant 0}$ should be equidistributed

...

Automatic sequences[2]

Example #1: Thue–Morse sequence

$$u_n = \begin{cases} 0 & \text{if the binary digit expansion of } n \text{ contains an even number of 1s} \\ 1 & \text{otherwise} \end{cases}$$

$$u_n = 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, \ldots$$

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Computing u_{23} with an automaton:

1. Write 23 in base 2 (little-endian convention): $\langle 23 \rangle_2 = 111010000\ldots$

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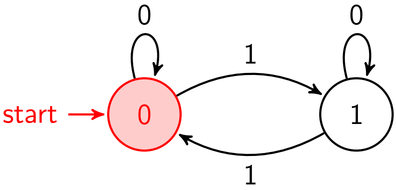

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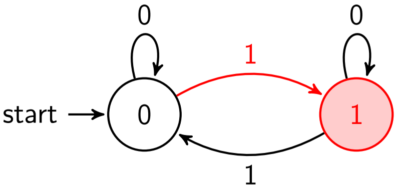

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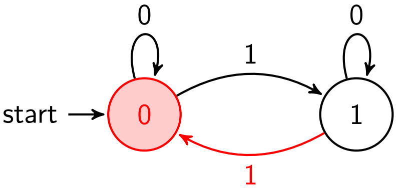

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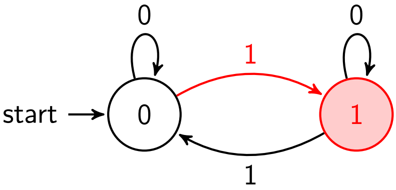

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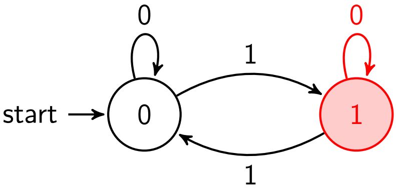

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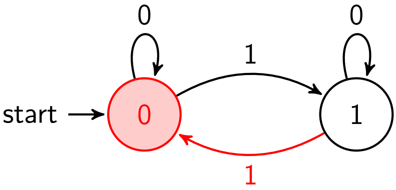

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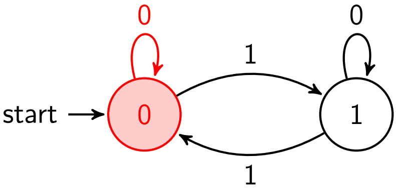

Automatic sequences[2]

Example #1: Thue–Morse sequence 2-automatic

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Computing u_{23} with an automaton: $u_{23} = 0$

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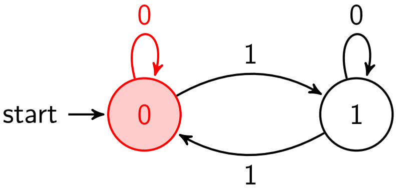

Automatic sequences

Example #2: Mod2

$$u_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{otherwise} \end{cases}$$

$$u_n = 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots$$

Automatic sequences

Example #2: Mod2

2-automatic

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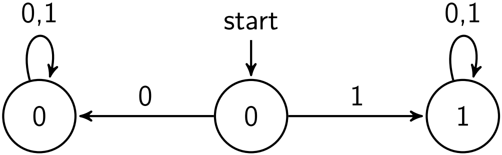
Computing u_n with an automaton:

1. Write n in base 2 (little-endian convention)
2. Feed $\langle n \rangle_2$ to the following automaton and output the state label you get stuck seeing



Automatic sequences

Example #3: Powers of 3

$$u_n = \begin{cases} 1 & \text{if } n \text{ is a power of } 3 \\ 0 & \text{otherwise} \end{cases}$$

$$u_n = 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, \ldots$$

Automatic sequences

Example #3: Powers of 3

3-automatic

$$u_n = \begin{cases} 1 & \text{if } n \text{ is a power of } 3 \\ 0 & \text{otherwise} \end{cases}$$

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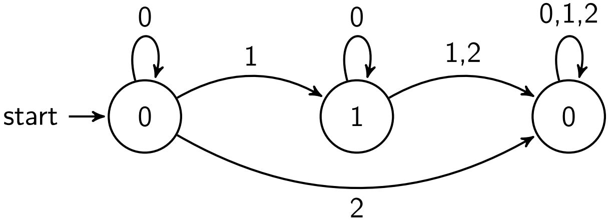
Computing u_n with an automaton:

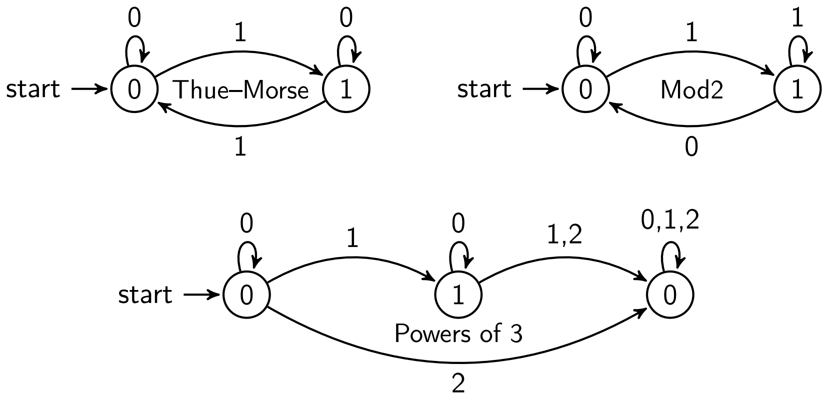
1. Write n in base 3 (little-endian convention)
2. Feed $\langle n \rangle_3$ to the following automaton and output the state label you get stuck seeing



Automatic sequences: Big-endian variant

Computing u_n with an automaton:

1. Write n in base k (big-endian convention): $\langle\langle 23 \rangle\rangle_2 = \ldots 000010111$
2. Feed $\langle\langle n \rangle\rangle_k$ to your favourite automaton and output the last state label you see



Automatic sequences and block-additive sequences[3,7]

Example #1: Thue–Morse sequence (in $\mathbb{Z}_2$)

$$u_n = \begin{cases} 0 & \text{if } n = 0 \\ u_{\lfloor n/2 \rfloor} & \text{if } n \geqslant 1 \text{ is even} \\ u_{\lfloor n/2 \rfloor} + 1 & \text{if } n \geqslant 1 \text{ is odd} \end{cases}$$

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Computing u_{23} with a sliding window:

1. Write 23 in base 2 (little-endian convention)
2. Feed $\langle 23 \rangle_2$ to the **size-1 window** with function $f : x \mapsto x$

$$\langle 23 \rangle_2 = \boxed{1} \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad \ldots$$

$$\Big\downarrow f$$

$$u_{23} = 1 +$$

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$$\downarrow f$$

$$u_{23} \quad = \quad 1 + 1 + 1 + 0 +$$

Automatic sequences and block-additive sequences[3,7]

Example #1: Thue–Morse sequence (in $\mathbb{Z}_2$) rank-1 block-additive

$$u_n = \begin{cases} 0 & \text{if } n = 0 \\ u_{\lfloor n/2 \rfloor} & \text{if } n \geqslant 1 \text{ is even} \\ u_{\lfloor n/2 \rfloor} + 1 & \text{if } n \geqslant 1 \text{ is odd} \end{cases}$$

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$$\downarrow f$$

$$u_{23} = 1 + 1 + 1 + 0 + 1$$

Automatic sequences and block-additive sequences

Example #2: Generalised Golay–Rudin–Shapiro sequence (in $\mathbb{Z}_p$)

$$u_n = \begin{cases} 0 & \text{if } n = 0 \\ u_{\lfloor n/p \rfloor} + ij & \text{if } n \geqslant 1 \text{ and } n \equiv i + pj \bmod p^2 \end{cases}$$

$u_n = 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, \ldots$

for $p = 2$

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$$u_n = 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, {\color{red}1}, 1, 1, 1, 0, 1, \ldots$$

for $p = 2$

Computing $u_{\color{red}23}$ with a sliding window:

1. Write ${\color{red}23}$ in base 2 (little-endian convention)
2. Feed $\langle 23 \rangle_2$ to the **size-2 window** with function $f : (x, y) \mapsto xy$

$$\langle 23 \rangle_2 = \boxed{1 \quad 1} \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad \ldots$$

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Automatic sequences and block-additive sequences

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$$u_{23} = 1 + 1 + 0 + 0 +$$

Automatic sequences and block-additive sequences

Example #2: Generalised Golay–Rudin–Shapiro sequence (in $\mathbb{Z}_p$)

rank-2 block-additive

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$$\downarrow f$$

$$u_{23} = \quad 1 + 1 + 0 + 0 + 0$$

Automatic sequences and block-additive sequences

Example #3: Mod2 (in $\mathbb{Z}_2$)

$$u_n = \begin{cases} 0 & \text{if } n = 0 \\ u_{\lfloor n/2 \rfloor} & \text{if } n \geqslant 1 \text{ and } n \equiv 0 \text{ or } 3 \bmod 4 \\ u_{\lfloor n/2 \rfloor} + 1 & \text{if } n \geqslant 1 \text{ and } n \equiv 1 \text{ or } 2 \bmod 4 \end{cases}$$

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Computing u_{23} with a sliding window:

1. Write 23 in base 2 (little-endian convention)
2. Feed $\langle 23 \rangle_2$ to the **size-2 window** with function $f : (x, y) \mapsto x + y$

$$\langle 23 \rangle_2 = \boxed{1 \quad 1} \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad \ldots$$

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Automatic sequences and block-additive sequences

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Automatic sequences and block-additive sequences

Example #3: Mod2 (in $\mathbb{Z}_2$) rank-2 block-additive

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Computing u_{23} with a sliding window: $u_{23} = 1$

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$$\Big\downarrow f$$

$$u_{23} = 0 + 0 + 1 + 1 + 1$$

Automatic sequences: Take-away home. . .

Automatic sequence

A sequence $(u_n)_{n \geqslant 0}$ is ***k*-automatic** if there exists a labelled DFA that, upon reading the base-k digits of n, gets stuck in states labelled u_n.

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Block-additive sequence

A sequence $(u_n)_{n \geqslant 0}$ is rank-r block-additive in $\mathbb{Z}_k$ if there exists a function $\varphi \colon \mathbb{Z}_{k^r} \mapsto \mathbb{Z}_k$ such that

$$u_n = \begin{cases} 0 & \text{if } n = 0 \\ u_{\lfloor n/k \rfloor} + \varphi(n \bmod k^r) & \text{if } n \geqslant 0 \end{cases}$$

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Examples: Mod2, Thue–Morse, Generalised GRS, Non-multiples of 3

Counter-examples: Powers of 3, Multiples of 3

Proposition: Every block-additive sequence is automatic.

What makes a deterministic sequence look random?

You should have no idea of what you will find!

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Equidistributed terms

A sequence $(u_n)_{n \geqslant 0} \in S^{\mathbb{N}}$ is **1-uncorrelated** if

$$|S| \cdot |\{k \leqslant n \colon u_k = s\}| \sim n$$

for all symbols $s \in S$.

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Thue–Morse	$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$
Generalised GRS (for all p)	$0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1, \ldots$

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Counter-examples:

Squares	$1,1,0,0,1,0,0,0,0,1,0,0,0,0,0,0,\ldots$
Multiples of 3	$1,0,0,1,0,0,1,0,0,1,0,0,1,0,0,1,\ldots$
Non-multiples of 3	$0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,\ldots$
Odd number of digits	$1,1,0,0,1,1,1,1,0,0,0,0,0,0,0,0,\ldots$

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Generalised GRS[4] (for all p) $0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1, \ldots$

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Counter-examples:

Mod2 $\mathbb{P}[01] = 1/2$

Thue–Morse $\mathbb{P}[00] = 1/3$

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Equidistributed tuples

A sequence $(u_n)_{n \geqslant 0} \in S^{\mathbb{N}}$ is **ℓ-uncorrelated** if

$$|S^\ell| \cdot |\{k \leqslant n \colon (u_{k+a_1}, u_{k+a_2}, \ldots, u_{k+a_\ell}) = (s_1, s_2, \ldots, s_\ell)\}| \sim n$$

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Example for $\ell = 3$:
Generalised GRS[8] (for $p = 2$) $0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1, \ldots$

Counter-example for $\ell = 3$:
Generalised GRS (for $p = 3$) $\mathbb{P}[000] = 5/81$

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Counter-example for $\ell = 4$:
Generalised GRS (for $p = 2$) $\quad \mathbb{P}[0000] = 3/32$

Avoiding small-ℓ correlations

Theorem[2]

Every non-constant k-automatic sequence with an s-state big-endian DFA is k^{s+1}-correlated.

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Let $\ell = k^{s+1}$. Our sequence $(u_n)_{n \geqslant 0}$ has at most ℓs^2 distinct subsequences of length ℓ :

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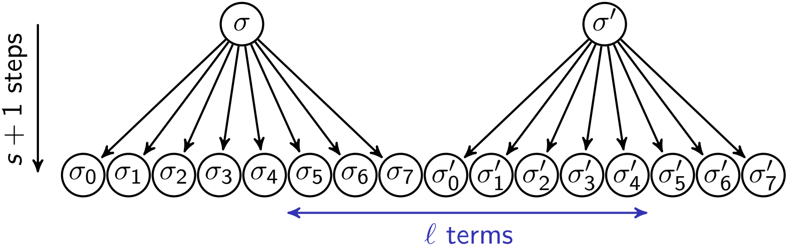

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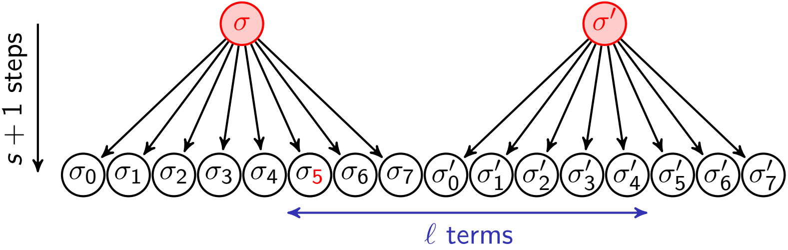

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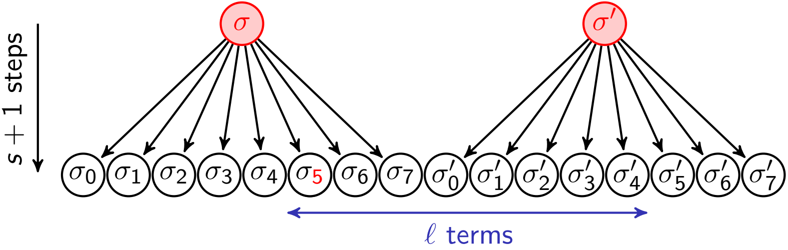

Thus, at least one of the $k^\ell \geqslant 2^\ell > \ell s^2$ sequences of length ℓ is missing.

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Results so far:

Simple criteria for 2-/3-uncorrelated rank-3 block-additive sequences in $\mathbb{Z}_2$.

Rank-5 block-additive sequences in $\mathbb{Z}_2$ are 4-correlated.

Rank-3 block-additive sequences in $\mathbb{Z}_3$ are 3-correlated.

What is coming next?

- Extending our criteria to all 2-uncorrelated block-additive sequences in $\mathbb{Z}_2$.
- Finding a 4-uncorrelated block-additive sequence or proving none exists.
- Finding a 4-uncorrelated automatic sequence or proving none exists.
- Deciding whether an automaton $\mathcal{A}$ gives an ℓ-uncorrelated sequence.

Some references

[1] *Une nouvelle propriété des suites de Rudin-Shapiro*, Queffélec (1987)
[2] *Automatic sequences*, Allouche & Shallit (2003)
[3] *Block-additive functions on the Gaussian integers*, Drmota, Grabner & Liardet (2008)
[4] *Bounds for the discrete correlations of infinite sequences on k symbols and generalized Rudin-Shapiro sequences*, Grant, Shallit & Stoll (2009)
[5] *On a Golay-Shapiro-like sequence*, Allouche (2016)
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[7] *Discrete correlations of order 2 of generalized Golay-Shapiro sequences: a combinatorial approach*, Marcovici, Stoll & Tahay (2021)
[8] *Finding automatic sequences with few correlations* Jugé & Marcovici (2022)


THANK YOU FOR LISTENING!
DO YOU HAVE EASY QUESTIONS?