Galloping in fast-growth natural merge sorts

Elahe Ghasemi[2,3], Vincent Jugé[2] & Ghazal Khalighinejad[1,3]

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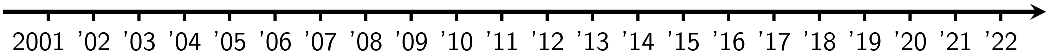
3: Sharif University of Technology

08/07/2022

Library sorting algorithms in a few languages (for composite-type arrays)



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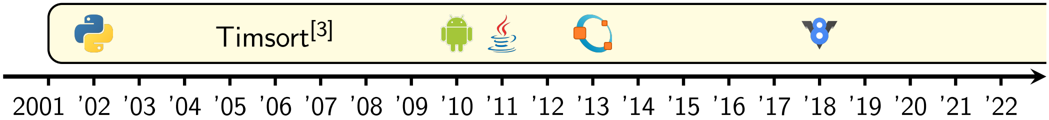

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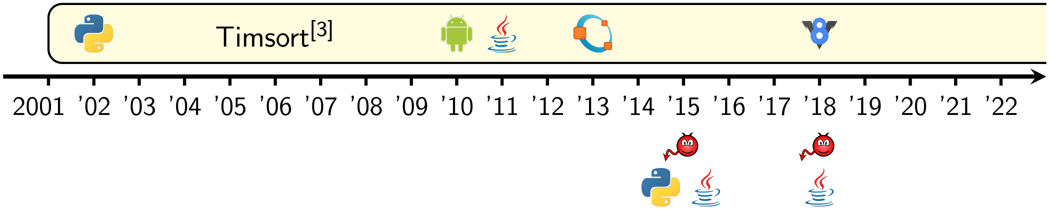

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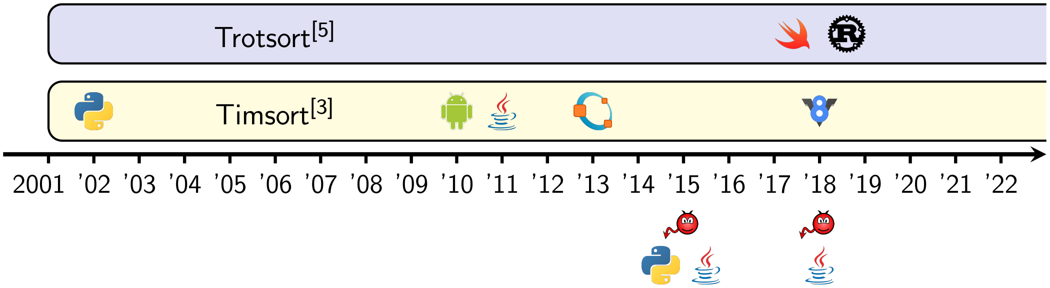

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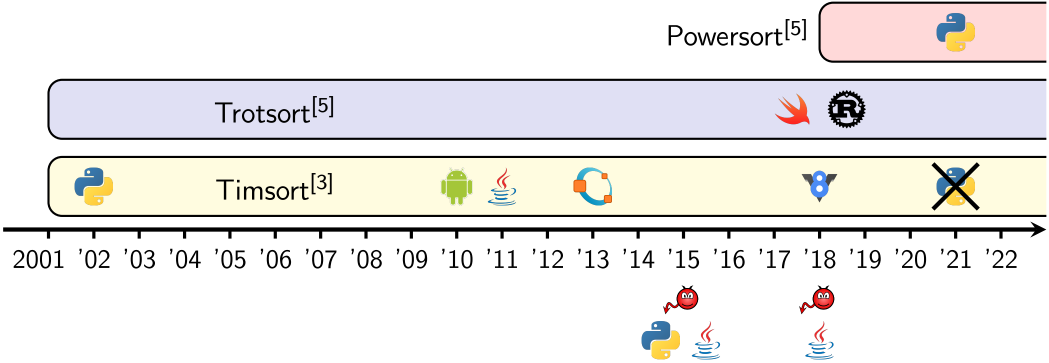

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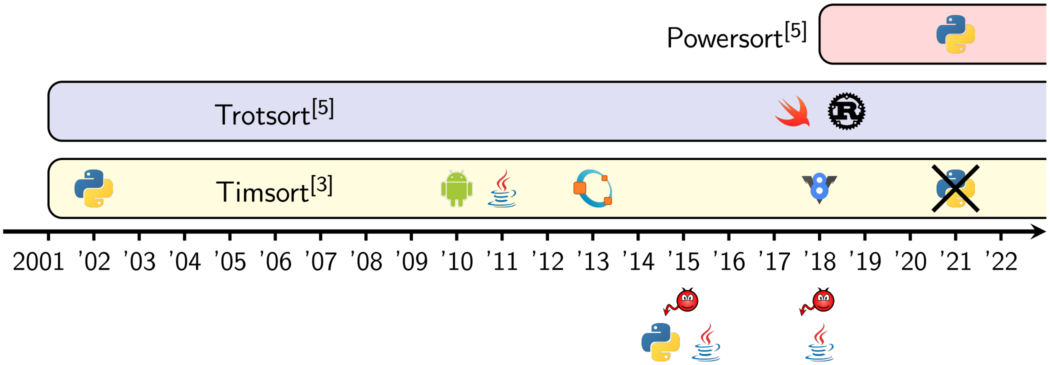
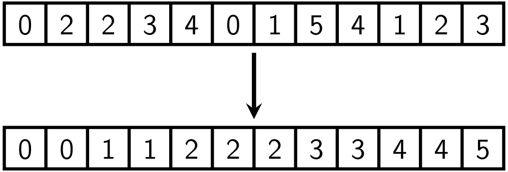


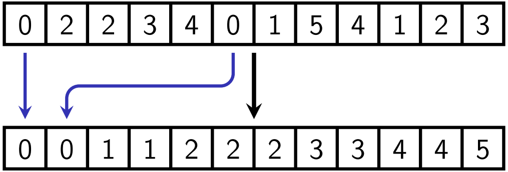
Why don't people just use plain (DualPivot)QuickSort + Heapsort?

Sorting data



Heapsort and **Mergesort** have a **worst-case time complexity** of $\mathcal{O}(n \log(n))$ and we cannot do better, even on average...

Sorting data in a stable manner



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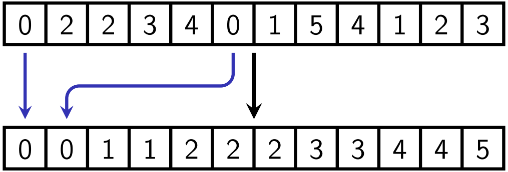

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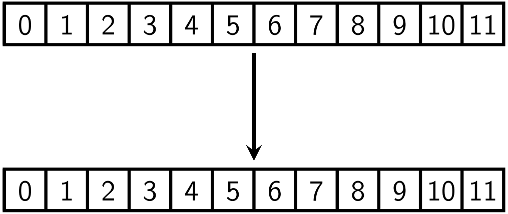
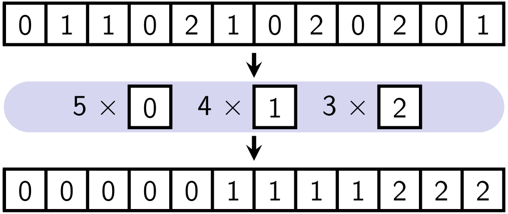
and we cannot do better, even on average...
But, sometimes, **we can!**





Let us do better!

0	3	4	4	3	2	1	4	3	2	0	5

1. Subdivide your array in **monotonic** (non-decreasing or decreasing) **runs**.

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4 runs of lengths 4, 3, 4 and 1

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Theorem[5]

Powersort uses $\mathcal{O}(n + n\,\mathcal{H})$ element moves and $\mathcal{O}(n) + n\,\mathcal{H}$ comparisons.

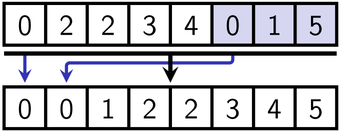
We cannot do better than $\mathcal{O}(n) + n\,\mathcal{H}$ comparisons![4]

There are X possible reorderings, with $X \geqslant 2^{1-\rho}\binom{n}{r_1 \ldots r_\rho} \geqslant 2^{(\mathcal{H}-5)n}$.

The principles of Timsort, Trotsort, Powersort et al.

Algorithms based on **merging** adjacent runs

☛ **Stable** algorithms
(good for **composite** types)



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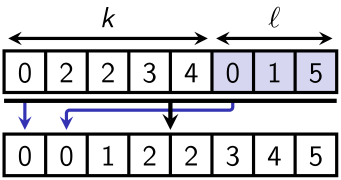

1. **Extend small runs** to save time $\mathcal{O}(n)$, and make them non-decreasing
2. **Run merging** sub-routine: naïve (Trotsort) or optimised (Timsort & Powersort)
   - time $\mathcal{O}(k+\ell)$
   - memory $\mathcal{O}(\min(k,\ell))$

   } **Merge cost:** $k+\ell \geqslant$ #comparisons
3. **Policy** for choosing runs to merge:
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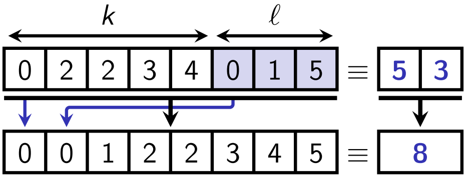

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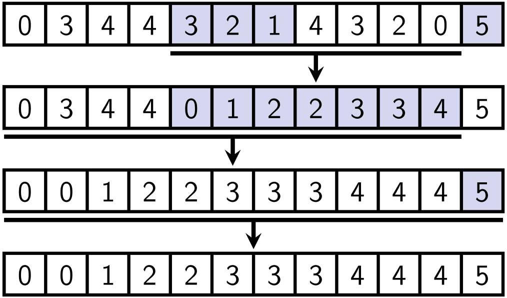
4. **Complexity analysis:**
   - ☛ Evaluate the **total merge cost**
   - ☛ Just work with **run lengths**

Merge trees and fast growth

Timsort merges



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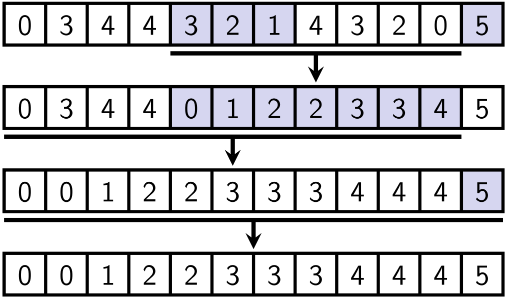
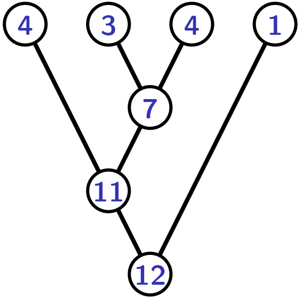


Timsort merge tree



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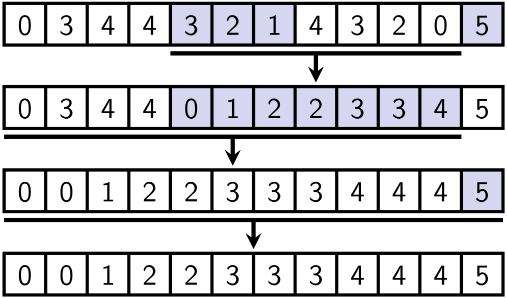

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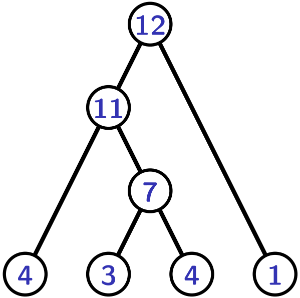

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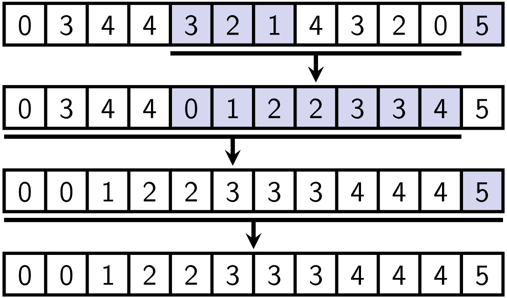

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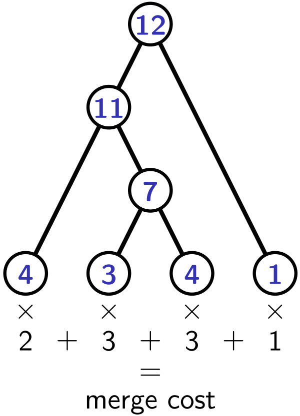

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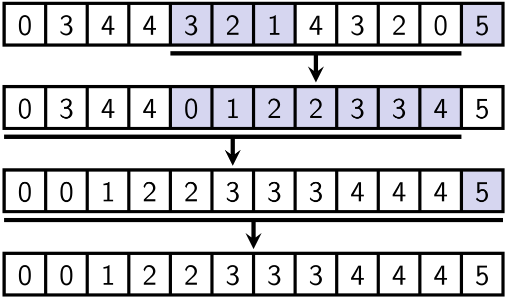
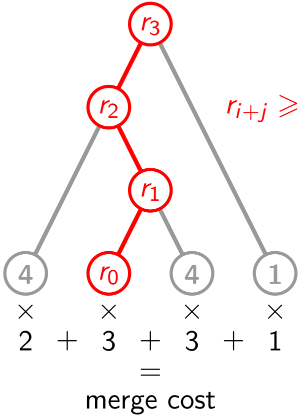


Timsort merge branch



$$r_{i+j} \geqslant (14/13)^{j-2} r_i$$

Fast growth and merge cost

Fast growth[6]

An natural merge sort is **fast-growing** if node sizes grow exponentially fast on its branches:

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- ☛ Such an algorithm has a merge cost $\leqslant \log_a(2) n \mathcal{H} + bn$.

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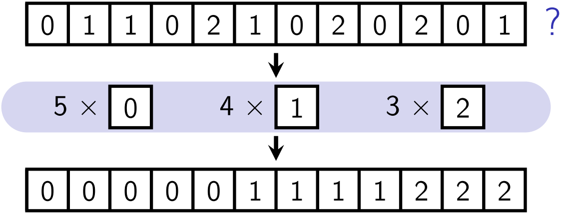
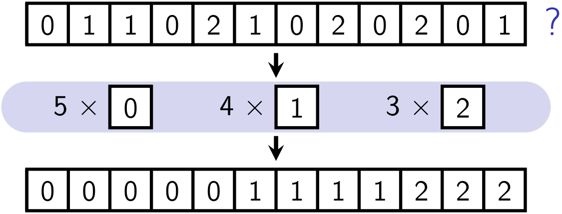
- **Fast-growing algorithms** work in time $\mathcal{O}(n + n\mathcal{H})$.

  Examples: Timsort, α-Mergesort, Powersort, Peeksort, adaptive Shiverssort
- **Powersort** performs no more than $n(\mathcal{H} + 4)$ comparisons (because $a = 2$ and $b = 4$).
- **Peeksort** and **adaptive Shiverssort** perform only $\mathcal{O}(n) + n\mathcal{H}$ comparisons (but $a > 2$).

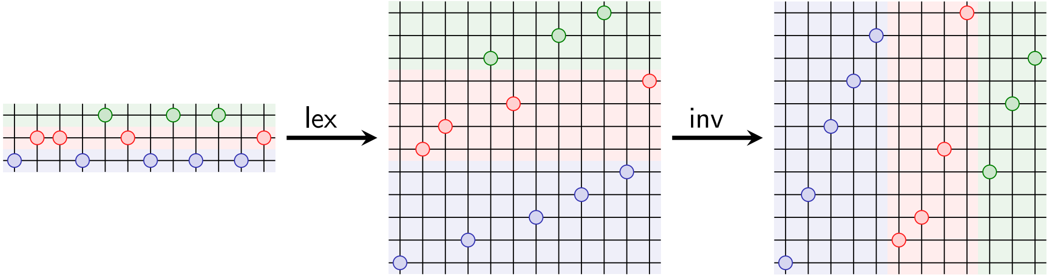
What about ?



What about 0 1 1 0 2 1 0 2 0 2 0 1 ?



Few **runs** vs few **values** vs few **dual runs**:



Let us do better, dually!

3 dual runs of lengths **5**, **4** and **3**

0	1	1	0	2	1	0	2	0	2	0	1

1. Subdivide your data in non-decreasing, non-overlapping **dual runs**
2. New parameters: **Number of dual runs** ($\rho^\star$) and their **lengths** ($r_i^\star$)

Dual-run entropy: $\mathcal{H}^\star = \sum_{i=1}^{\rho^\star} (r_i^\star/n) \log_2(n/r_i^\star) \leqslant \log_2(\rho^\star) \leqslant \log_2(n)$

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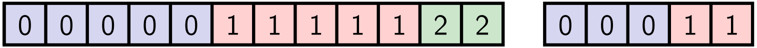
Theorem[6]

Fast-growing merge sorts require $\mathcal{O}(n + n\,\mathcal{H}^\star)$ comparisons if they use Timsort's **galloping** run-merging routine*.
*we are slightly cheating

and we cannot use less than $\mathcal{O}(n) + n\,\mathcal{H}^\star$ comparisons in general.

Galloping merging procedure

Merging runs ≈ finding an integer (several times)[1,2]



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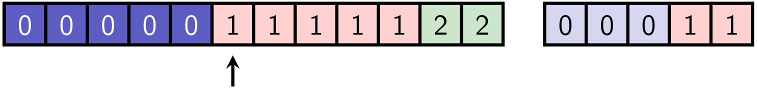

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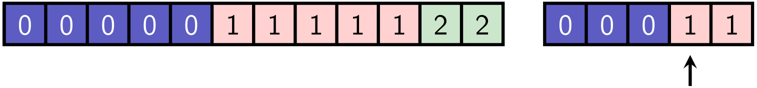

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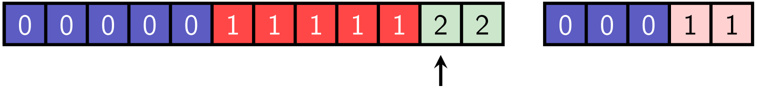

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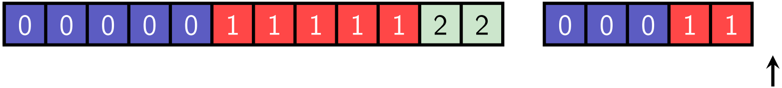

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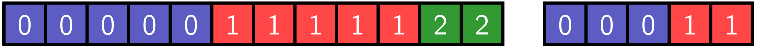

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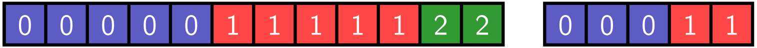

Finding an integer $\boldsymbol{x}$ by asking y and being told whether $y \geqslant x$:

1. Ask $y = 1, 2, 3, 4 \ldots$ (time x)

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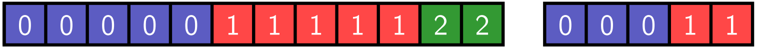

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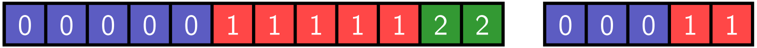

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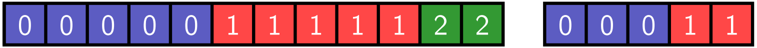

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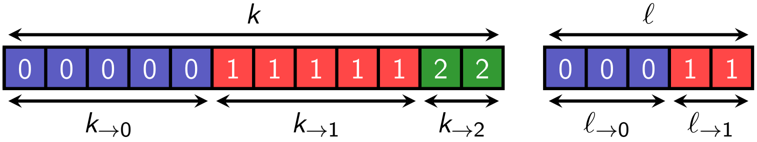

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Timsort merging procedure ≈ methods 1 + 2 with threshold **t**[2,3]:

4. Ask $y = 1, 2, \ldots, \mathbf{t}+1, \mathbf{t}+2, \mathbf{t}+4, \mathbf{t}+8, \ldots$, then find the bits of $x - \mathbf{t}$

☛ **Merge cost:** $\sum_i \min\{(1+\mathbf{t}^{-1})(k_{\to i} + \ell_{\to i}), 6\mathbf{t} + 4\log_2(k_{\to i} + \ell_{\to i} + 1)\} \geqslant$ #comparisons

Conclusions (after a few more computations)

- ☛ For fixed thresholds **t**, fast-growth natural merge sorts require $\mathcal{O}(n + n\mathcal{H}^\star)$ comparisons.
- ☛ Choosing adequate choices of **t**, **Powersort** requires $\mathcal{O}(n) + (1 + o(1))n\mathcal{H}^\star$ comparisons.

Choose $\mathbf{t} \approx \log(k + \ell)$ to merge runs of lengths k and ℓ

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[1] *An almost optimal algorithm for unbounded searching*, Bentley & Yao (1976)
[2] *Optimistic Sorting and Information Theoretic Complexity*, McIlroy (1993)
[3] *Description of TimSort*, Peters
`svn.python.org/projects/python/trunk/Objects/listsort.txt` (2001)
[4] *On compressing permutations and adaptive sorting*, Barbay & Navarro (2013)
[5] *Nearly-optimal mergesorts*, Munro & Wild (2018)
[6] *Galloping in natural merge sorts*, Ghasemi, Jugé & Khalighinejad (2022)


THANK YOU FOR YOUR ATTENTION!
DO YOU HAVE ANY EASY QUESTIONS?