Combinatorics of braids

Vincent Jugé

Université Paris Diderot (IRIF)

05/02/2016

A PhD about braids?

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Some questions of interest. . .

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3. How do complicated braids **typically** behave?

A PhD about braids?

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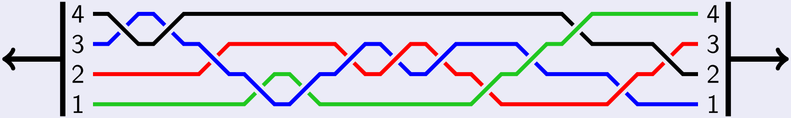
   Choose a **dynamic framework**/**probability measure**.

What are braids? – Algebra

Isotopy classes of braid diagrams (Artin 1926)



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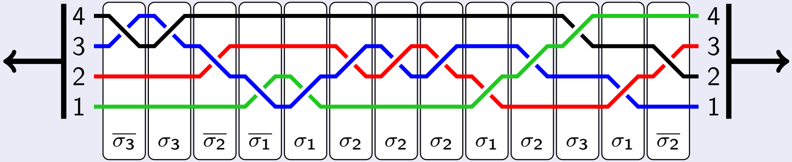

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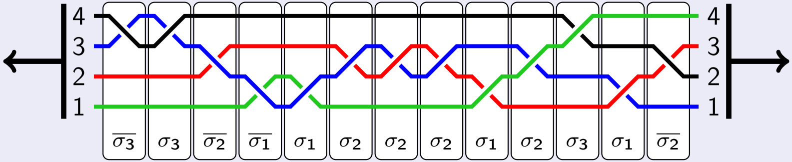

Finitely generated monoid

$$\mathcal{B}_n = \left\langle \quad \middle| \quad \right\rangle.$$

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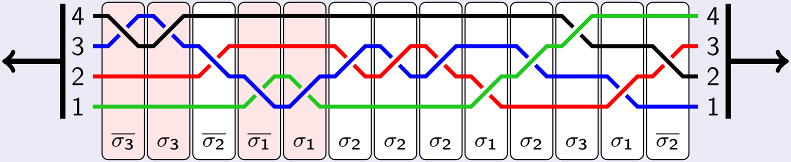

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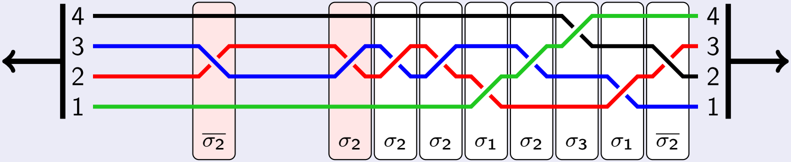

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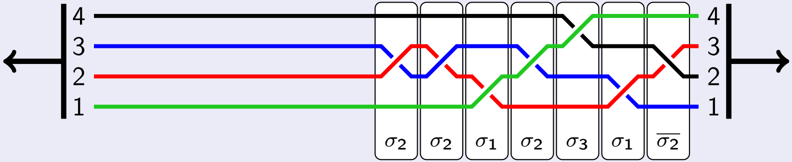

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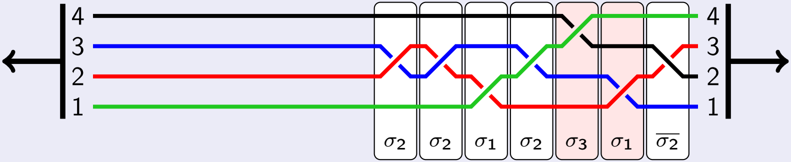

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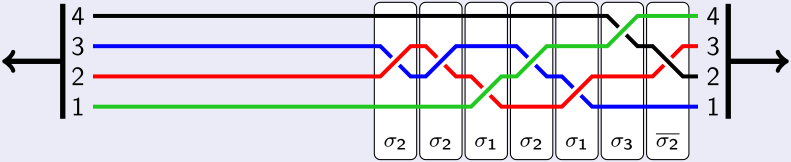

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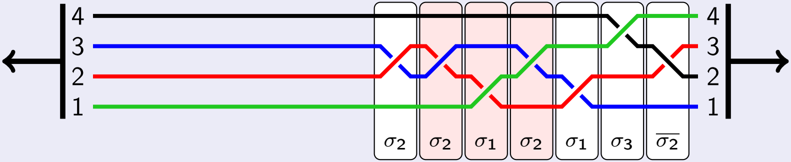

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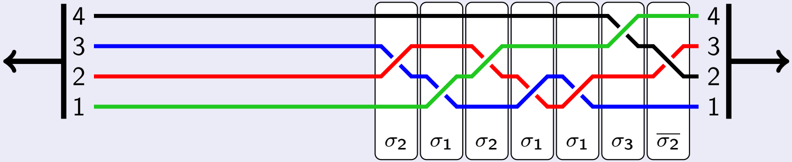

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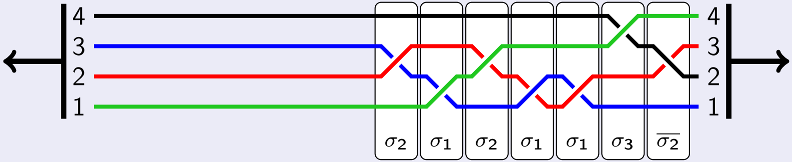

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Isotopy classes of laminations of the punctured plane (Birman 1975)

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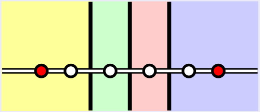

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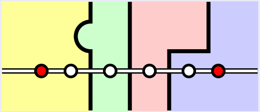

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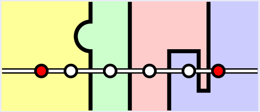

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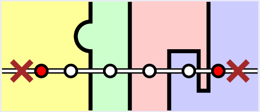

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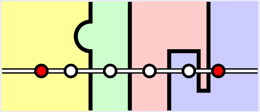

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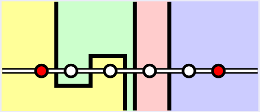

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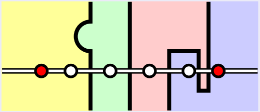

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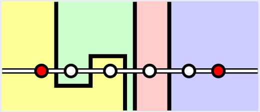
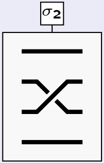


Braid acting on a lamination



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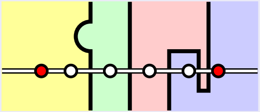

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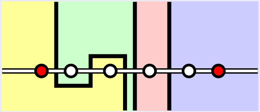

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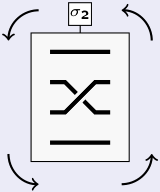

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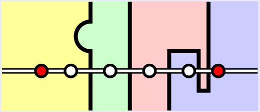

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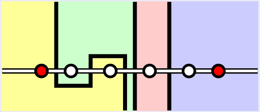

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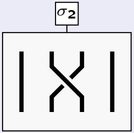

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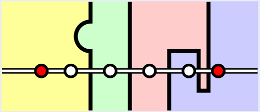

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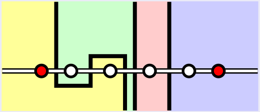

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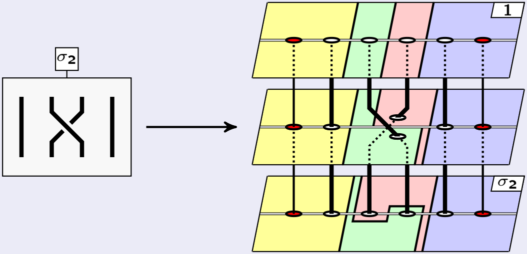

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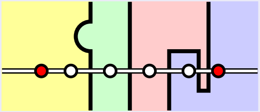

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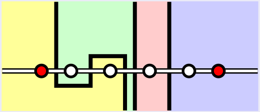

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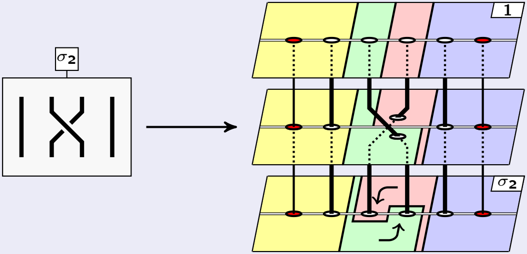
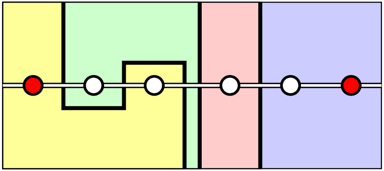


What are braids? – Geometry

Braids are isotopy classes of **which** laminations?

Open lamination



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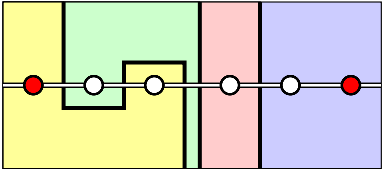

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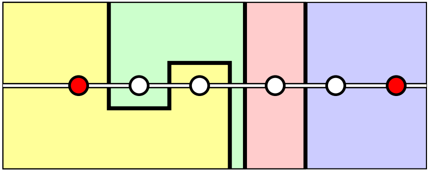

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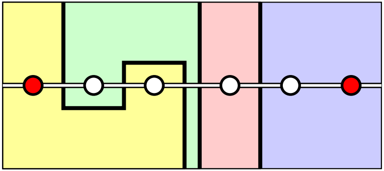

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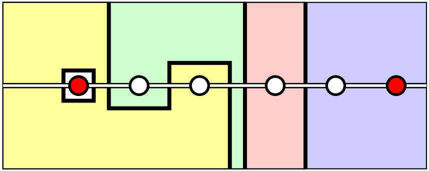

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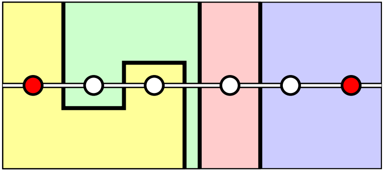

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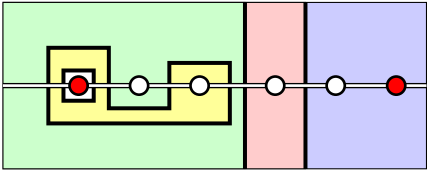

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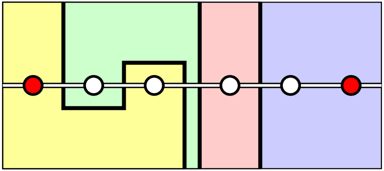

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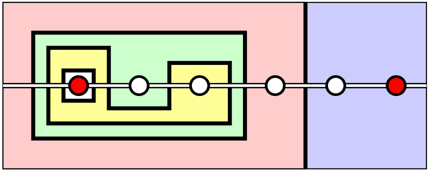

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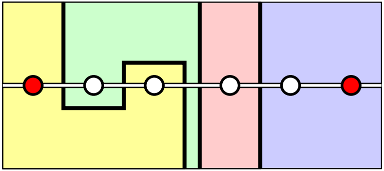

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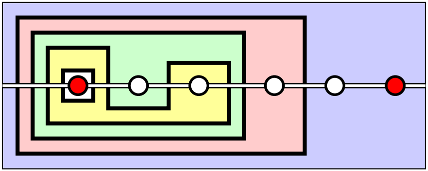

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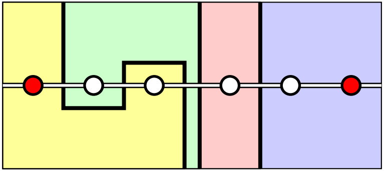

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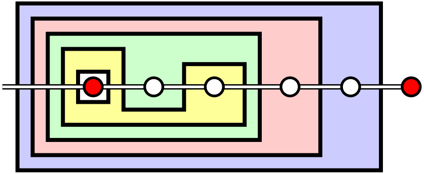

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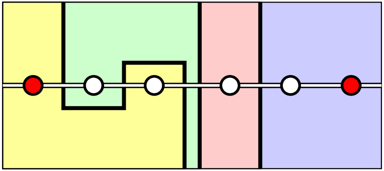

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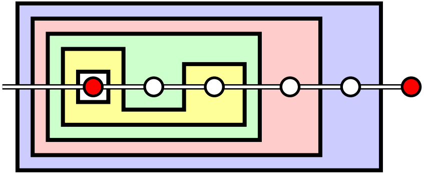
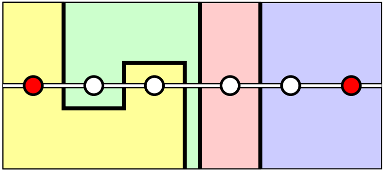


Curve diagram



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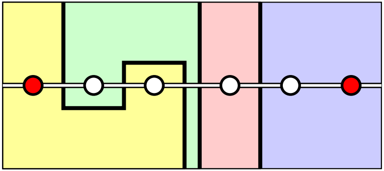

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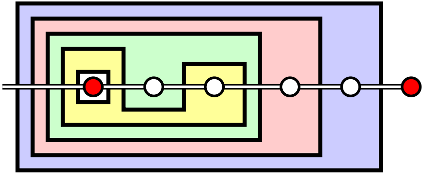

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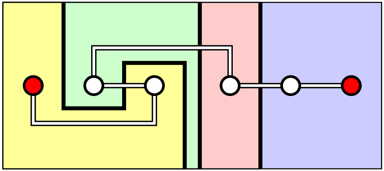

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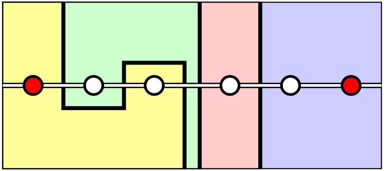

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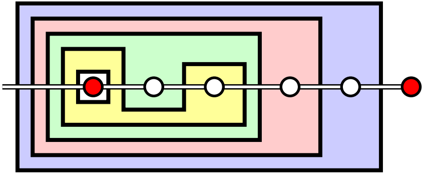

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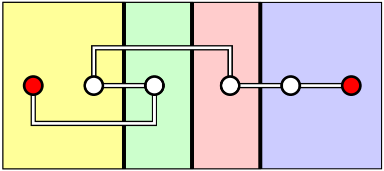

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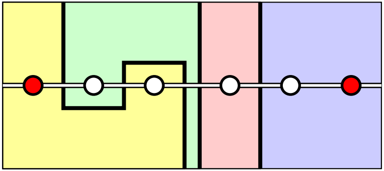

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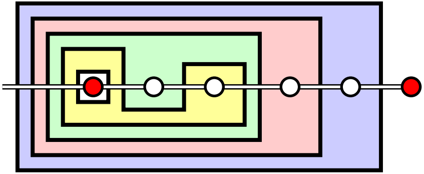

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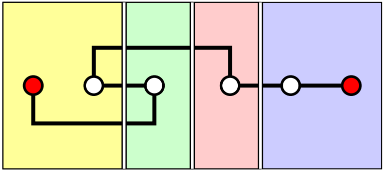

What are braids? – Checking braid equality

Garside normal form (Garside 1969, Adian 1984)

1. The monoid of **positive** braids $\mathcal{B}_n^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle^+$ is a **lattice** for the divisibility ordering $\preccurlyeq$. $(\alpha \preccurlyeq \beta \Leftrightarrow \exists \gamma \in \mathcal{B}_n^+, \alpha\gamma = \beta)$

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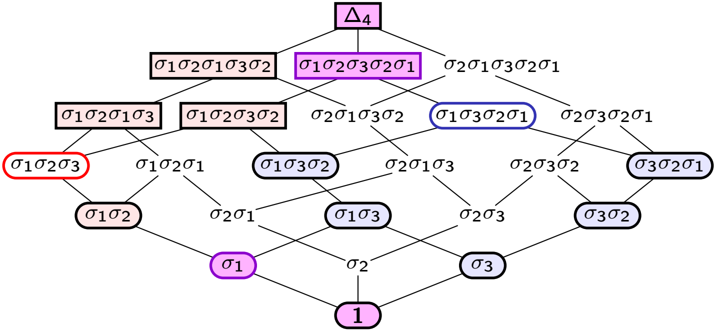

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2. There exists a **Garside element** $\Delta_n = \bigvee\{\sigma_1, \ldots, \sigma_{n-1}\}$.
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   - $\alpha = a_1 a_2 \ldots a_k$;
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The Garside normal form:

- can be extended to the **group** $\mathcal{B}_n$;
- is **automatic**: for all $i \in \{1, \ldots, n-1\}$, the languages
  - $\{(\mathbf{Gar}(\alpha), \mathbf{Gar}(\alpha\sigma_i)) : \alpha \in \mathcal{B}_n\}$;
  - $\{(\mathbf{Gar}(\alpha), \mathbf{Gar}(\sigma_i\alpha)) : \alpha \in \mathcal{B}_n\}$

  are regular;
- solves the **equality problem**: $\alpha = \beta$ iff $\mathbf{Gar}(\alpha) = \mathbf{Gar}(\beta)$.

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Tight laminations/curve diagrams

A lamination/curve diagram is **tight** if it minimises crossings ┿ or ╫ .

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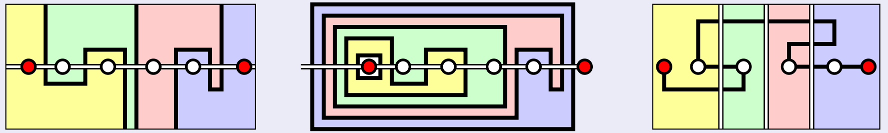

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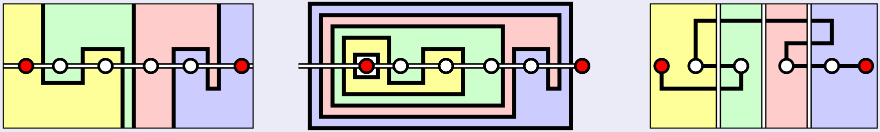

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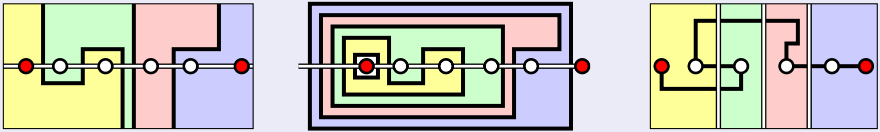

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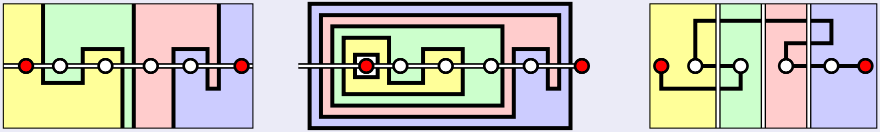

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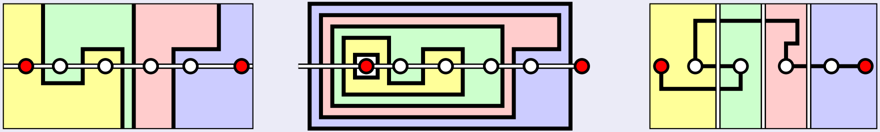

- Two tight laminations/curve diagrams represent the same braid iff they are **visibly isotopic** to each other. (≈ normal form)

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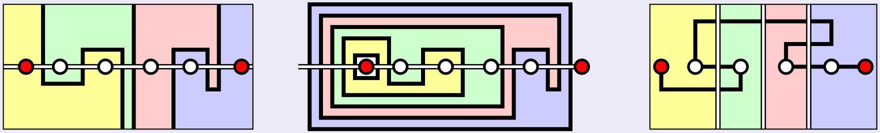

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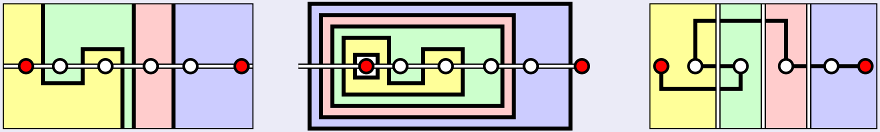

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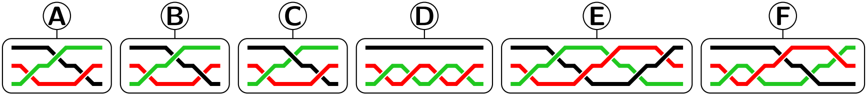
What are braids? – Checking complexity

Which braid is the most complicated?



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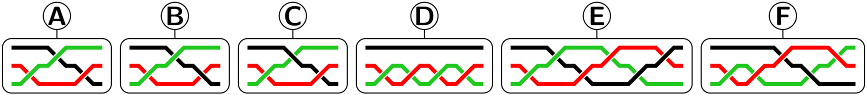

Several approaches to braid complexity

- "Naive" Artin length: **E** $>$ **F** $>$ **D** $>$ **A** $\approx$ **B** $\approx$ **C**;

- **A** $= \sigma_1\sigma_2\sigma_1$: $|\mathbf{A}| = 3$;
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- **C** $= \sigma_1\overline{\sigma_2}\sigma_1$: $|\mathbf{C}| = 3$;
- **D** $= \sigma_1^4$: $|\mathbf{D}| = 4$;
- **E** $= (\sigma_1\sigma_2)^3$: $|\mathbf{E}| = 6$;
- **F** $= \sigma_1^2\sigma_2\overline{\sigma_1\sigma_2}$: $|\mathbf{F}| = 5$.

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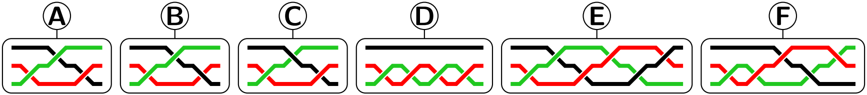

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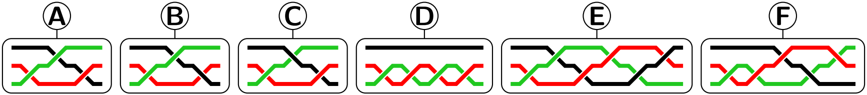

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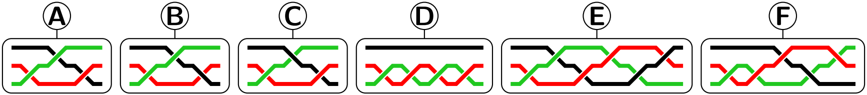

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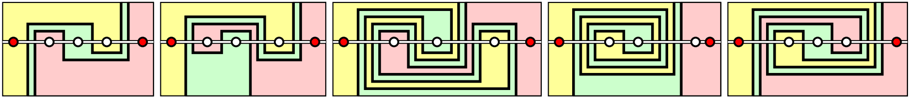

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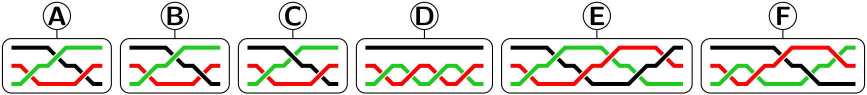

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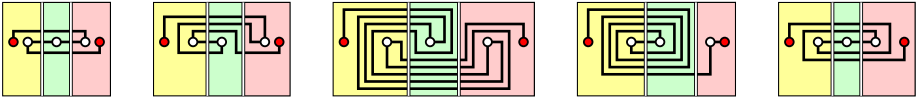

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How fast can you compute the complexity of a braid $\alpha \in \mathcal{B}_n$ of length k?

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Geometry

Braid

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Garside normal form (regular)

Class of drawings

$\mathcal{B}_n = \{$tight drawings$\}$

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Part 1/4

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Garside: $\sum_{\alpha \in \mathcal{B}_n} z^{|\alpha|}$ rational

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Artin: $\sum_{\alpha \in \mathcal{B}_{n \geqslant 4}} z^{|\alpha|}$?

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Geometric: $\sum_{\alpha \in \mathcal{B}_3} z^{|\alpha|}$ $\neg$rational, $\neg$algebraic, $\neg$holonomic
Part 2/4

Geometric: $\sum_{\alpha \in \mathcal{B}_{n \geqslant 4}} z^{|\alpha|}$?

Depth-first exploration

Width-first exploration

Braids of large size

Random walk

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(Vershik, 2000)

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Part 4/4 (with S. Abbes, S. Gouëzel & J. Mairesse)

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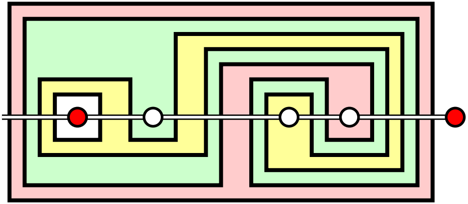

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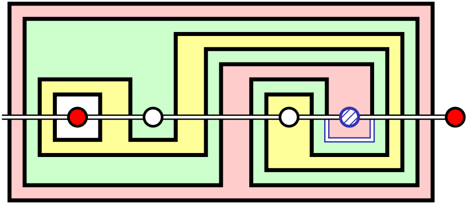

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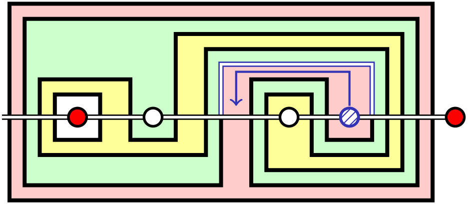

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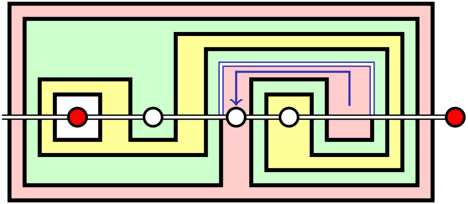

Moves performed:
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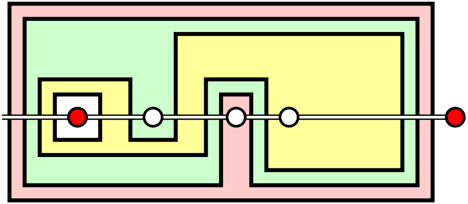

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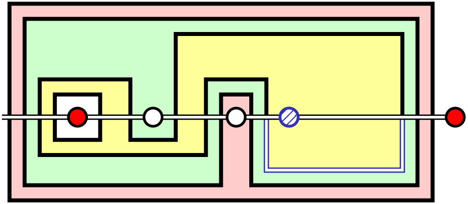

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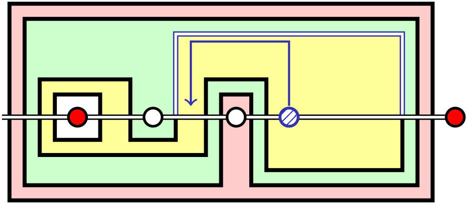

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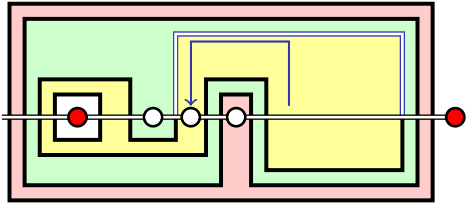

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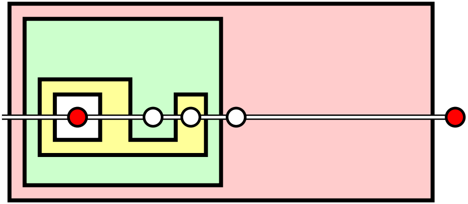

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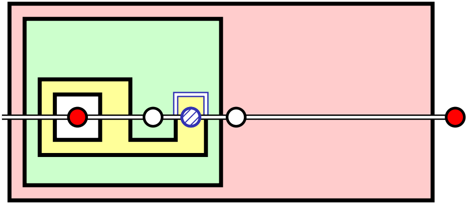

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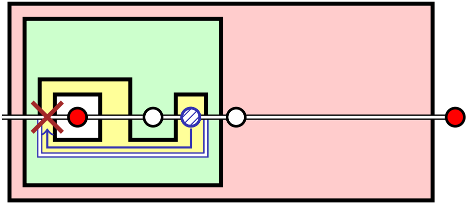

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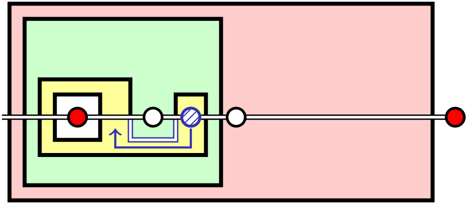
$[2 \curvearrowleft 3][2 \curvearrowleft 3]$

While your lamination is not trivial:

1. Select the rightmost (mobile) puncture that lies inside a **bigon**;
2. Slide it along its **right neighbour arc** (and remember it);
3. Relax your diagram!

What is the right relaxation normal form? (by S. Caruso & B. Wiest)

Move your rightmost tensed puncture and relax!



Moves performed:
$[2 \leftharpoondown 3][2 \leftharpoondown 3]$

While your lamination is not trivial:

1. Select the rightmost (mobile) puncture that lies inside a **bigon**;
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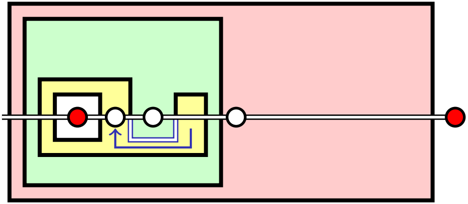

Moves performed:
[2 ↼ 3][2 ↼ 3][1 ↽ 2]

While your lamination is not trivial:

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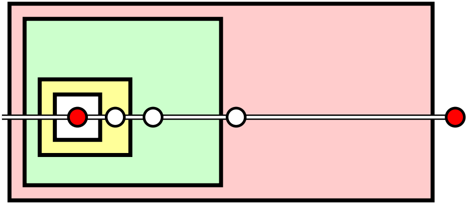
Trivial lamination!



Moves performed:
$[2 \leftharpoondown 3][2 \leftharpoondown 3][1 \leftharpoondown 2]$

While your lamination is not trivial:

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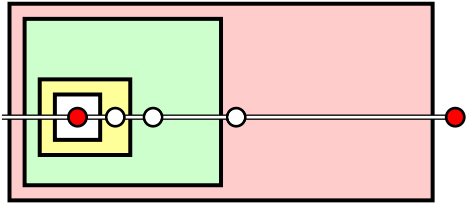

Moves performed:
$[2 \leftharpoondown 3][2 \leftharpoondown 3][1 \leftharpoondown 2]$

Relaxation normal form (**RNF**):
$[1 \rightharpoondown 2] \cdot [2 \rightharpoonup 3] \cdot [2 \rightharpoonup 3]$

$[k \rightharpoondown \ell] = \sigma_k \ldots \sigma_{\ell-1}$
$[k \rightharpoonup \ell] = \overline{\sigma_k} \ldots \overline{\sigma_{\ell-1}}$

While your lamination is not trivial:

1. Select the rightmost (mobile) puncture that lies inside a **bigon**;
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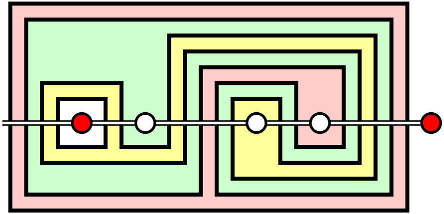
3. Relax your diagram!

Tight closed lamination, cell map and lamination/arc trees

Tight closed lamination



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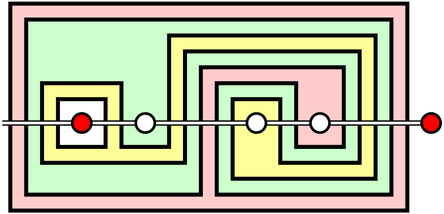
Tight closed lamination



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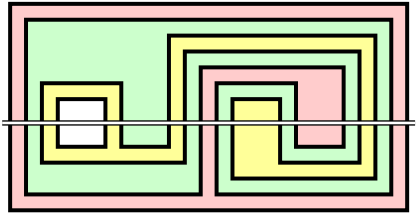

Tight closed lamination, cell map and lamination/arc trees

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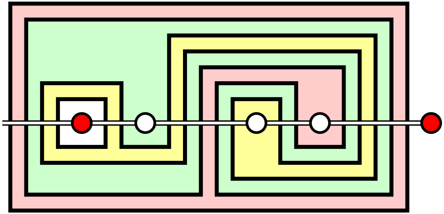
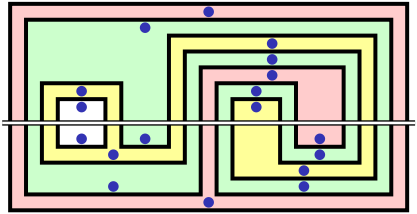
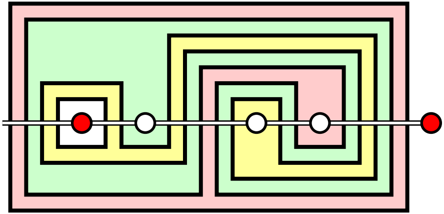


Cell map

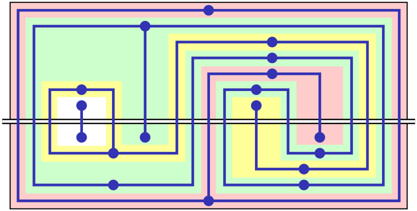


Tight closed lamination, cell map and lamination/arc trees

Tight closed lamination



Cell map



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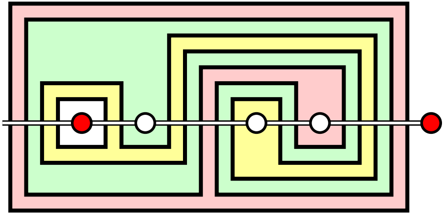

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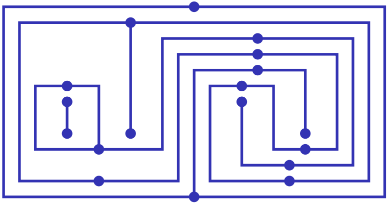
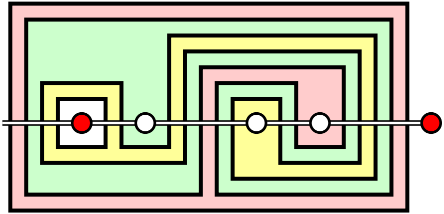


Tight closed lamination, cell map and lamination/arc trees

Tight closed lamination



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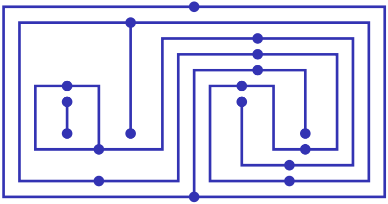
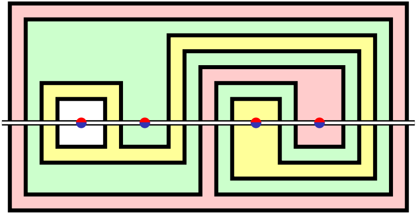
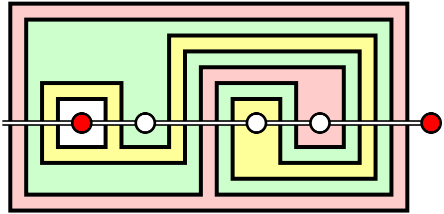


Lamination trees (**LT**)



Tight closed lamination, cell map and lamination/arc trees

Tight closed lamination



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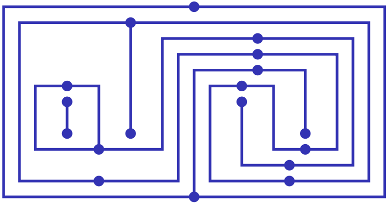

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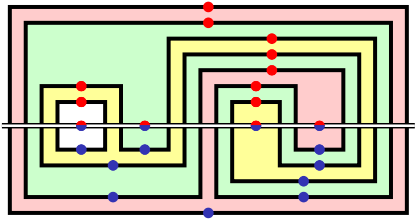
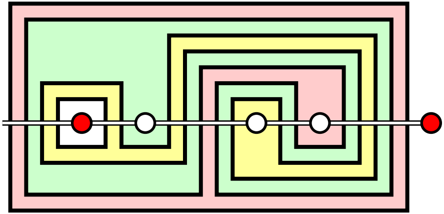


Tight closed lamination, cell map and lamination/arc trees

Tight closed lamination



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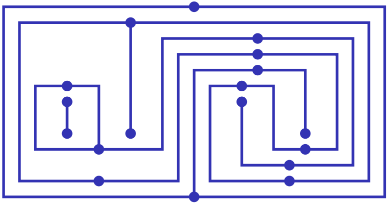

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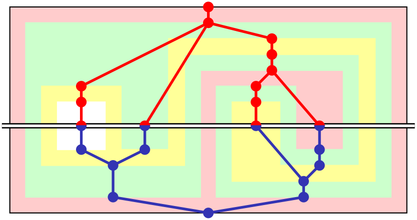

Tight closed lamination, cell map and lamination/arc trees

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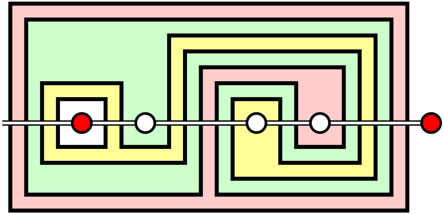

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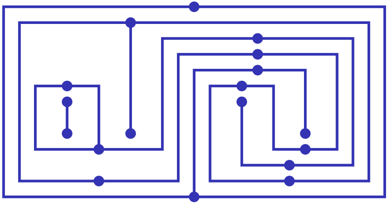

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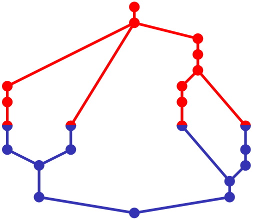

Tight closed lamination, cell map and lamination/arc trees

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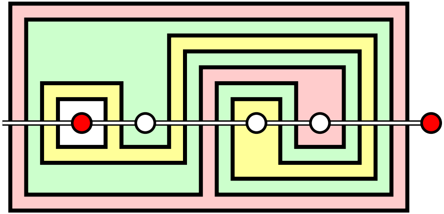

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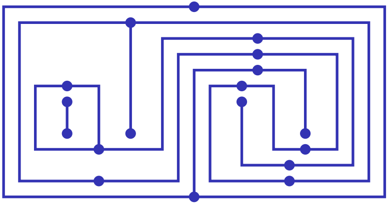

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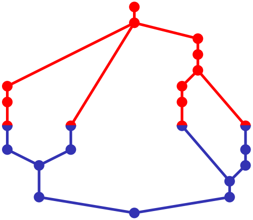
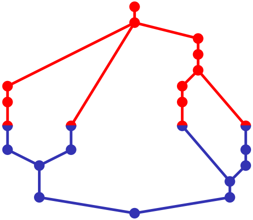


Arc trees



Tight closed lamination, cell map and lamination/arc trees

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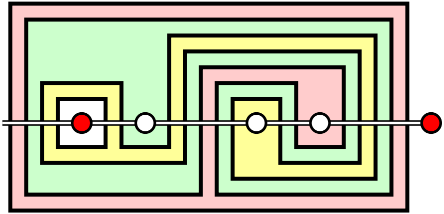

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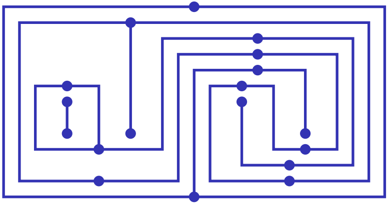

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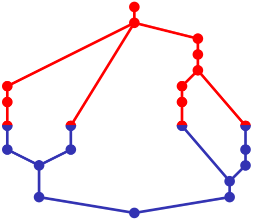

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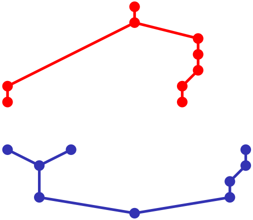

Tight closed lamination, cell map and lamination/arc trees

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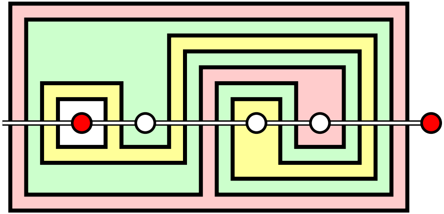

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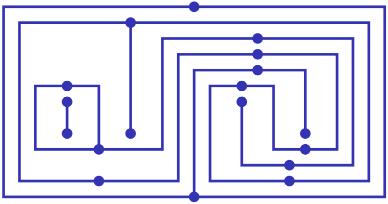

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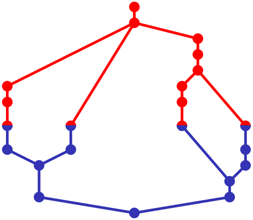

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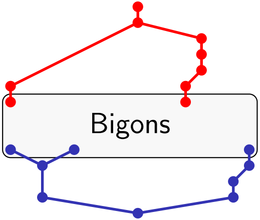

Tight closed lamination, cell map and lamination/arc trees

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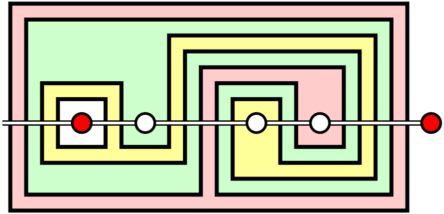

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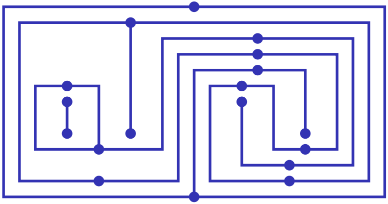

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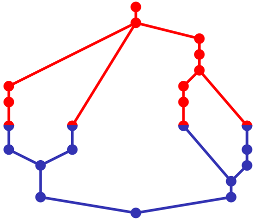
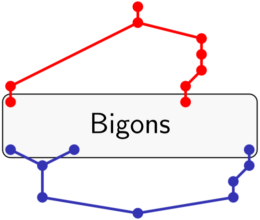


(unary-binary trees + 0/1 leaf)

Arc trees



(unary-binary trees)

After a careful case analysis...

Given two braids $\alpha \in \mathcal{B}_n$ and $[k \rightharpoondown \ell]$, remembering small-size subtrees $\mathbf{lt}(\alpha)$ of $\mathbf{LT}(\alpha)$ is enough to:

- check whether $\mathbf{RNF}(\alpha[k \rightharpoondown \ell]) = \mathbf{RNF}(\alpha) \cdot [k \rightharpoondown \ell]$;

After a careful case analysis. . .

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- compute analogous subtrees $\mathbf{lt}(\alpha[k \rightharpoondown \ell])$ (if needed).

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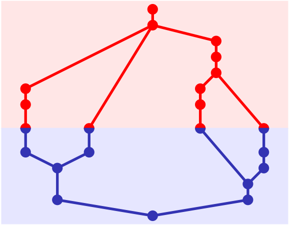

After a careful case analysis. . .

Given two braids $\alpha \in \mathcal{B}_n$ and $[k \rightharpoonup \ell]$, remembering small-size subtrees $\mathbf{lt}(\alpha)$ of $\mathbf{LT}(\alpha)$ is enough to:

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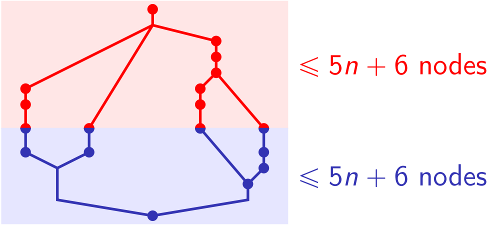
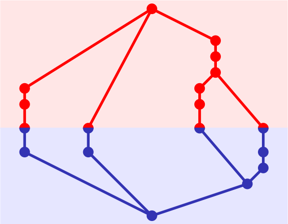


After a careful case analysis. . .

Given two braids $\alpha \in \mathcal{B}_n$ and $[k \frown \ell]$, remembering small-size subtrees $\mathbf{lt}(\alpha)$ of $\mathbf{LT}(\alpha)$ is enough to:

- check whether $\mathbf{RNF}(\alpha[k \frown \ell]) = \mathbf{RNF}(\alpha) \cdot [k \frown \ell]$;
- compute analogous subtrees $\mathbf{lt}(\alpha[k \frown \ell])$ (if needed).



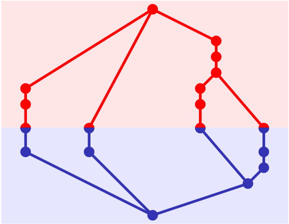
$\leqslant 5n + 6$ nodes

$\leqslant 5n + 6$ nodes

After a careful case analysis...

Given two braids $\alpha \in \mathcal{B}_n$ and $[k \rightharpoondown \ell]$, remembering small-size subtrees $\mathbf{lt}(\alpha)$ of $\mathbf{LT}(\alpha)$ is enough to:

- check whether $\mathbf{RNF}(\alpha[k \rightharpoondown \ell]) = \mathbf{RNF}(\alpha) \cdot [k \rightharpoondown \ell]$;
- compute analogous subtrees $\mathbf{lt}(\alpha[k \rightharpoondown \ell])$ (if needed).



$\leqslant 5n + 6$ nodes

$\leqslant 5n + 6$ nodes

Theorem (J. 2015)

The right relaxation normal form is **regular**.

Going further

Some additional results

- Memory requirements: nearly optimal (up to a ratio $\leqslant 20$);
- Dehornoy ordering: σ-positivity $\Leftrightarrow$ **RNF** in a regular language.

Going further

Some additional results

- Memory requirements: nearly optimal (up to a ratio $\leqslant 20$);
- Dehornoy ordering: σ-positivity $\Leftrightarrow$ **RNF** in a regular language.

and open questions

- Is the right relaxation normal form (bi-)automatic? (Yes if $n \leqslant 3$)
- Regularity of other transmission-relaxation normal forms? (wide open)

Contents

Which complexity should we look at?

Knowing the open laminated complexity of α, can we compute its:

- closed laminated complexity?

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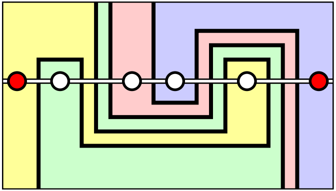
Tight open lamination



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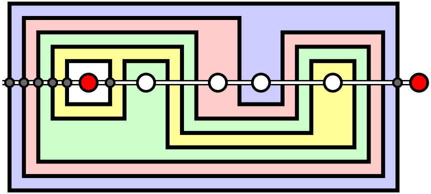

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Knowing the open laminated complexity of α, can we compute its:

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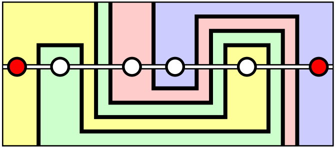
- diagrammatic complexity?

Which complexity should we look at?

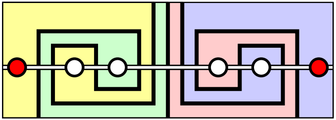
Knowing the open laminated complexity of α, can we compute its:

- closed laminated complexity? Yes: $|\alpha|_c = |\alpha|_o + n + 3$
- diagrammatic complexity? No for $n \geqslant 4$

Tight open laminations

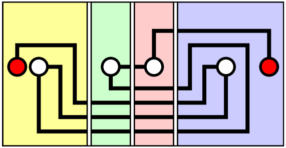


$|\sigma_1\overline{\sigma_2\sigma_3}|_o = 11$

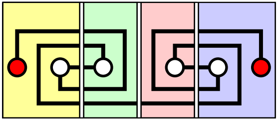


$|\sigma_1^2\overline{\sigma_3}^2|_o = 11$

Tight curve diagrams



$|\sigma_1\overline{\sigma_2\sigma_3}|_d = 13$



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Which complexity should we look at?

Knowing the open laminated complexity of α, can we compute its:

- closed laminated complexity? Yes: $|\alpha|_c = |\alpha|_o + n + 3$
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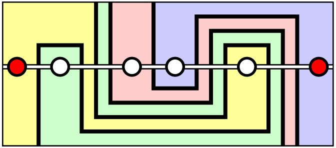
- **inverse** diagrammatic complexity?

Which complexity should we look at?

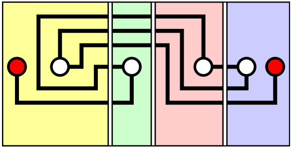
Knowing the open laminated complexity of α, can we compute its:

- closed laminated complexity? Yes: $|\alpha|_c = |\alpha|_o + n + 3$
- diagrammatic complexity? No for $n \geqslant 4$
- **inverse** diagrammatic complexity? Yes: $|\overline{\alpha}|_d = |\alpha|_o$

Tight open lamination



Inverse tight curve diagram



Which complexity should we look at?

Knowing the open laminated complexity of α, can we compute its:

- closed laminated complexity? Yes: $|\alpha|_c = |\alpha|_o + n + 3$
- diagrammatic complexity? No for $n \geqslant 4$
- **inverse** diagrammatic complexity? Yes: $|\overline{\alpha}|_d = |\alpha|_o$

Let us compute geometric generating functions!

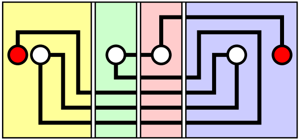
$$\mathcal{O}_n(z) = \sum_{\alpha \in \mathcal{B}_n} z^{|\alpha|_o} \qquad \mathcal{C}_n(z) = \sum_{\alpha \in \mathcal{B}_n} z^{|\alpha|_c} \qquad \mathcal{D}_n(z) = \sum_{\alpha \in \mathcal{B}_n} z^{|\alpha|_d}$$

$$\mathcal{C}_n(z) = z^{n+3}\mathcal{O}_n(z) = z^{n+3}\mathcal{D}_n(z)$$

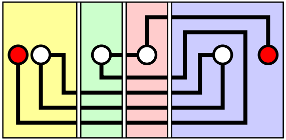
Generalised tight curve diagrams and coordinates

Generalising tight curve diagrams

Tight curve diagram



Tight generalised curve diagram



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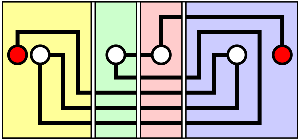

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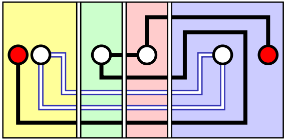

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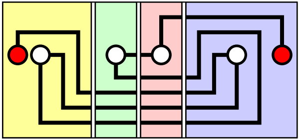

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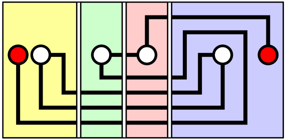

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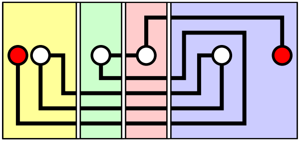

Coordinates: $\langle (x_0, x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle$

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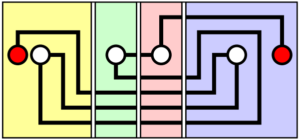

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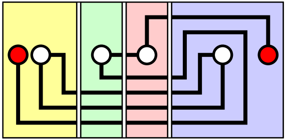

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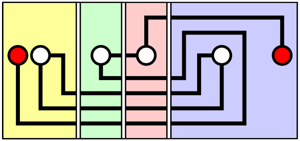

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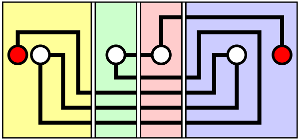

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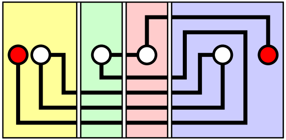

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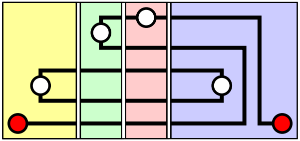

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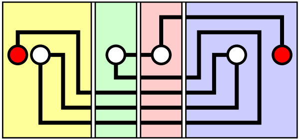

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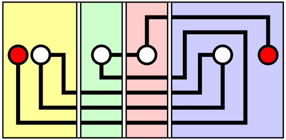

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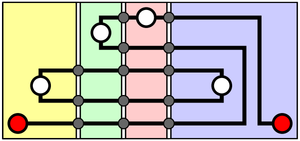

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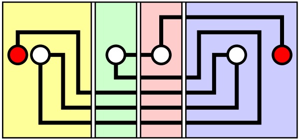

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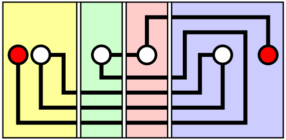

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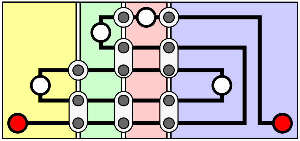

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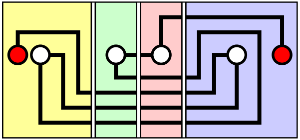

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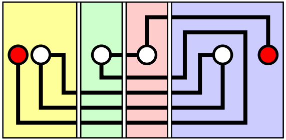

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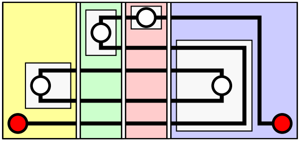

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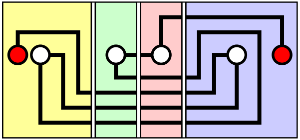

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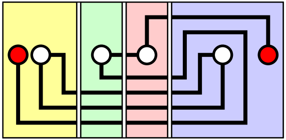

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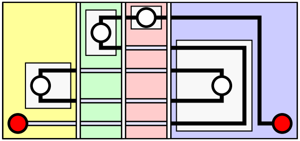

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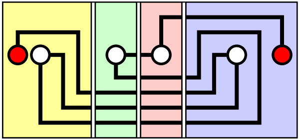

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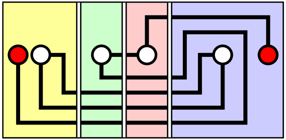

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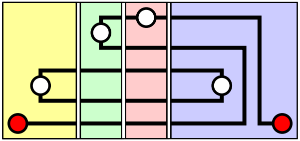

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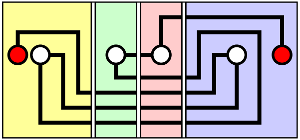

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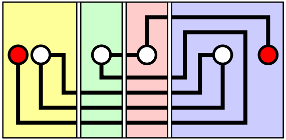

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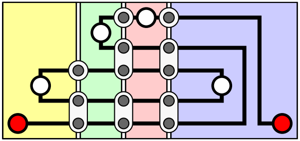

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$\sum_{i=1}^{n-1} x_i = \frac{|\alpha|_d + 1 - n}{2}$: Let us compute $\mathcal{G}_n(z) = \sum_{k \geqslant 0} g_{n,k} z^k = \sqrt{z}^{1-n} \mathcal{D}_n(\sqrt{z})$!

And finally...

Theorem (J. 2015)

In the 3-strand braid group $\mathcal{B}_3$, we have:

- $\mathcal{G}_3(z) = 2\frac{1+2z-z^2}{z^2(1-z^2)}\left(\sum_{k \geqslant 3} \varphi(k) z^k\right) + \frac{1-3z^2}{1-z^2}$ and
- $g_{3,k} = \mathbf{1}_{k=0} + 2\left(\varphi(k+2) - \mathbf{1}_{k \in 2\mathbb{Z}} + 2\sum_{i=1}^{\lfloor k/2 \rfloor} \varphi(k+3-2i)\right)\mathbf{1}_{k \geqslant 1}$,

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$\mathcal{G}_3(z)$ is **more complicated** than $\sum_{\alpha\in\mathcal{B}_3} z^{|\alpha|_{\mathsf{Artin}}} = \frac{(1+z)(1-z+z^2-2z^3)}{(2-z)(1-2z)(1-z-z^2)}$!

Contents

Combinatorics of positive braids

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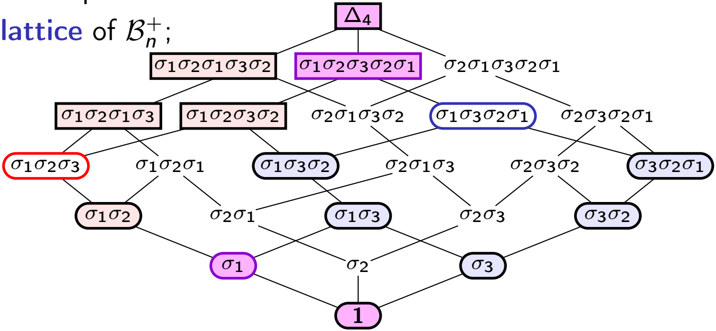

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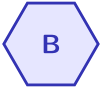
Consequence on Garside normal forms:

4. local **neighbouring criterion**: $w_1 \cdot w_2 \cdot \ldots \cdot w_k \in \mathbf{Gar}(\mathcal{B}_n^+)$ iff
   - $w_1, \ldots, w_k \in \mathcal{S}_n \backslash \{\mathbf{1}\}$;
   - $\mathbf{R}(w_i) \supseteq \mathbf{L}(w_{i+1})$ for all $i < k$.

($\mathbf{L}(\alpha) = \{\sigma_i : \sigma_i \preccurlyeq \alpha\}$ and $\mathbf{R}(\alpha) = \{\sigma_i : \alpha \succcurlyeq \sigma_i\}$)

Generalising braid monoids



- **Braid** monoid: $\langle \sigma_i \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i \rangle^+$;

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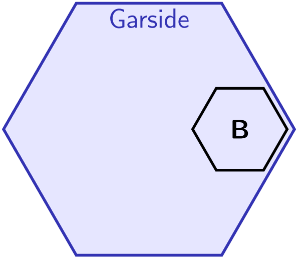

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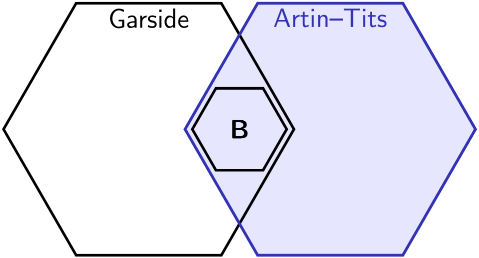
- **Garside** monoid: finite generating set Σ, cancellative, $\leqslant$- and $\geqslant$-lattice, with a **Garside element** Δ such that $\{x : x \leqslant \Delta\} = \{x : \Delta \geqslant x\} \supseteq \Sigma$;

Generalising braid monoids



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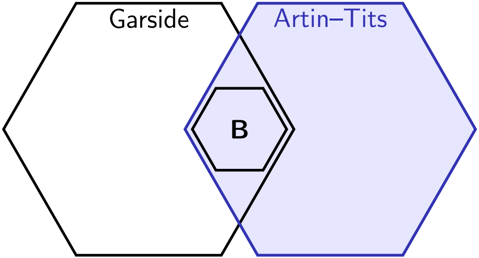
- **Artin–Tits** monoid: $\langle \sigma_i \mid [\sigma_i \sigma_j]^{\ell(i,j)} = [\sigma_j \sigma_i]^{\ell(i,j)} \rangle^+$;

$$\ell(i,j) = 2 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i$$

Generalising braid monoids



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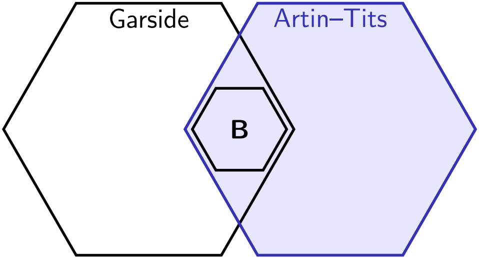
$$\ell(i,j) = 3 \Rightarrow \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$$

Generalising braid monoids



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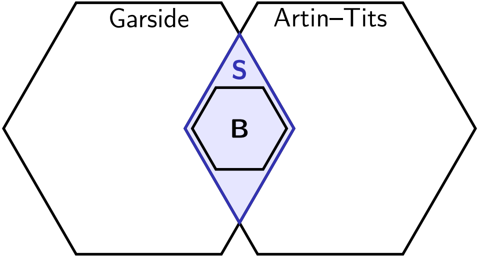
  $\ell(i,j) = +\infty \Rightarrow$ no relation!

Generalising braid monoids



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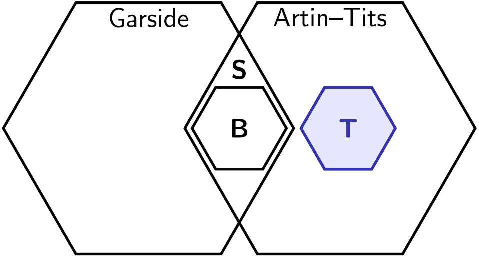
Generalising braid monoids



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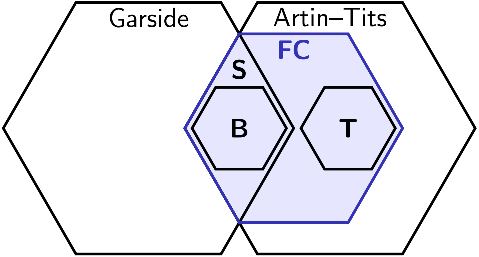
Generalising braid monoids



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Contents

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Chronological form: 

Left Garside normal form: 

Right Garside normal form: 

Left$^\Delta$ Garside normal form: 

1^{st} step

Random walk in a braid monoid (with J. Mairesse)

Random walk

1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ uniformly chosen in $\{\sigma_1, \ldots, \sigma_{n-1}\}$.
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Chronological form: 

Left Garside normal form: 

Right Garside normal form: 

Left$^\Delta$ Garside normal form: 

2[nd] step

Random walk in a braid monoid (with J. Mairesse)

Random walk

1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ uniformly chosen in $\{\sigma_1, \ldots, \sigma_{n-1}\}$.
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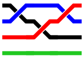
Chronological form: 

Left Garside normal form: 

Right Garside normal form: 

Left$^\Delta$ Garside normal form: 

3rd step

Random walk in a braid monoid (with J. Mairesse)

Random walk

1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ uniformly chosen in $\{\sigma_1, \ldots, \sigma_{n-1}\}$.
2. Random process $(X_k)_{k \geqslant 0}$ defined by $X_0 = \mathbf{1}$ and $X_{k+1} = X_k Y_k$.

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Chronological form: 

Left Garside normal form: 

Right Garside normal form: 

Left$^\Delta$ Garside normal form: 

4^{th} step

Random walk in a braid monoid

(with J. Mairesse)

Random walk

1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ uniformly chosen in $\{\sigma_1, \ldots, \sigma_{n-1}\}$.
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Several Garside normal forms:

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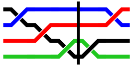
Chronological form:

Left Garside normal form:

Right Garside normal form:



Left$^\Delta$ Garside normal form:

5$^{\text{th}}$ step

Random walk in a braid monoid (with J. Mairesse)

Random walk

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1. $\mathbf{Gar}_\ell(\beta) = \beta_1 \cdot \ldots \cdot \beta_k$ with $\mathbf{R}(\beta_i) \supseteq \mathbf{L}(\beta_{i+1})$;
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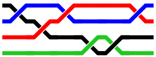
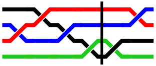
Chronological form:

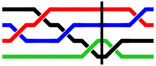


Left Garside normal form:

Right Garside normal form:



Left$^\Delta$ Garside normal form:



6^{th} step

Random walk in a braid monoid (with J. Mairesse)

Random walk

1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ uniformly chosen in $\{\sigma_1, \ldots, \sigma_{n-1}\}$.
2. Random process $(X_k)_{k \geqslant 0}$ defined by $X_0 = \mathbf{1}$ and $X_{k+1} = X_k Y_k$.

Several Garside normal forms:

1. $\mathbf{Gar}_\ell(\beta) = \beta_1 \cdot \ldots \cdot \beta_k$ with $\mathbf{R}(\beta_i) \supseteq \mathbf{L}(\beta_{i+1})$;
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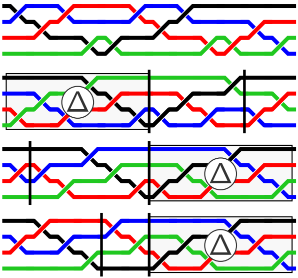
Chronological form:

Left Garside normal form:

Right Garside normal form:

Left$^\Delta$ Garside normal form:



12^{th} step

Random walk in a braid monoid (with J. Mairesse)

Random walk

1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ uniformly chosen in $\{\sigma_1, \ldots, \sigma_{n-1}\}$.
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Several Garside normal forms:

1. $\mathbf{Gar}_\ell(\beta) = \beta_1 \cdot \ldots \cdot \beta_k$ with $\mathbf{R}(\beta_i) \supseteq \mathbf{L}(\beta_{i+1})$;
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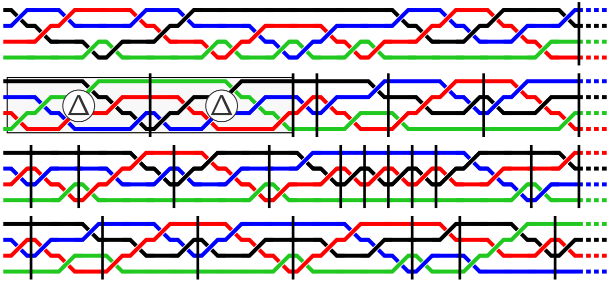
Chronological form:

Left Garside normal form:

Right Garside normal form:

Left$^\Delta$ Garside normal form:



50^{th} step

Random walk in a braid monoid (with J. Mairesse)

Random walk

1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ uniformly chosen in $\{\sigma_1, \ldots, \sigma_{n-1}\}$.
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Several Garside normal forms:

1. $\mathbf{Gar}_\ell(\beta) = \beta_1 \cdot \ldots \cdot \beta_k$ with $\mathbf{R}(\beta_i) \supseteq \mathbf{L}(\beta_{i+1})$;
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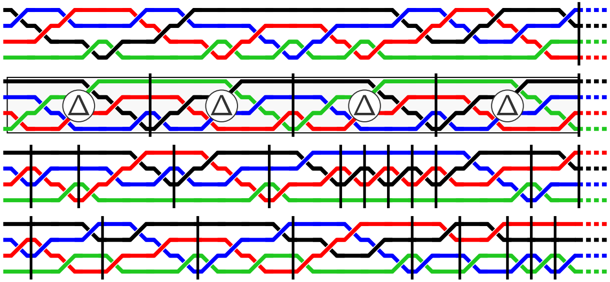
Chronological form:

Left Garside normal form:

Right Garside normal form:

Left$^\Delta$ Garside normal form:



100$^{\text{th}}$ step

Random walk in a braid monoid (with J. Mairesse)

Random walk

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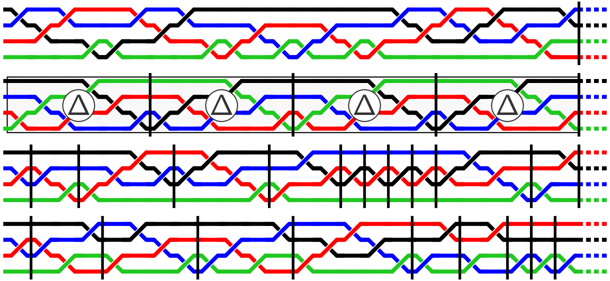
Chronological form:

Left Garside normal form:

Right Garside normal form:

Left$^\Delta$ Garside normal form:



150^{th} step

Random walk in a braid monoid (with J. Mairesse)

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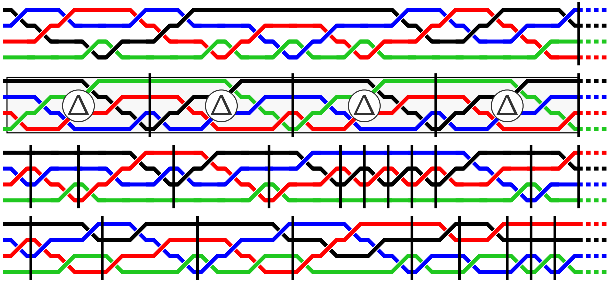
Chronological form:

Left Garside normal form:

Right Garside normal form:

Left$^\Delta$ Garside normal form:



∞^{th} step

Blocking patterns: Going to infinity…

Blocking pattern

Braid $\mathbf{P} \in \mathcal{B}_n^+$ such that, for all $\alpha, \beta \in \mathcal{B}_n^+$ such that $\Delta_n \not\leqslant \alpha\mathbf{P}\beta$:

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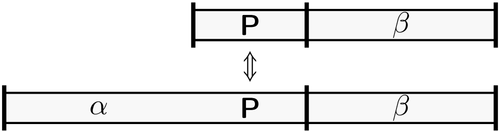

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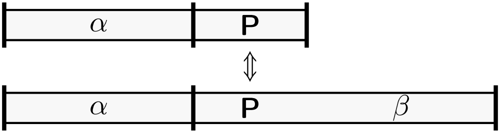

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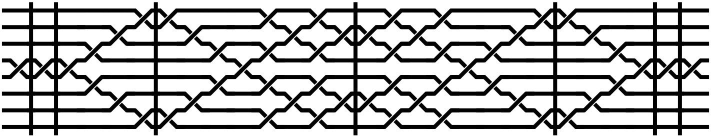
Blocking patterns exist in all braid monoids! (Caruso & Wiest 2012)



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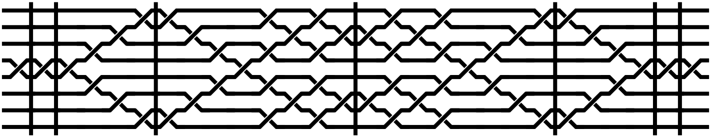

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Some properties of blocking patterns

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Theorem (J. & Mairesse 2016$^+$)

Prefixes of the words $\mathbf{Gar}_r(X_k)_{k\geqslant 0}$ almost surely converge.

. . . and beyond!

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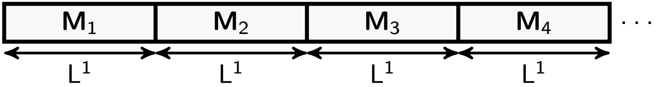

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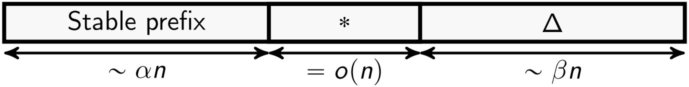

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Going even further

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(with S. Abbes,
S. Gouëzel &
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Another approach to uniform sampling

Smoothening uniform measures on spheres

1. Choose $\mu_p : \alpha \mapsto H_n(p) p^{|\alpha|}$ $\qquad (H_n(p) = \sum_{I \subseteq \{\sigma_1, \ldots, \sigma_{n-1}\}} (-1)^{\# I} p^{|\Delta_I|})$;

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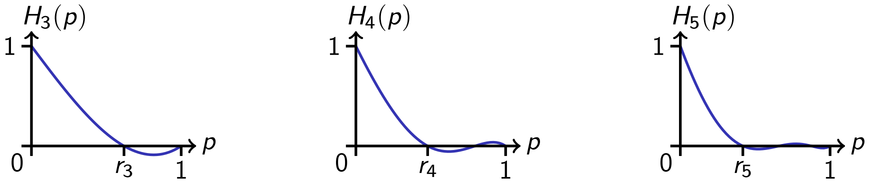
What if $H_n(p) \to 0$? (i.e. $p \to r_n$)



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Theorem (Abbes, Gouëzel, J. & Mairesse 2016$^+$)

Uniform probability measures on $\mathcal{B}_n^k$ **converge weakly** towards μ_∞ when $k \to +\infty$.

μ_∞ is a uniform probability measure on infinite braids!

Going further

Stable region conjectures (Gebhardt & Tawn 2014)

1. The words $\{\mathbf{Gar}_\ell(\beta) \mid \beta \in \partial\mathcal{B}_n^+\}$ contain a geometric number of Δ_n;

$\mathbf{Gar}_\ell(\beta)$:	Δ	Δ	Δ	$\lambda^1(\beta)$	$\lambda^2(\beta)$	$\lambda^3(\beta)$	$\lambda^4(\beta)$

$$\mathbb{P}[\#\Delta_n = k] = r_n^{k|\Delta_n|}$$

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Going to higher dimensions

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Going further

Stable region conjectures (Gebhardt & Tawn 2014)

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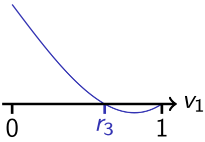
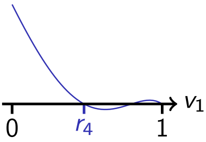
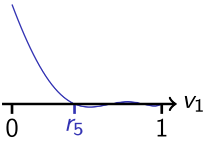


Monoid $\mathcal{B}_3^+$



Monoid $\mathcal{B}_4^+$



Monoid $\mathcal{B}_5^+$

Going further

Stable region conjectures (Gebhardt & Tawn 2014)

1. The words $\{\mathbf{Gar}_\ell(\beta) \mid \beta \in \partial\mathcal{B}_n^+\}$ contain a geometric number of Δ_n;
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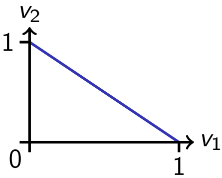
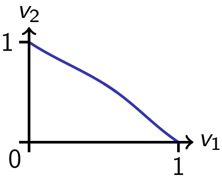
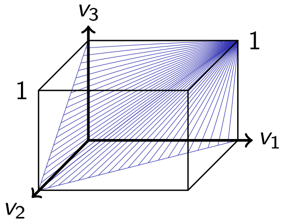


Monoid $\mathbb{N} * \mathbb{N}$



Monoid $\langle a, b \mid abab = baba\rangle^+$



Monoid $\langle a, b, c \mid ac = ca\rangle^+$

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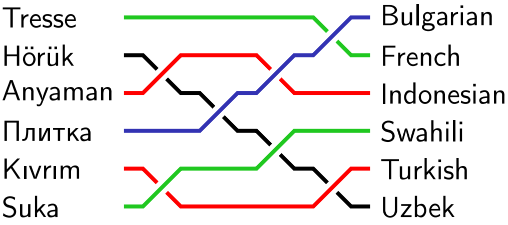
Uniform measures

4. Uniform measures on positive spheres converge towards a simple critical Markov process (for all irreducible A–T monoids of FC type).



Do you have questions?

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