Combinatorics of braids and Garside normal forms

Vincent Jugé

ALEA Young Researchers' Workshop 2017

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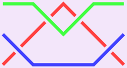

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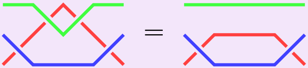

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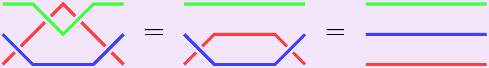

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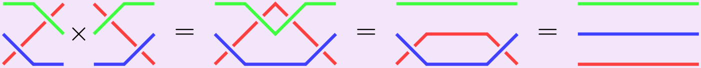

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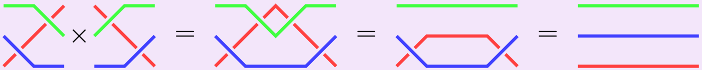
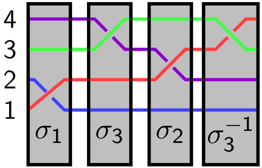


Useful notations:



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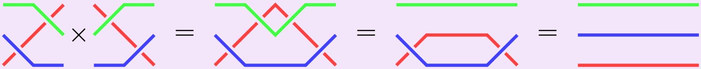

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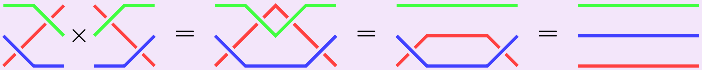

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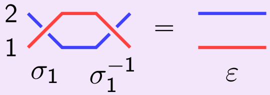

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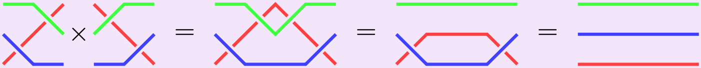

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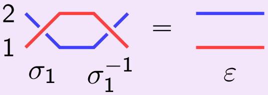

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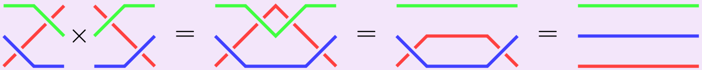

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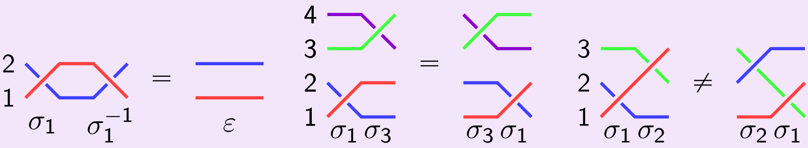

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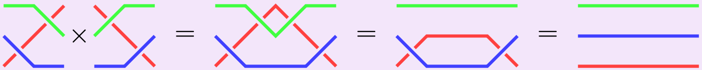

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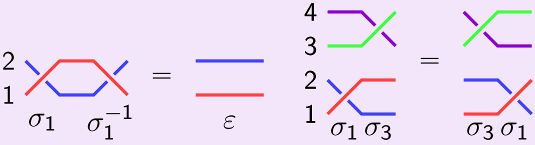

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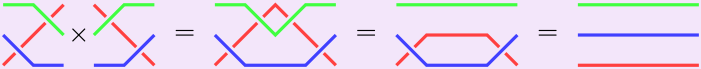

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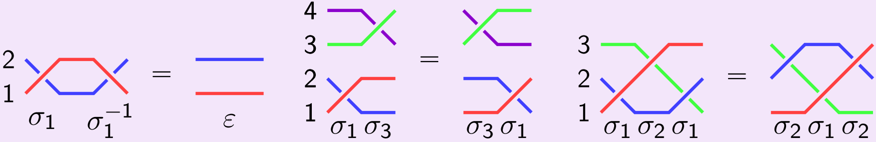

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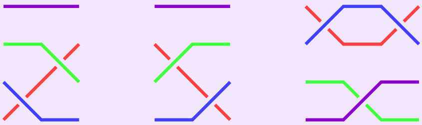

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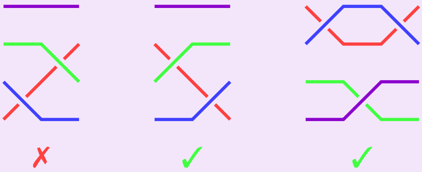

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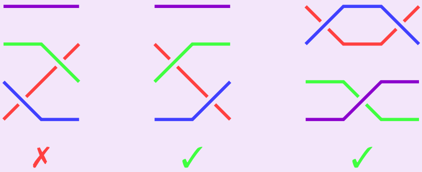
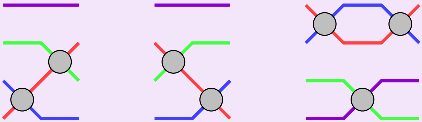


What are permutations?

2. Braids where we do not know which strand is in the foreground ($\sigma_i = \sigma_i^{-1}$)



Divisibility and positive braids

Divisibility in non-negative integers: $a \mid b$

An integer a divides an integer b iff $\exists$ an integer c such that $b = a \times c$.

$3 \mid 12$, $2 \mid 14$, $7 \mid 0$ and $0 \mid 0$ but $4 \nmid 7$

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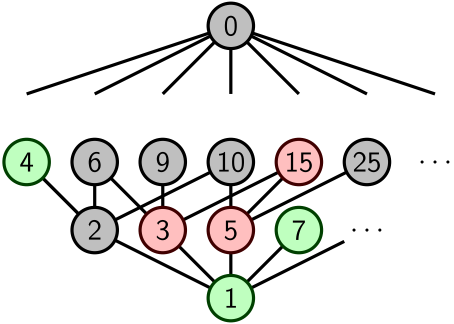
Divisibility in positive braids: $\alpha \leqslant_\ell \beta$ and $\beta \geqslant_r \alpha$

1. The braid α **left-**divides the braid β iff $\exists$ a braid γ s.t. $\beta = \alpha \times \gamma$.
2. The braid α **right-**divides the braid β iff $\exists$ a braid γ s.t. $\beta = \gamma \times \alpha$.

$$\sigma_1 \leqslant_\ell \sigma_1\sigma_2\sigma_1,\ \sigma_1 \leqslant_\ell \sigma_2\sigma_1\sigma_2,\ \sigma_1\sigma_2\sigma_1 \geqslant_r \sigma_1 \text{ and } \sigma_2 \leqslant_\ell \sigma_2\sigma_1 \text{ but } \sigma_1 \not\leqslant_\ell \sigma_2\sigma_1$$

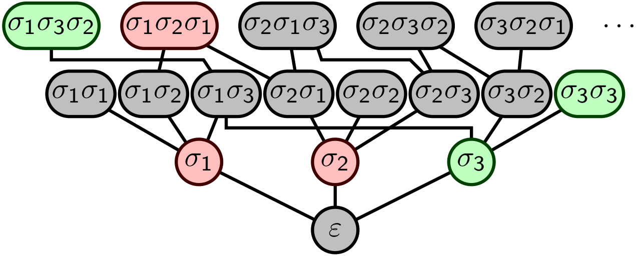
Divisibility diagrams: GCD and LCM

In non-negative integers: $\mathrm{GCD}(4,7) = 1$ and $\mathrm{LCM}(3,5) = 15$



Divisibility diagrams: GCD and LCM

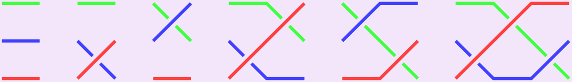
In 4-strand positive braids (**left** division):
$\mathsf{GCD}(\sigma_1\sigma_3\sigma_2, \sigma_3\sigma_3) = \sigma_3$ and $\mathsf{LCM}(\sigma_1, \sigma_2) = \sigma_1\sigma_2\sigma_1$



Showing that there exist GCDs and LCMs: simple braids

What are simple braids?

Braids whose strands cross at most once.



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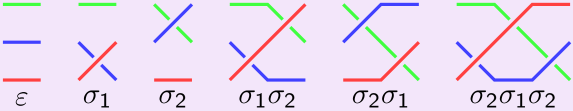

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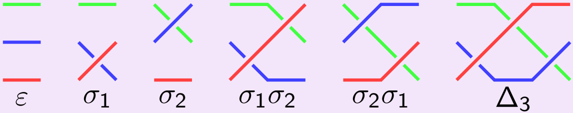

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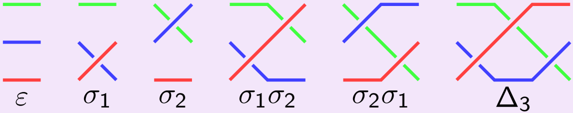

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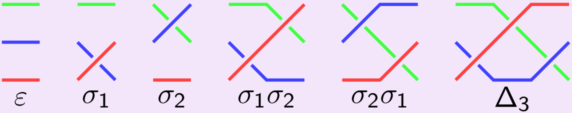

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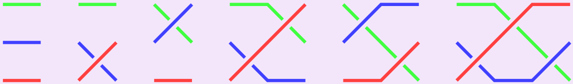

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4	4
3	3
2	2
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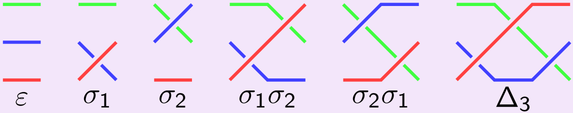

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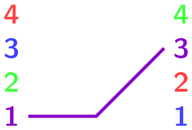

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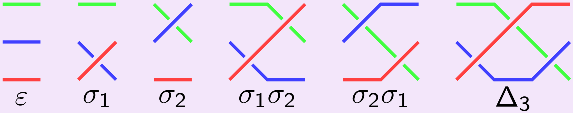

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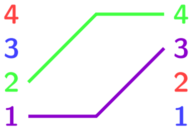

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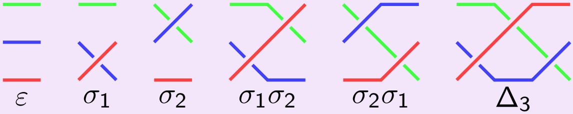

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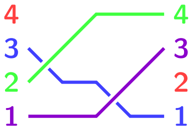

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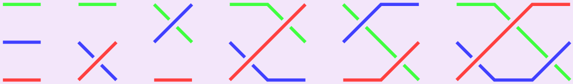

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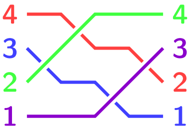

Divisibility of simple braids

For a simple β, let $\mathcal{L}(\beta) = \{(i, j) : i < j, \textbf{strand}_{i\rightarrow} \text{ crosses } \textbf{strand}_{j\rightarrow}\}$.

If you please – draw me a $\mathcal{L}(\beta)$

The set **S** belongs to $\{\mathcal{L}(\beta) \mid \beta \text{ is simple}\}$ if and only if, for all $i < j < k$:

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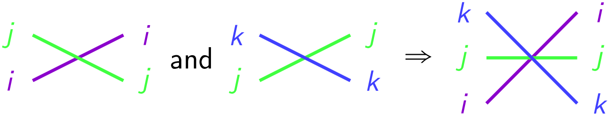

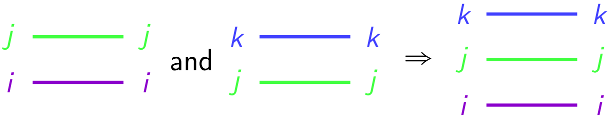

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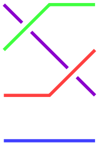

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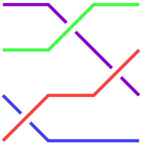
4. We have $\mathbf{S} = \mathcal{L}(\sigma_i\gamma)$!

$$\mathbf{S} = \{(1,2),(1,4),(3,4)\}$$



Divisibility of simple braids

For a simple β, let $\mathcal{L}(\beta) = \{(i,j) : i < j, \textbf{strand}_{i\rightarrow} \text{ crosses } \textbf{strand}_{j\rightarrow}\}$.

If you please – draw me a $\mathcal{L}(\beta)$

The set $\mathbf{S}$ belongs to $\{\mathcal{L}(\beta) \mid \beta \text{ is simple}\}$ if and only if, for all $i < j < k$:

- $(i,j) \in \mathbf{S}$ and $(j,k) \in \mathbf{S} \Rightarrow (i,k) \in \mathbf{S}$, and
- $(i,j) \notin \mathbf{S}$ and $(j,k) \notin \mathbf{S} \Rightarrow (i,k) \notin \mathbf{S}$.

1. Choose a pair $(i, i+1)$ in $\mathbf{S}$ and delete it.
2. Swap i's and $(i+1)$'s.
3. Build a braid γ by induction on $|\mathbf{S}|$.
4. We have $\mathbf{S} = \mathcal{L}(\sigma_i \gamma)$!

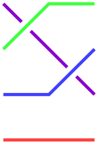
Bonus:

$\beta \leqslant_\ell \gamma$ iff $\mathcal{L}(\beta) \subseteq \mathcal{L}(\gamma)$

Obtaining GCDs and LCMs 1/3

GCDs and LCMs for simple braids

1. Simple braids have LCMs.



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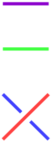

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Obtaining GCDs and LCMs 1/3

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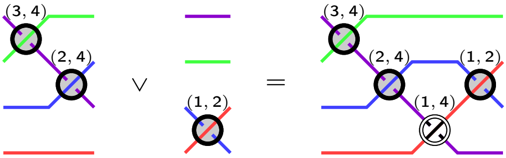

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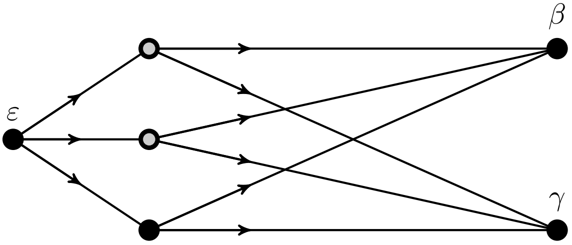
2. Simple braids have GCDs:
   $\mathbf{GCD}(\beta,\gamma) = \mathbf{LCM}(\{\delta \mid \delta \leqslant_\ell \beta \text{ and } \delta \leqslant_\ell \gamma\})$.



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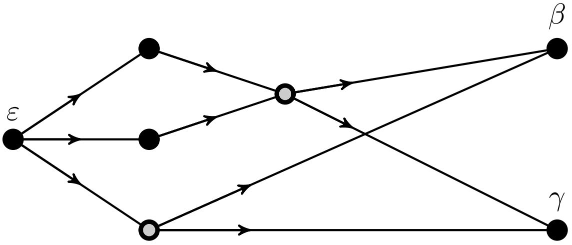

Obtaining GCDs and LCMs 1/3

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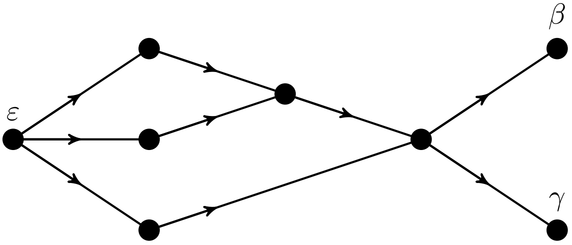
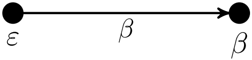


Obtaining GCDs and LCMs 2/3

The greatest simple divisor

β and γ are simple braids

1. Complement $\mathbf{C}(\beta, \gamma)$



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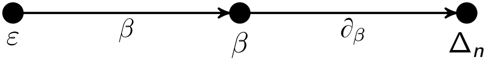

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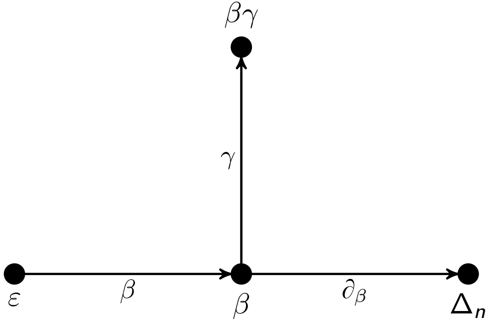

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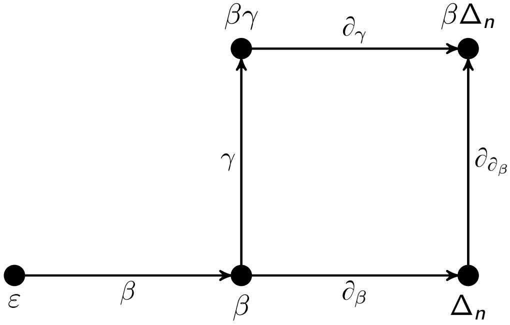

Obtaining GCDs and LCMs 2/3

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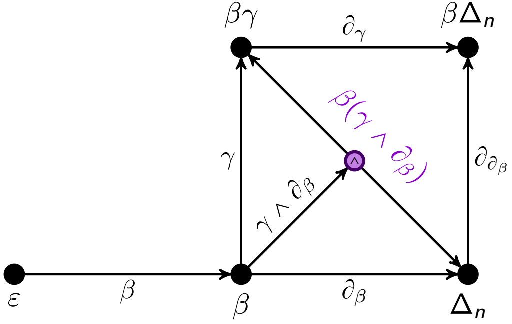

Obtaining GCDs and LCMs 2/3

The greatest simple divisor — $x_1, x_2, \ldots, x_k, \beta$ and γ are simple braids

1. Complement $\mathbf{C}(\beta, \gamma) = \beta(\gamma \wedge \partial_\beta) = \bigvee\{x \text{ simple} \mid \beta \leqslant_\ell x \leqslant_\ell \beta\gamma\}$
2. Head $\mathbf{H}(x_1, x_2, \ldots, x_k) = \mathbf{C}(x_1, \mathbf{H}(x_2, \ldots, x_k))$ — $\mathbf{H}(\cdot) = \varepsilon$

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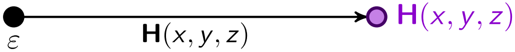

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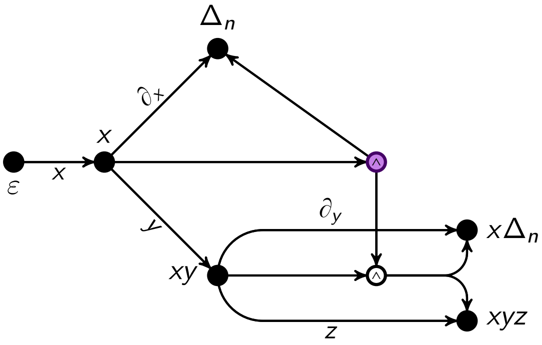

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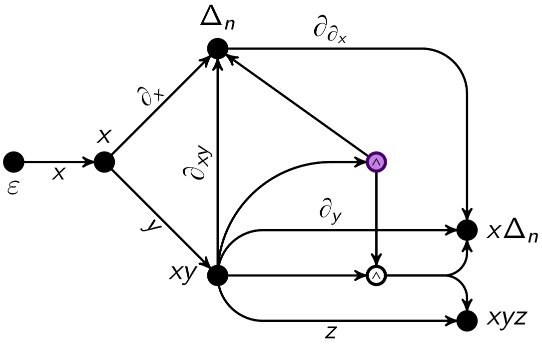

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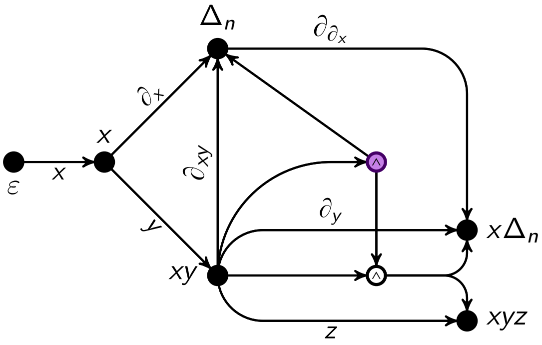

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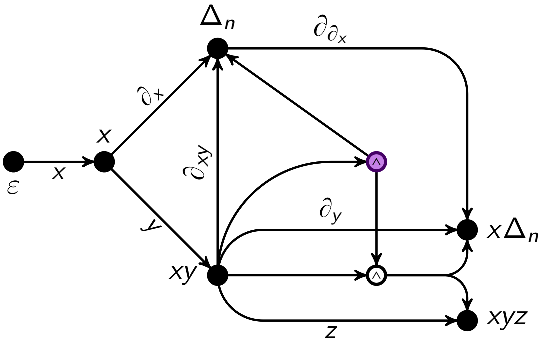

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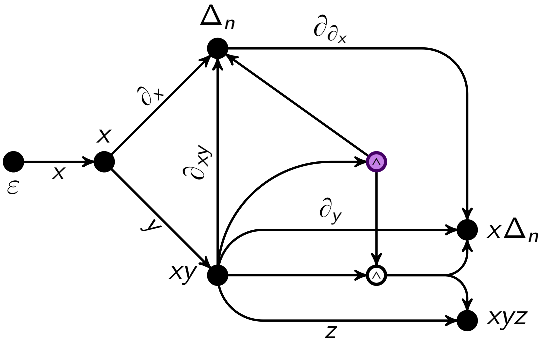
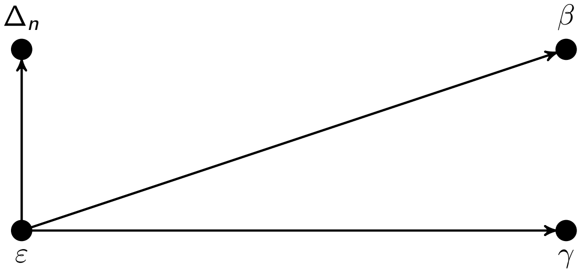


Obtaining GCDs and LCMs 3/3

GCDs and LCMs for everybody

1. Positive braids have GCDs.



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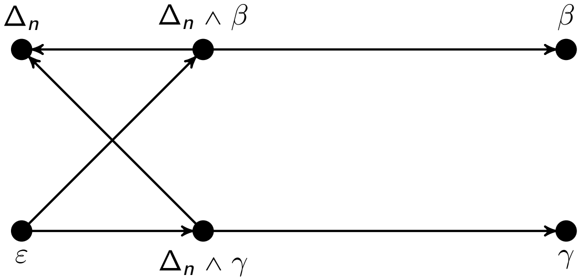

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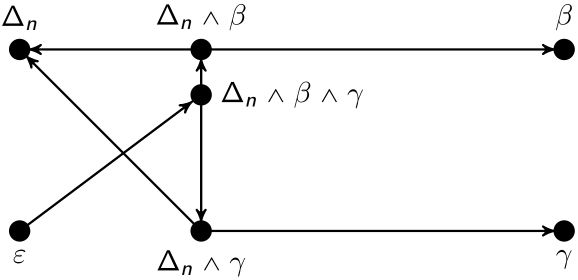

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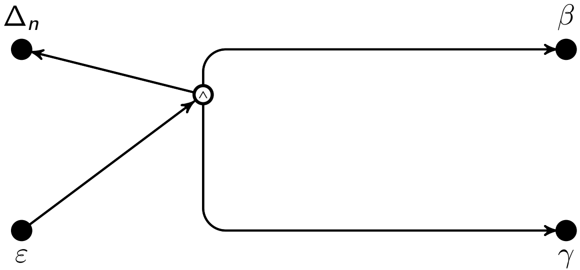

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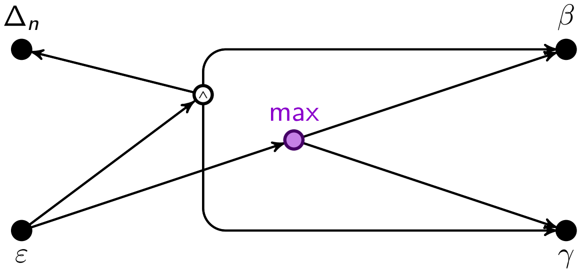

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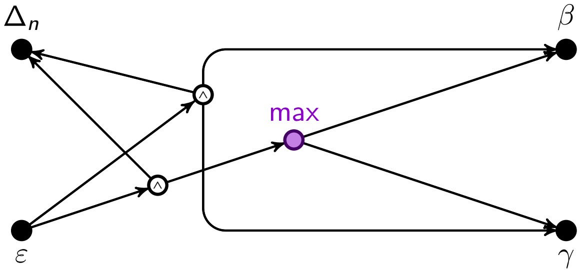

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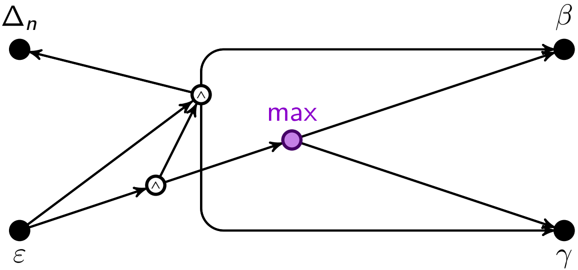

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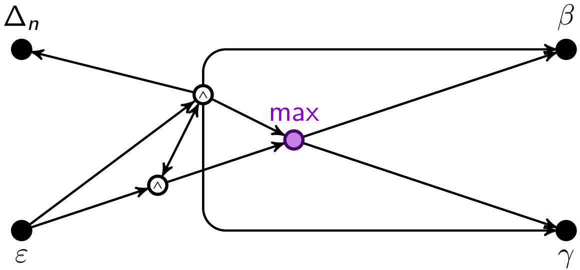

Obtaining GCDs and LCMs 3/3

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1. Positive braids have GCDs.
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Three lemmas

3. $\sigma_i \Delta_n \Delta_n = \Delta_n \Delta_n \sigma_i$ for all i

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3. $\sigma_i \Delta_n \Delta_n = \Delta_n \Delta_n \sigma_i$ for all i
4. $\beta \Delta_n^2 = \Delta_n^2 \beta$ for all β

$$\sigma_1 \sigma_3 \sigma_2 \sigma_1 \Delta_n^2 = \sigma_1 \sigma_3 \sigma_2 \Delta_n^2 \sigma_1 = \ldots = \Delta_n^2 \sigma_1 \sigma_3 \sigma_2 \sigma_1$$

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Three lemmas

3. $\sigma_i \Delta_n \Delta_n = \Delta_n \Delta_n \sigma_i$ for all i
4. $\beta \Delta_n^2 = \Delta_n^2 \beta$ for all β
5. $\beta \leqslant_\ell \Delta_n^{2|\beta|}$ for all β

$$\sigma_1\sigma_3\sigma_2\sigma_1 \leqslant_\ell \sigma_1\sigma_3\sigma_2\Delta_n^2 = \Delta_n^2\sigma_1\sigma_3\sigma_2 \leqslant_\ell \Delta_n^2\sigma_1\sigma_3\Delta_n^2 \leqslant_\ell \ldots \leqslant_\ell \Delta_n^8$$

Thank you!