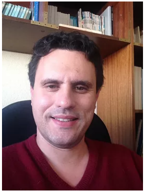
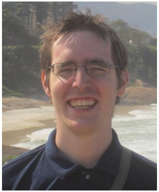
Growth rates of braid monoids with many generators

Ramón Flores[1], Juan González-Meneses[1] & Vincent Jugé[2]





1: Universidad de Sevilla – 2: Université Gustave Eiffel (LIGM)

15/04/2020

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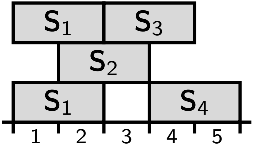

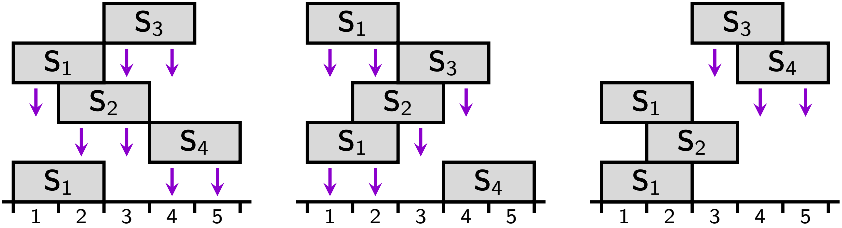

Heaps of pieces vs Trace monoids

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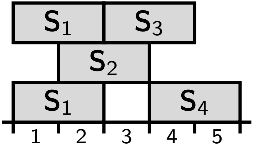
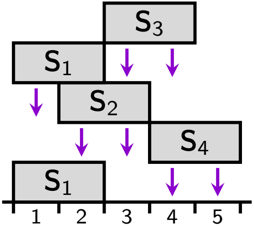


Trace monoid

$$\mathsf{T}_4 = \left\langle \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3, \mathsf{S}_4 \;\middle|\; \begin{array}{l} \mathsf{S}_i\mathsf{S}_j = \mathsf{S}_j\mathsf{S}_i \\ \text{if } i \neq j \pm 1 \end{array} \right\rangle^+$$



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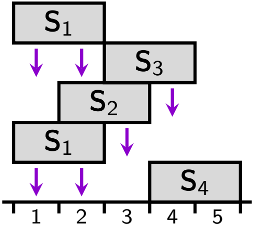

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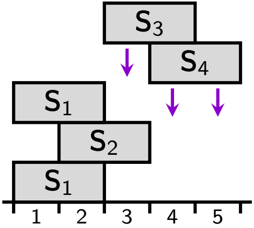

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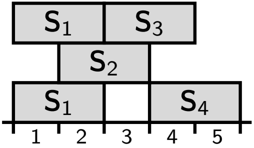

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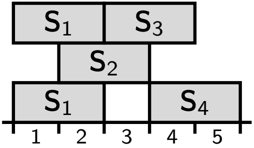

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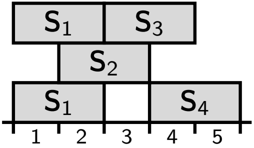

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How does $\lambda_{4,k}$ behave when $k \to +\infty$?

Growth rate of a finitely generated monoid

In a monoid **M** generated by a finite family $\mathcal{F}$,

- $|\tau| = \#$generators (in $\mathcal{F}$) needed to write τ
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Proof: If $m_k = 0$ for some $k \geqslant 0$, then $m_\ell = 0$ for all $\ell \geqslant k$.
Otherwise, set $x_k = (\log m_k)/k \geqslant 0$ and $\mathbf{X}_k = \max\{x_1, \ldots, x_k\}$.
For all $\ell \leqslant k$ and $q \geqslant 1$, we have

$$x_{qk+\ell} \leqslant (k\,q\,x_k + \ell\,x_\ell)/(q\,k + \ell) \leqslant x_k + \mathbf{X}_k/q \xrightarrow{q\to\infty} x_k$$

and thus $\limsup_{k\to\infty} x_k \leqslant x_k$.

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How does $\lambda_{4,k}$ precisely behave when $k \to +\infty$?

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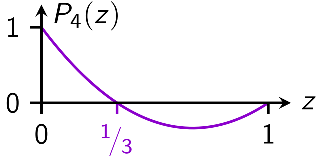
Möbius polynomial

$$P_4(z) = \mathcal{G}_4(z)^{-1} = 1 - 4z + 3z^2$$



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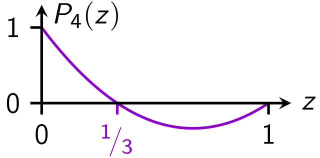

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Corollary: $\lambda_{4,k}^{1/k} \to 1/\rho_4 = 3$

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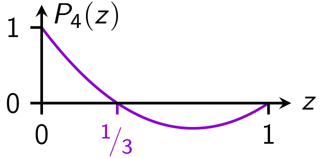
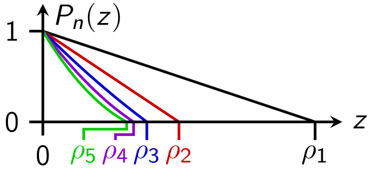


Growth rates of wide trace monoids

How does ρ_n behave when $n \to +\infty$?

Recurrence equation

$$P_{-1}(z) = P_0(z) = 1$$
$$P_n(z) = P_{n-1}(z) - zP_{n-2}(z) \quad \text{if } n \geqslant 1$$



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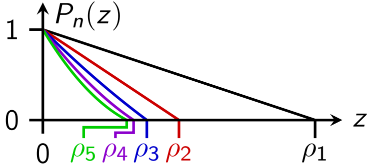

$$\rho_n \to \rho_\infty \geqslant 0$$

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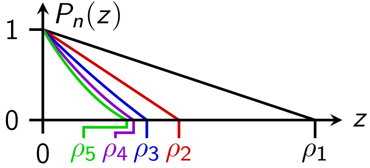
$$\rho_n = \frac{1}{4\cos(\frac{\pi}{n+2})^2}$$



$$\rho_n \to \rho_\infty = 1/4$$

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How did this proof work?

— Part #1: Introducing Möbius polynomials[3,7] —

- Define the **Möbius** polynomial $P_n(z)$
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— Part #3: Conclusion —

- Compute the limit ρ_∞ of roots ρ_n

Part #1: Introducing Möbius polynomials (1/2)

A few preliminary properties...

1. **length** is **additive**: $|\tau| + |\sigma| = |\tau \cdot \sigma|$
2. $\mathbf{T}_n$ is **left-cancellative**: $\tau \cdot \sigma = \tau \cdot \sigma' \Leftrightarrow \sigma = \sigma'$
3. $(\mathbf{T}_n, \preccurlyeq)$ is a **lower-semilattice**: GCDs exist $(\tau \preccurlyeq \tau \cdot \sigma)$
   - **Corollary:** when a set S has a common multiple, it has a LCM
4. $\mathcal{F}_n = \{\mathbf{S}_1, \ldots, \mathbf{S}_n\}$ is **parabolic**: for all $\mathcal{F}' \subseteq \mathcal{F}_n$,
   - $\mathbf{T}_{\mathcal{F}'} = \{\tau \in \mathbf{T}_n :$ all factors $\mathbf{S}_j$ of τ belong to $\mathcal{F}'\}$ is a sub-monoid
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and key objects:

5. **left set** of a heap: $\mathbf{L}(\tau) = \{\mathbf{S}_i \colon \mathbf{S}_i \preccurlyeq \tau\}$
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This also holds in braid monoids!

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This also holds in all Artin-Tits monoids!

Part #1: Introducing Möbius polynomials (2/2)

Theorem: In the ring $\mathbb{Z}[\mathbf{T}_n]$, (Δ_X =LCM(X))

$$(\mathbf{S}_1 + \mathbf{S}_2 + 2\mathbf{S}_3) \cdot (\mathbf{S}_1 - \mathbf{S}_3) = \mathbf{S}_1^2 + \mathbf{S}_2 \cdot \mathbf{S}_1 + 2\mathbf{S}_3 \cdot \mathbf{S}_1 - \mathbf{S}_1 \cdot \mathbf{S}_3 - \mathbf{S}_2 \cdot \mathbf{S}_3 - 2\mathbf{S}_3^2$$

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Corollary: In the ring $\mathbb{Z}[z]$,

$$\sum_{X \in \mathbf{L}_n} (-1)^{|X|} z^{|\Delta_X|} \cdot \sum_{\tau \in \mathbf{T}_n} z^{|\tau|} = 1 = P_n(z) \cdot \mathcal{G}_n(z)$$

Parts #2 & #3: Computing Möbius polynomials and ρ_∞

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- $P_n(z) = 0$ iff $z = \dfrac{1}{4\cos(k\pi/(n+2))^2}$ for some $k \in \mathbb{Z}$ (and $z \neq 1/4$)

Conclusion: $\rho_n = \dfrac{1}{4\cos(\pi/(n+2))^2} \to \rho_\infty = 1/4$

Contents

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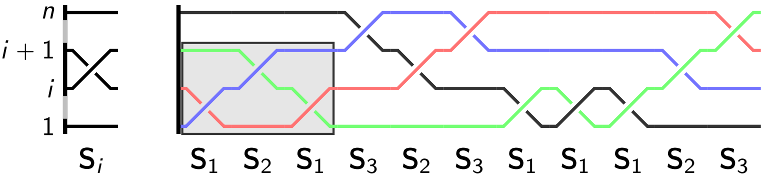

Artin braid diagram

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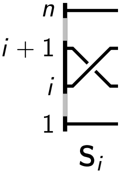

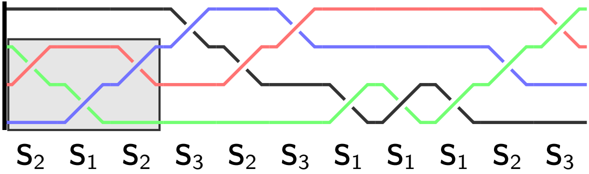

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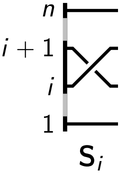

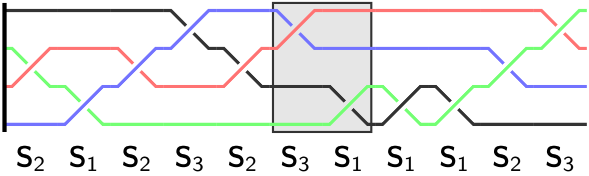

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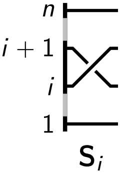

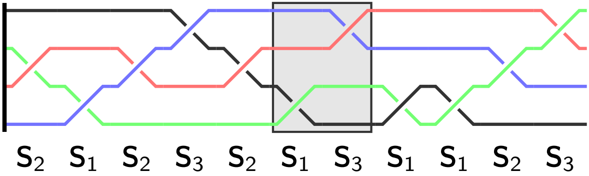

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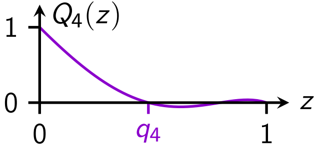

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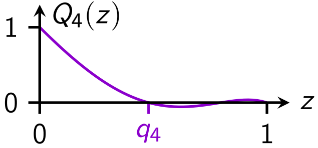

$q_n \to q_\infty \geqslant 0$: **What is q_∞?**

Möbius polynomials in braid monoids

Theorem[6]: $Q_{-1}(z) = Q_0(z) = 1$ and

$$Q_n(z) = \sum_{j=0}^{n} (-1)^j z^{j(j+1)/2} Q_{n-1-j}(z) \text{ if } n \geqslant 1$$

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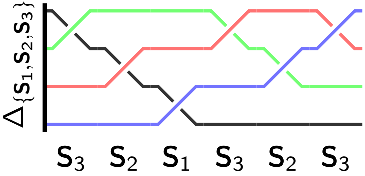

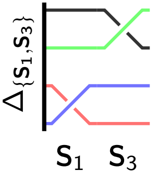

Möbius polynomials in braid monoids

Theorem[6]: $Q_n(z) = 0$ if $n \leqslant -2$, $Q_{-1}(z) = 1$ and

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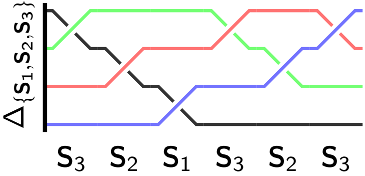

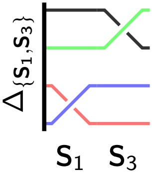

Computing Möbius polynomials in braid monoids (1/2)

The sequence $(Q_n(z))_{n \geqslant -1}$ is recurrent linear of **infinite** order, with $Q_n(z) = 0$ if $n \leqslant -2$ and $Q_{-1}(z) = 0$

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(because $r_i \approx z^{-i}$)

Computing Möbius polynomials in braid monoids (2/2)

and, some (ugly) computations later...

Theorem [10, 11]

$q_\infty \approx 0.30904\ldots$ is the least real $z \geqslant 0$ such that $\mathcal{Q}_z$ has a double root

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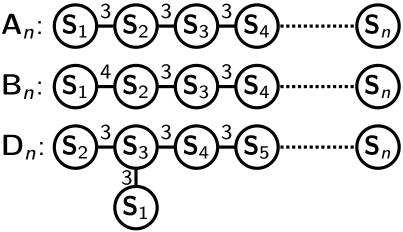
Theorem [11]

The same result holds in monoids of type **B** and **D**



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Let us find another proof!

Growth rates of trace monoids: an algebraic proof

Useful tools...

Direct limit of monoids: $\mathbf{T}_0 \subseteq \mathbf{T}_1 \subseteq \mathbf{T}_2 \subseteq \ldots \subseteq \mathbf{T}_\infty$

Enbedding $\mathbf{T}_\infty$ into $\mathbb{Z}[z, \mathbf{T}_\infty]$: $S \leftrightarrow \sum_{\tau \in S} \tau \leftrightarrow S(z) = \sum_{\tau \in S} z^{|\tau|}$

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and associated results:

- $\mathcal{L}_{i+1}^{n+1} = \mathrm{sh}(\mathcal{L}_i^n) \cdot \mathcal{L}_1^{n+1}$
- $\mathcal{L}_1^n = \mathcal{L}_0^n + \mathbf{S}_1 \cdot \mathcal{L}_2^n$

Growth rates of trace monoids: an algebraic proof

Useful tools...

Direct limit of monoids: $\mathbf{T}_0 \subseteq \mathbf{T}_1 \subseteq \mathbf{T}_2 \subseteq \ldots \subseteq \mathbf{T}_\infty$

Enbedding $\mathbf{T}_\infty$ into $\mathbb{Z}[z, \mathbf{T}_\infty]$: $S \leftrightarrow \sum_{\tau \in S} \tau \leftrightarrow S(z) = \sum_{\tau \in S} z^{|\tau|}$

Shift endomorphism: $\mathrm{sh}\colon \mathbf{S}_i \mapsto \mathbf{S}_{i+1}$

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Growth rates of braid monoids: an algebraic proof (1/2)

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- No proof that $q_\infty \leqslant \mathrm{rad}\,\mathcal{L}_1^\infty$!

Growth rates of braid monoids: an algebraic proof (2/2)

Go into $\mathbb{Z}[z, \Theta, \mathbf{A}_\infty]$ and study $Q_\infty = \sum_{n \geqslant 0} \sum_{T \in \mathbf{L}_{n-1}} \Theta^n (-1)^{|T|} \Delta_T$

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$$= 1 + \sum_{k \geqslant 1} \sum_{\hat{T} \in \mathbf{L}_\infty} \sum_{n \geqslant \max \hat{T}} \Theta^{k+n+1} (-1)^{k-1+|\hat{T}|} \Delta_{\{\mathbf{S}_1, \ldots, \mathbf{S}_{k-1}\}} \cdot \mathrm{sh}^k(\Delta_{\hat{T}})$$

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Proof:

$$
\begin{aligned}
Q_\infty &= \sum_{n \geqslant 0} \sum_{T \in \mathbf{L}_{n-1}} \Theta^n (-1)^{|T|} \Delta_T = \sum_{T \in \mathbf{L}_\infty} \sum_{n \geqslant \max T} \Theta^{n+1} (-1)^{|T|} \Delta_T \\
&= 1 + \sum_{k \geqslant 1} \sum_{\hat{T} \in \mathbf{L}_\infty} \sum_{n \geqslant \max \hat{T}} \Theta^{k+n+1} (-1)^{k-1+|\hat{T}|} \Delta_{\{\mathbf{S}_1, \ldots, \mathbf{S}_{k-1}\}} \cdot \mathrm{sh}^k(\Delta_{\hat{T}}) \\
&= 1 - \sum_{k \geqslant 1} (-1)^k \Theta^k \Delta_{\{\mathbf{S}_1, \ldots, \mathbf{S}_{k-1}\}} \cdot \mathrm{sh}^k(Q_\infty)
\end{aligned}
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$$\begin{aligned}
Q_\infty &= \sum_{n \geqslant 0} \sum_{T \in \mathbf{L}_{n-1}} \Theta^n (-1)^{|T|} \Delta_T = \sum_{T \in \mathbf{L}_\infty} \sum_{n \geqslant \max T} \Theta^{n+1} (-1)^{|T|} \Delta_T \\
&= 1 + \sum_{k \geqslant 1} \sum_{\hat{T} \in \mathbf{L}_\infty} \sum_{n \geqslant \max \hat{T}} \Theta^{k+n+1} (-1)^{k-1+|\hat{T}|} \Delta_{\{\mathbf{S}_1, \ldots, \mathbf{S}_{k-1}\}} \cdot \mathrm{sh}^k(\Delta_{\hat{T}}) \\
&= 1 - \sum_{k \geqslant 1} (-1)^k \Theta^k \Delta_{\{\mathbf{S}_1, \ldots, \mathbf{S}_{k-1}\}} \cdot \mathrm{sh}^k(Q_\infty)
\end{aligned}$$

$$Q_\infty(z) = 1 - \sum_{k \geqslant 1} (-1)^k \Theta^k z^{k(k-1)/2} Q_\infty(z)$$

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Theorem: $\mathcal{Q}_z(\Theta) \cdot Q_\infty(z) = \mathcal{Q}_z(\Theta) \cdot \sum_{n \geqslant 0} \Theta^n Q_n(z) = 1$

Corollary: If $\operatorname{rad} \mathcal{L}_1^\infty < z < q_\infty$, then $\mathcal{Q}_z(\Theta) \cdot Q_\infty(z) = 1$ for all $\Theta \in \mathbb{C}$.

Proof: $P_n(z)/P_{n+1}(z) = \mathcal{L}_{n+1}^{n+1}(z)/\mathcal{L}_n^n(z) = \mathcal{L}_1^{n+1}(z) \to \mathcal{L}_1^\infty(z) = +\infty$, hence $\operatorname{rad} \mathcal{Q}_z = \operatorname{rad}_\Theta Q_\infty(z) = +\infty$.

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hence $\mathrm{rad}\, \mathcal{Q}_z = \mathrm{rad}_\Theta\, Q_\infty(z) = +\infty$.

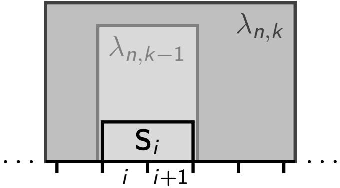
Theorem [11]

In monoids of type **A**, **B** and **D**, we have $q_\infty = \mathrm{rad}\, \mathcal{L}_1^\infty \approx 0.30904\ldots$

Contents

Open problems & future research

- Generating random **infinitely wide & tall** heaps: $\mathbb{P}[\mathbf{S}_i \cdots] = 1/4$



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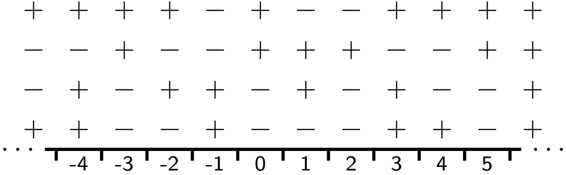

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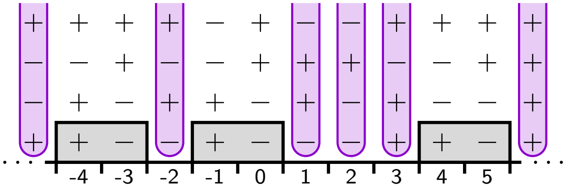

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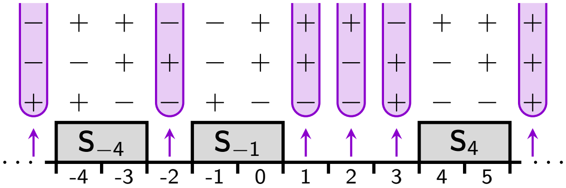

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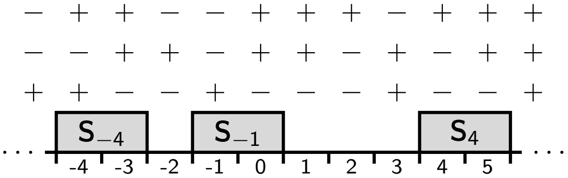

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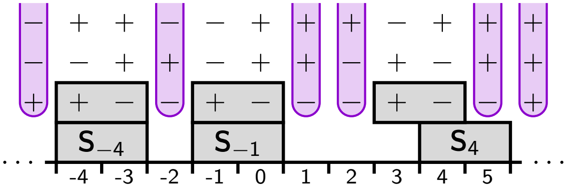

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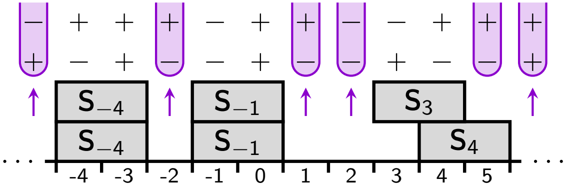

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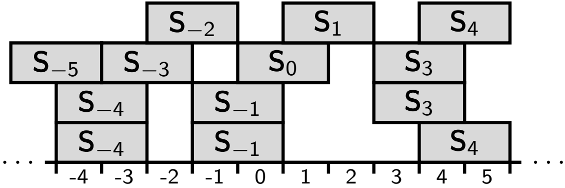

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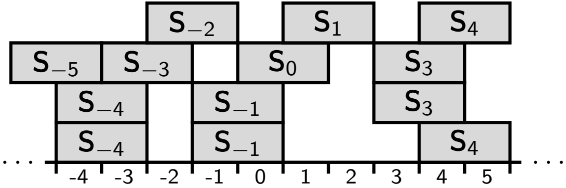

- What about **infinitely wide & tall** braids? $\mathbb{P}[\mathbf{S}_i \cdots] = q_\infty \approx 0.30904$

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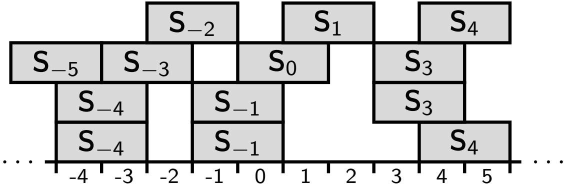

- What about **infinitely wide & tall** braids? $\mathbb{P}[\mathbf{S}_i \cdots] = q_\infty \approx 0.30904$

- Investigating other classes of **Artin-Tits** monoids:

  Coincidence or not? $\mathcal{Q}_z(\Theta) \cdot Q_\infty(z) = 1$ and $\mathcal{Q}_z(\mathcal{L}_1^\infty(z)) = 0$

  Coincidence or not? $q_\infty = \operatorname{rad} \mathcal{L}_1^\infty$

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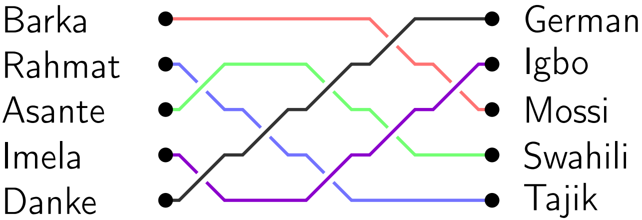
Thank you very much for your attention!



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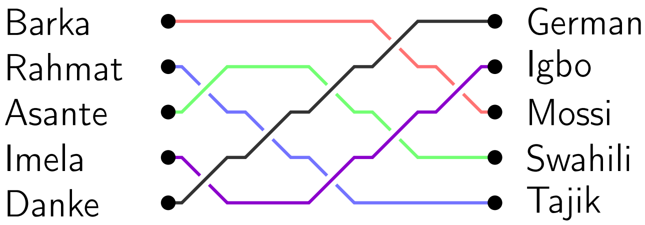

Questions?