Drawing heaps uniformly at random

Samy Abbes[1], Sébastien Gouëzel[2,3], Vincent Jugé[2,4] & **Jean Mairesse**[2,5]

1: Paris 7 (IRIF) — 2: CNRS — 3: Nantes (LMJL) — 4: ENS Cachan (LSV) — 5: Paris 6 (LIP6)

25/05/2016

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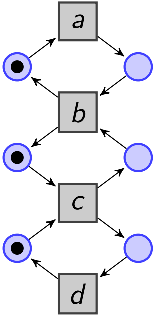
Petri nets and dependency graphs

Consider your favorite one-bounded Petri net with ...



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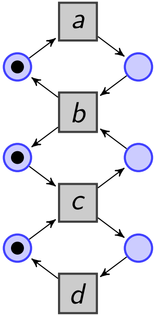

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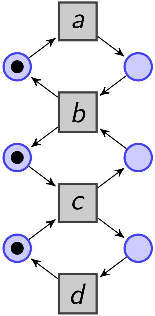

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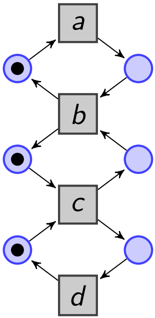

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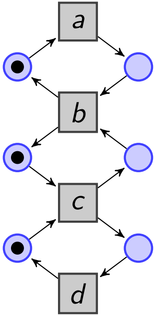

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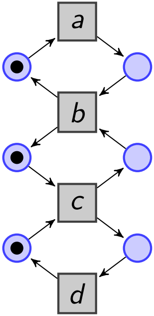

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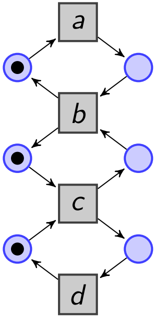

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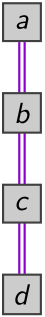

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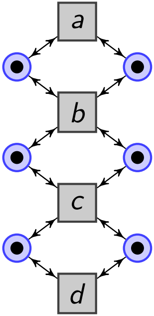

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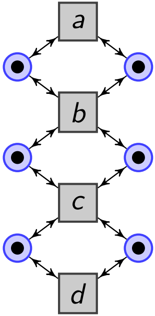

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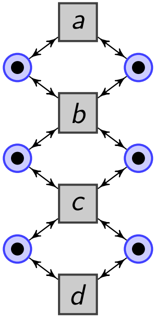

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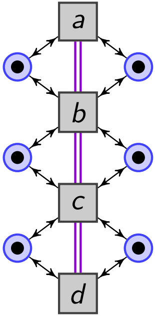

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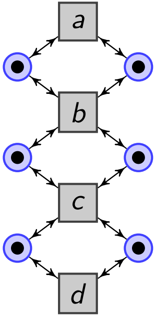

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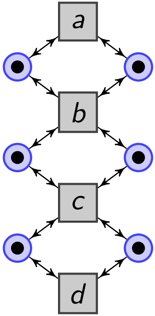

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Petri nets and dependency graphs

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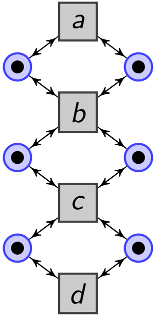

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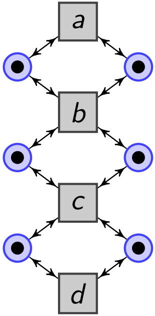

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Contents

Heaps of pieces and trace monoids

Heap of pieces

- Pieces:

Trace monoid

- Alphabet:

$$\Sigma = \{a, b, c, d\}$$

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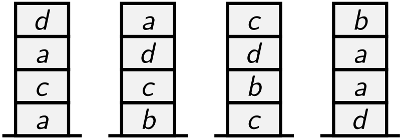
Heap of pieces

- Pieces:

- Purely vertical heaps:



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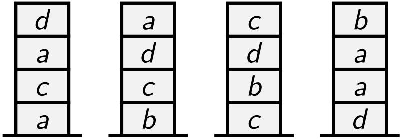

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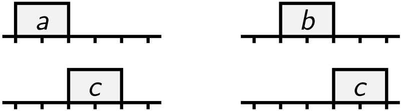

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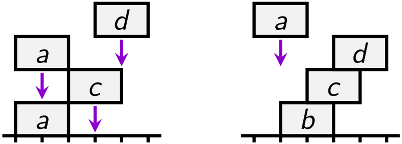

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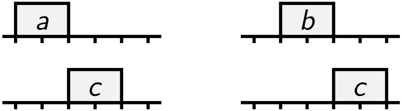

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Dependency graph

Heaps of pieces and trace monoids

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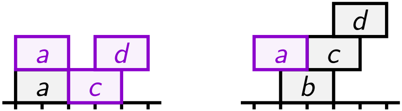

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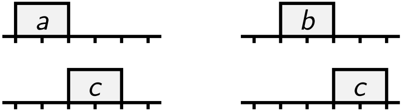

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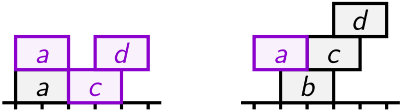

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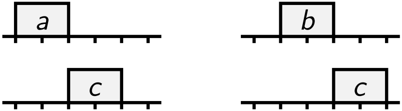

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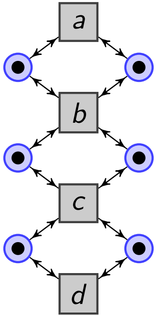
Dependency graph

Heaps of pieces viewed from their places

Petri net



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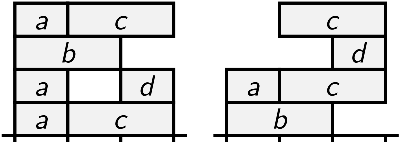

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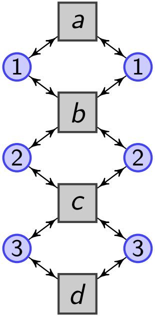

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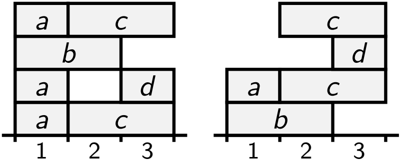

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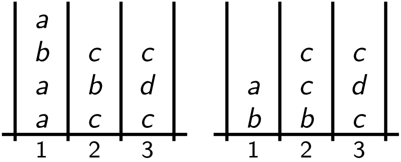

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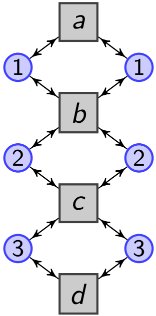

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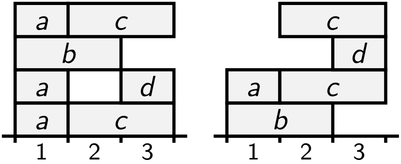

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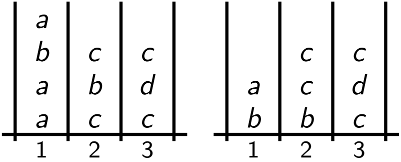

Heap of pieces ⇔ **Consistent** place views

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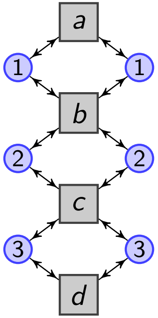

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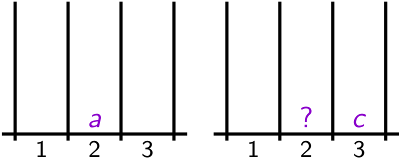

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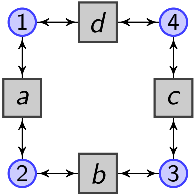

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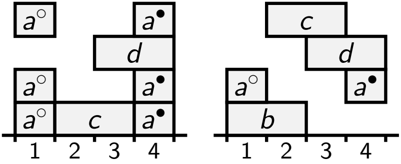

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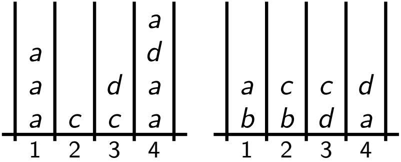

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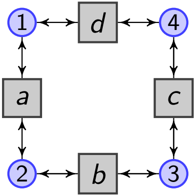

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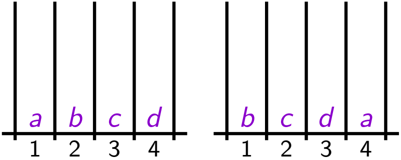

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Heaps of pieces and Cartier-Foata normal forms

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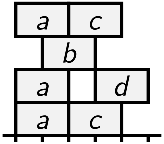

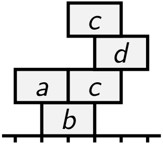

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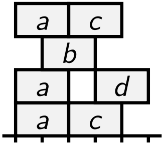

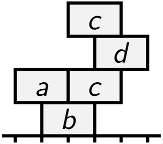

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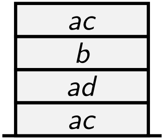
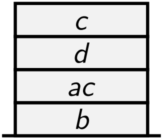
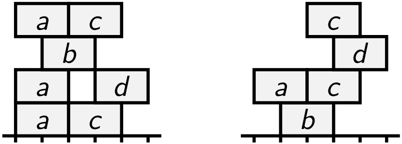




Heaps of pieces and Cartier-Foata normal forms

Heap of pieces

- Vertical heaps of pieces:



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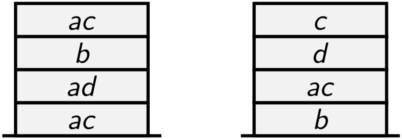

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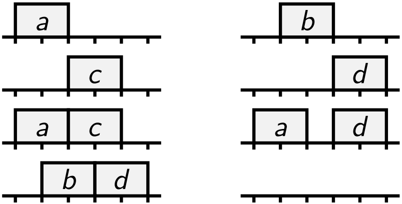

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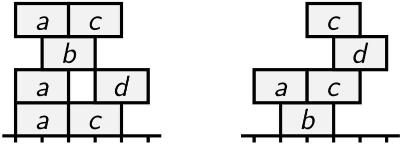

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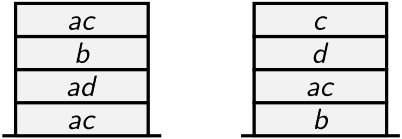

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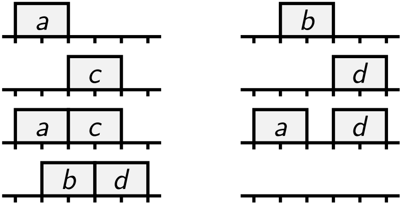

- **Local conditions** on consecutive cliques in heaps

Heaps of pieces and left divisibility

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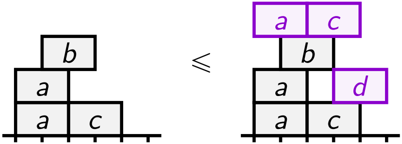

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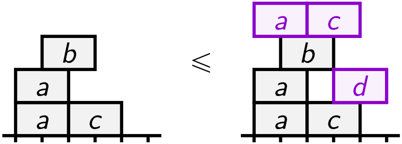

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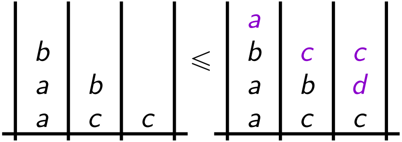

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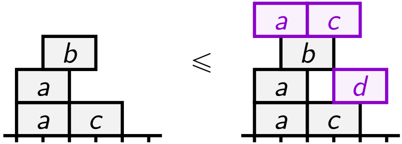

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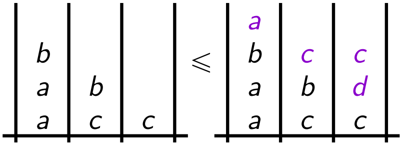

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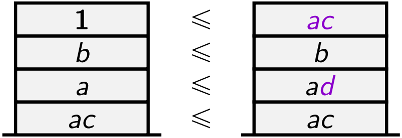

+ upper commutativity

$(bd \in \mathcal{C})$

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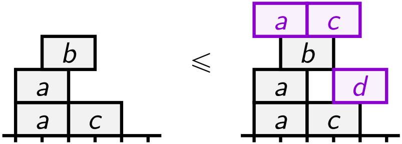

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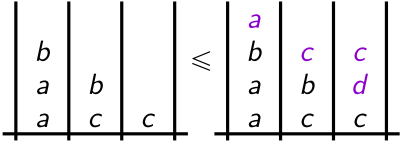

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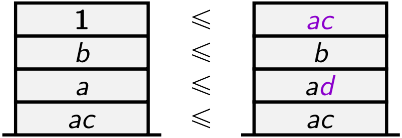

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Combinatorial properties

- $\mathbf{a} \wedge \mathbf{b}$ (and $\mathbf{a} \vee \mathbf{b}$) exist
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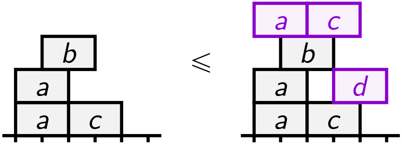

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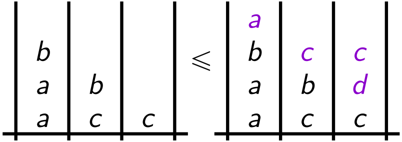

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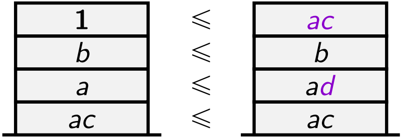

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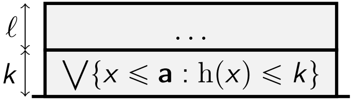

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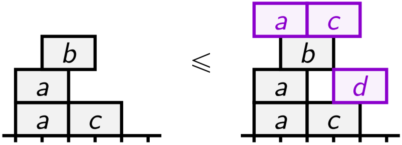

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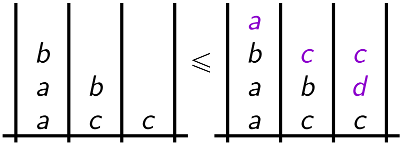

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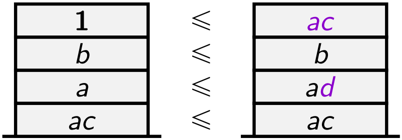

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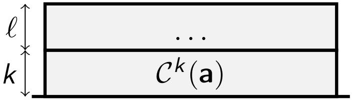

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Theorem (S. Abbes & J. Mairesse 2015)

The uniform distribution on $\mathcal{M}_k$ converges weakly in $\overline{\mathcal{M}^+}$ when $k \to +\infty$

Weak convergence: length($\mathbf{a}$) = $|\mathbf{a}|$

Generating series and Möbius polynomial

$$\mathcal{G}(z) = \sum_{\alpha \in \mathcal{M}^+} z^{|\alpha|} = \sum_{k \geqslant 0} \lambda_k z^k \text{ and } \mathcal{H}(z) = \sum_{\gamma \in \mathcal{C}} (-z)^{|\gamma|}$$

Proposition (P. Cartier & D. Foata 1969)

$$\mathcal{G}(z)\mathcal{H}(z) = 1$$

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$$\mathcal{G}(z)\mathcal{H}(z) = 1$$

Corollary (D. Krob, J. Mairesse & I. Michos 2001)

$\mathcal{H}(z)$ has a smallest positive root p such that:

- $(\mathcal{H}(z) = 0 \wedge |z| \leqslant p) \Leftrightarrow z = p$
- $0 < p \leqslant 1$

and there exists constants $\Lambda > 0$ and $\ell \in \mathbb{N}$ such that $\lambda_k \sim \Lambda p^{-k} k^{\ell}$

Weak convergence: length($\mathbf{a}$) = $|\mathbf{a}|$

Proof of the theorem – length($\mathbf{a}$) = $|\mathbf{a}|$

1. $\mu_k : S \mapsto \frac{\#(S \cap \mathcal{M}_k)}{\lambda_k}$

Weak convergence: length($\mathbf{a}$) = $|\mathbf{a}|$

Proof of the theorem – length(a) = |a|

1. $\mu_k : S \mapsto \frac{\#(S \cap \mathcal{M}_k)}{\lambda_k}$
2. $\mathbf{x} \mapsto \mathbf{ax}$ maps $\mathcal{M}_k$ to $(\Uparrow \mathbf{a}) \cap \mathcal{M}_{k+|\mathbf{a}|}$ bijectively

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4. $\mathcal{M}^+$ is compact and $\{\varnothing\} \cup \{\Uparrow \mathbf{a}\}$ is closed under $\cap$

Weak convergence: length($\mathbf{a}$) $= |\mathbf{a}|$ and length($\mathbf{a}$) $= \mathsf{h}(\mathbf{a})$

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Proof of the theorem – length(a) $= \mathsf{h}(\mathrm{a})$

5. Split $\Uparrow \mathbf{a}$ into sets $\mathcal{M}^+(\mathbf{b}) = \{x : \mathbf{b} = \mathcal{C}^{\mathsf{h}(\mathbf{a})}(x)\}$

Weak convergence: length($\mathbf{a}$) = $|\mathbf{a}|$ and length($\mathbf{a}$) = h($\mathbf{a}$)

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5. Split $\Uparrow \mathbf{a}$ into sets $\mathcal{M}^+(\mathbf{b}) = \{x : \mathbf{b} = \mathcal{C}^{\mathrm{h}(\mathbf{a})}(x)\}$
6. Prove that $\#(\mathcal{M}^+(\mathbf{b}) \cap \mathcal{M}_k) \sim \Lambda_{\mathbf{b}} q_{\mathbf{b}}^k k^{\ell_{\mathbf{b}}}$ for some $\Lambda_{\mathbf{b}}$, $q_{\mathbf{b}}$ and $\ell_{\mathbf{b}}$

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6. Prove that $\#(\mathcal{M}^+(\mathbf{b}) \cap \mathcal{M}_k) \sim \Lambda_{\mathbf{b}} q_{\mathbf{b}}^k k^{\ell_{\mathbf{b}}}$ for some $\Lambda_{\mathbf{b}}$, $q_{\mathbf{b}}$ and $\ell_{\mathbf{b}}$
7. Complete the proof as above

Caution: $\lim \mu_k(\Uparrow \mathbf{a})$ does not depend only on h($\mathbf{a}$)!

Contents

Bernoulli distributions

A distribution μ on $\mathcal{M}^+$ is . . .

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- $\mu(\partial\mathcal{M}^+) = 0$
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Bernoulli distributions

Proving that finite uniform $\Leftrightarrow \mu(\{\mathsf{x}\}) = \mathcal{H}(\nu)\nu^{|\mathsf{x}|}$

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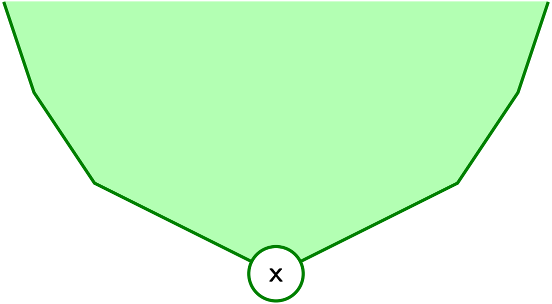

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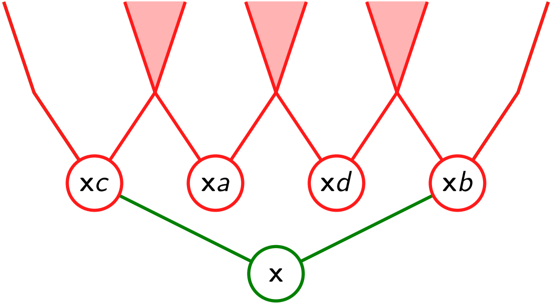

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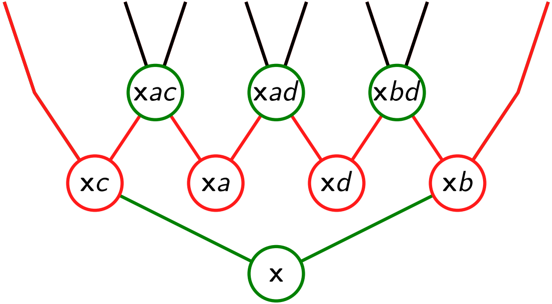

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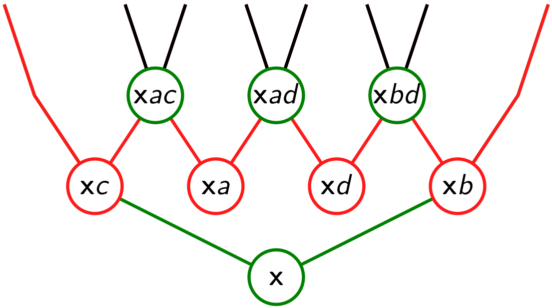

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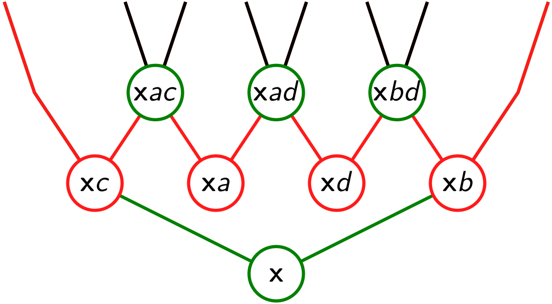

Simulating finite, uniform Bernoulli distributions

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Approach #1: Pick the length first

- Pick a **target length** k with probability $\lambda_k \nu^k \mathcal{H}(\nu)$
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Finite uniform Bernoulli distributions as Markov chains

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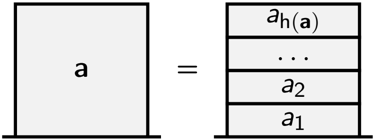

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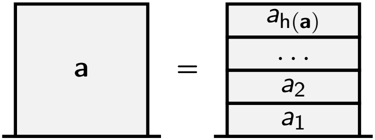

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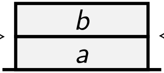
$a \to b \Leftrightarrow$  $\Leftrightarrow a = \mathcal{C}^1(ab)$

Infinite uniform Bernoulli distributions as Markov chains

Critical parameter: $\nu = p$

Infinite uniform Bernoulli distributions as Markov chains

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- M^ν and M^p are **stochastic Perron matrices** if $\mathcal{M}^+$ is irreducible

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- Collection of parameters: $(\nu_a) \in (0,1]^n$
- Multiplicative function: $\overline{\nu} : a_1 \ldots a_k \mapsto \nu_{a_1} \ldots \nu_{a_k}$
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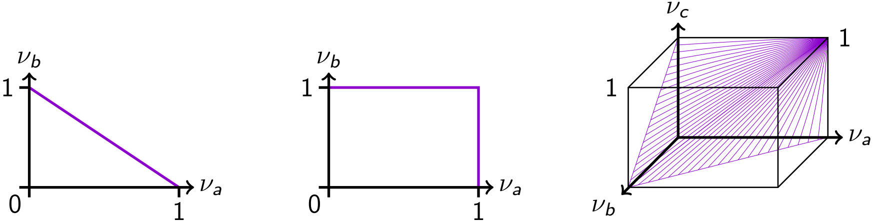

$\mathbb{N} * \mathbb{N} = \langle a, b \rangle^+$ $\quad$ $\mathbb{N}^2 = \langle a, b \mid ab = ba \rangle^+$ $\quad$ $\langle a, b, c \mid ac = ca \rangle^+$

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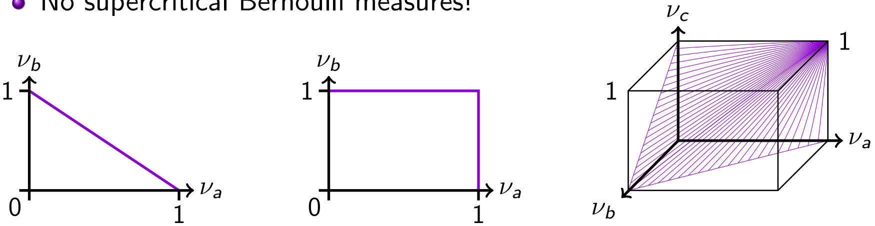
- No supercritical Bernoulli measures!



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From weak convergence to central limit theorems

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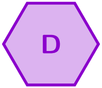
Central limit Theorem (S. A., S. G., V. J. & J. M. 2016$^+$)

There exists constants ρ and $\sigma^2 > 0$ such that

1. $\frac{A_k}{k} \xrightarrow{\mathcal{L}} \rho$
2. $\sqrt{k}\left(\frac{A_k}{k} - \rho\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$ if $\mathcal{M}^+$ is irreducible

From trace monoids to Artin–Tits and left-Garside monoids



- **Dimer** monoid: $\langle \sigma_i \mid i \neq j \pm 1 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i \rangle^+$

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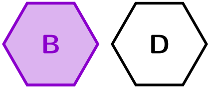

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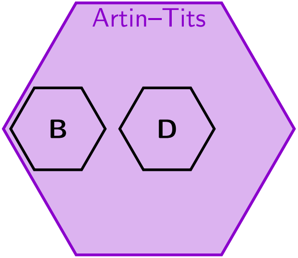

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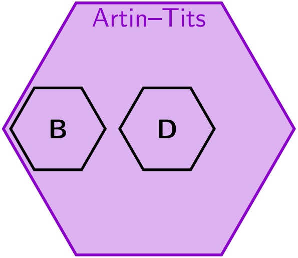
- **Artin–Tits** monoid: $\langle \sigma_i \mid [\sigma_i \sigma_j]^{\ell(i,j)} = [\sigma_j \sigma_i]^{\ell(i,j)} \rangle^+$

$$\ell(i,j) = 2 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i$$

From trace monoids to Artin–Tits and left-Garside monoids



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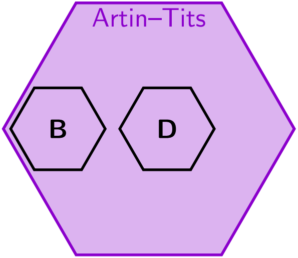
$$\ell(i,j) = 3 \Rightarrow \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$$

From trace monoids to Artin–Tits and left-Garside monoids



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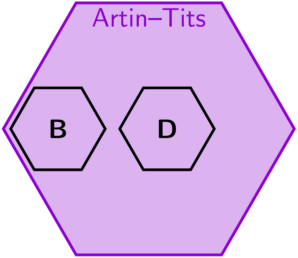
From trace monoids to Artin–Tits and left-Garside monoids



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$\ell(i,j) = +\infty \Rightarrow$ no relation!

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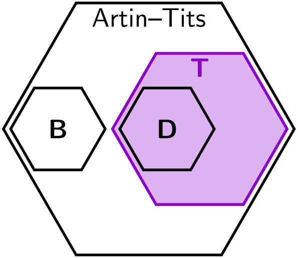

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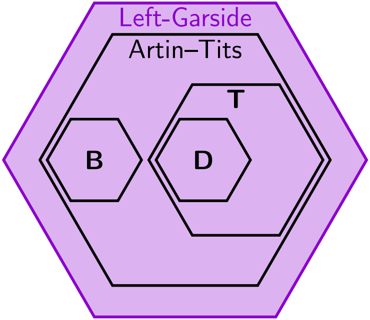
- **Trace** monoid: Artin–Tits with $\ell(i,j) \in \{2, +\infty\}$

From trace monoids to Artin–Tits and left-Garside monoids



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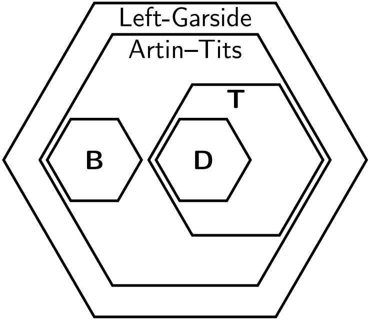

Theorem (— 2016$^+$)

1. **Weak conv.** in **all** A–T monoids
2. **CLT 1** in **all** A–T monoids
3. **CLT 2** in **irreducible** A–T monoids

+some extensions to left-Garside monoids

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And then?

Some directions of research

- Generalisation to **all** left-Garside monoids
- Generalisation to trace **groups**
- Sampling elements in **regular languages** $L \cap \mathcal{M}_k$ instead of $\mathcal{M}_k$
- Identifying nice Markov chains when $\mathrm{length}(\mathbf{a}) = \mathrm{h}(ba)$
- Your favorite one (tell me now!)