Random braids: Stabilizing the Garside normal form

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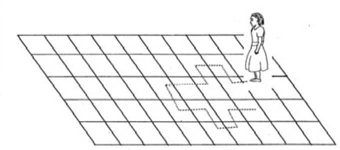
Questions & motivations

Consider a uniform random walk in a braid monoid



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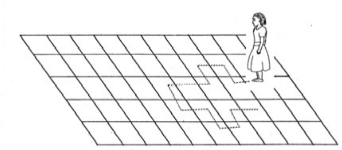

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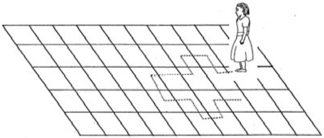

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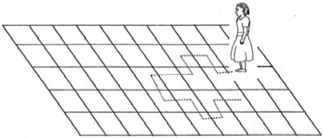

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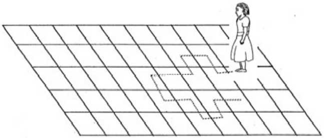

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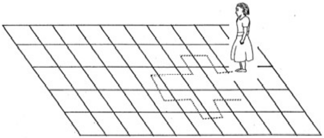

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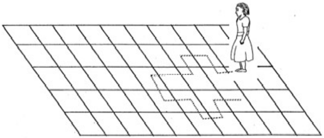
- "It works with them too!" (V. J. & J. M., this talk)



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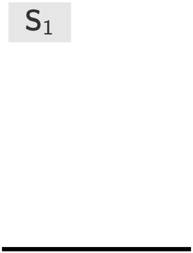

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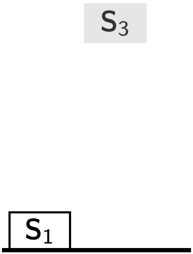

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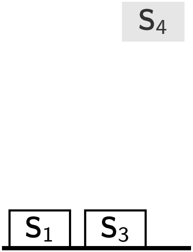

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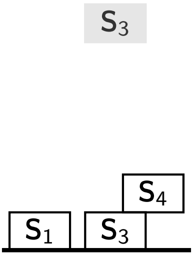

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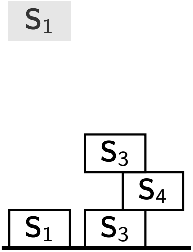

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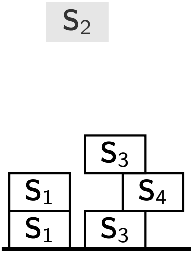

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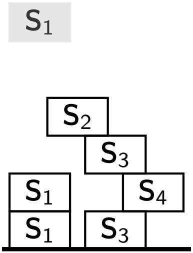

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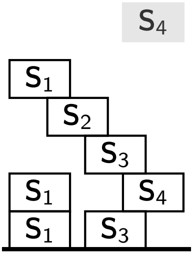

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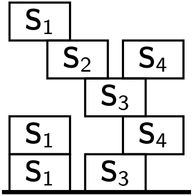

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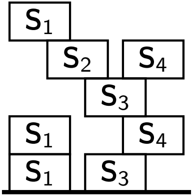

Viennot heap diagrams

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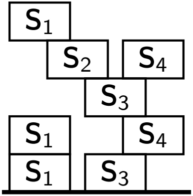

Viennot heap diagrams

Dimer monoid

$$\mathcal{M}_n^+ = \langle \mathbf{S}_1, \ldots, \mathbf{S}_n \mid |i-j| \geqslant 2 \Rightarrow \mathbf{S}_i\mathbf{S}_j = \mathbf{S}_j\mathbf{S}_i \rangle^+$$

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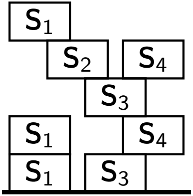

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Cartier-Foata normal form: $\mathsf{S}_1\mathsf{S}_3 \cdot \mathsf{S}_1\mathsf{S}_4 \cdot \mathsf{S}_3 \cdot \mathsf{S}_2\mathsf{S}_4 \cdot \mathsf{S}_1$

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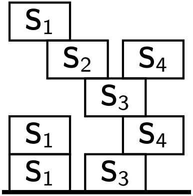

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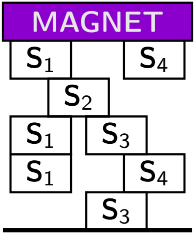

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Right Garside normal form: $\mathbf{S}_3 \cdot \mathbf{S}_1\mathbf{S}_4 \cdot \mathbf{S}_1\mathbf{S}_3 \cdot \mathbf{S}_2 \cdot \mathbf{S}_1\mathbf{S}_4$

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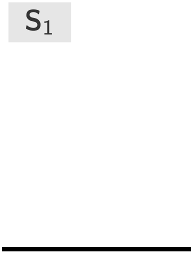

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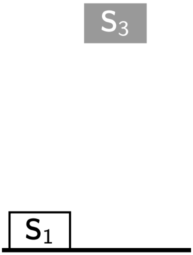

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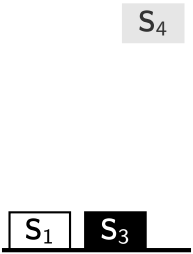

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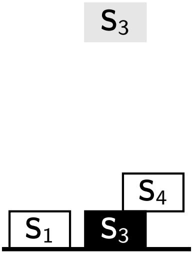

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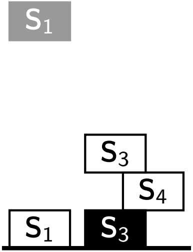

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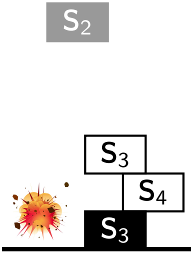

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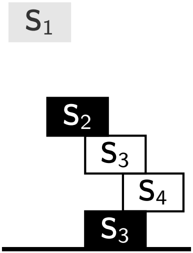

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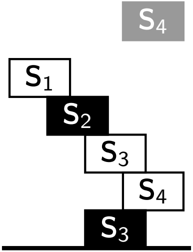

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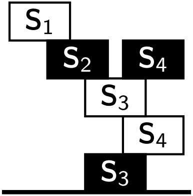

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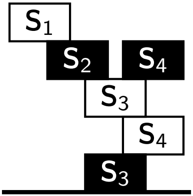

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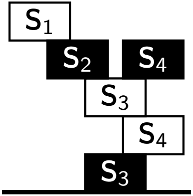

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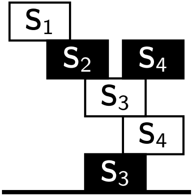

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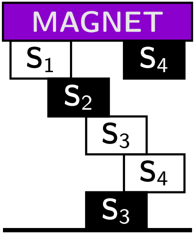

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Theorem [Folklore]

Convergence of the words

	$\mathbf{Gar_L}(X_k)_{k \geqslant 0}$	$\mathbf{Gar_R}(X_k)_{k \geqslant 0}$
prefix-	✓	✓
suffix-	✗	✗

Random walk in irreducible trace monoids

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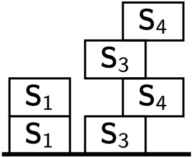
Suffix-divergence

Lemma: for all $x \in \mathcal{M}_n^+$,

$$\mathbf{suffix}(\mathbf{Gar_L}(x))$$



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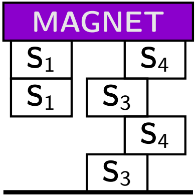

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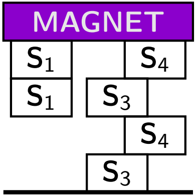

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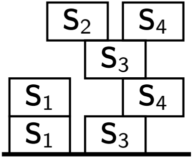

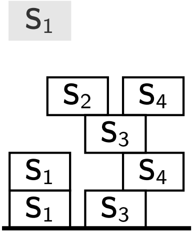

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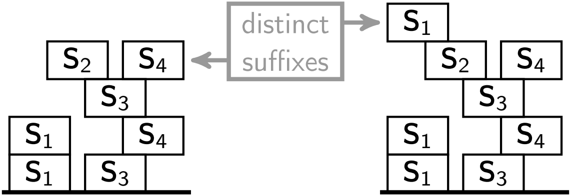

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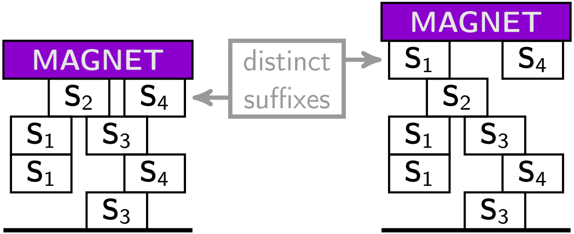

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Consequence: if $Y_k = \mathbf{S}_{i\pm1}$ for some $\mathbf{S}_i \in \mathbf{suffix}(\mathbf{Gar_L}(X_k))$ then

- $\mathbf{suffix}(\mathbf{Gar_L}(X_k)) \neq \mathbf{suffix}(\mathbf{Gar_L}(X_{k+1}))$;
- $\mathbf{suffix}(\mathbf{Gar_R}(X_k)) \neq \mathbf{suffix}(\mathbf{Gar_R}(X_{k+1}))$.

Prefix-convergence of $\mathbf{Gar_L}(X_k)_{k\geqslant0}$

Lemma: for all $x \in \mathcal{M}_n^+$, $\mathbf{prefix}(\mathbf{Gar_L}(x)) = \{\mathbf{S}_i : \mathbf{S}_i \leqslant x\}$.

Random walk in dimer monoids

Suffix-divergence

Lemma: for all $x \in \mathcal{M}_n^+$,

$$\mathbf{suffix}(\mathbf{Gar_L}(x)) \subseteq \mathbf{suffix}(\mathbf{Gar_R}(x)) = \{\mathbf{S}_i : x \geqslant \mathbf{S}_i\}.$$

Consequence: if $Y_k = \mathbf{S}_{i\pm 1}$ for some $\mathbf{S}_i \in \mathbf{suffix}(\mathbf{Gar_L}(X_k))$ then

- $\mathbf{suffix}(\mathbf{Gar_L}(X_k)) \neq \mathbf{suffix}(\mathbf{Gar_L}(X_{k+1}))$;
- $\mathbf{suffix}(\mathbf{Gar_R}(X_k)) \neq \mathbf{suffix}(\mathbf{Gar_R}(X_{k+1}))$.

Prefix-convergence of $\mathbf{Gar_L}(X_k)_{k \geqslant 0}$

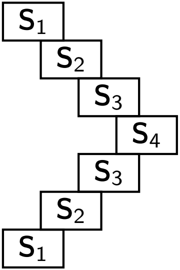
Lemma: for all $x \in \mathcal{M}_n^+$, $\quad \mathbf{prefix}(\mathbf{Gar_L}(x)) = \{\mathbf{S}_i : \mathbf{S}_i \leqslant x\}$.

Consequence: $\mathbf{prefix}(\mathbf{Gar_L}(X_k)) \subseteq \mathbf{prefix}(\mathbf{Gar_L}(X_{k+1}))$.

Random walk in dimer monoids

Prefix-convergence of $\mathrm{Gar}_{\mathbf{R}}(X_k)_{k \geqslant 0}$

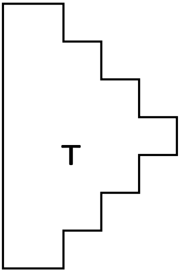
Key ingredient: blocking traces $\mathbf{T} = \mathbf{S}_1\mathbf{S}_2 \ldots \mathbf{S}_n\mathbf{S}_{n-1} \ldots \mathbf{S}_1$



Random walk in dimer monoids

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

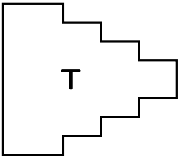
Key ingredient: blocking traces $\mathbf{T} = \mathbf{S}_1\mathbf{S}_2\ldots\mathbf{S}_n\mathbf{S}_{n-1}\ldots\mathbf{S}_1$



Random walk in dimer monoids

Prefix-convergence of $\mathsf{Gar}_{\mathbf{R}}(X_k)_{k \geqslant 0}$

Key ingredient: blocking traces $\mathbf{T} = \mathbf{S}_1 \mathbf{S}_2 \ldots \mathbf{S}_n \mathbf{S}_{n-1} \ldots \mathbf{S}_1$

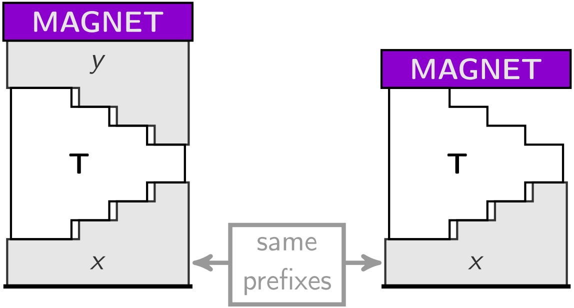


Random walk in dimer monoids

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

Key ingredient: blocking traces $\mathbf{T} = \mathbf{S}_1\mathbf{S}_2 \ldots \mathbf{S}_n\mathbf{S}_{n-1} \ldots \mathbf{S}_1$

Lemma: for all $x, y \in \mathcal{M}_n^+$, $\mathbf{prefix}(\mathbf{Gar_R}(x\mathbf{T}y)) = \mathbf{prefix}(\mathbf{Gar_R}(x\mathbf{T}))$.



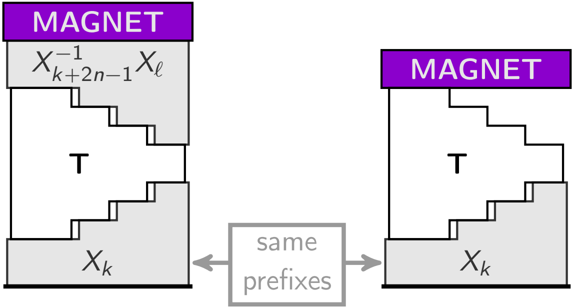
Random walk in dimer monoids

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

Key ingredient: blocking traces $\mathbf{T} = \mathbf{S}_1\mathbf{S}_2\ldots\mathbf{S}_n\mathbf{S}_{n-1}\ldots\mathbf{S}_1$

Lemma: for all $x, y \in \mathcal{M}_n^+$, $\mathbf{prefix}(\mathbf{Gar_R}(x\mathbf{T}y)) = \mathbf{prefix}(\mathbf{Gar_R}(x\mathbf{T}))$.

Consequence: if $Y_k \ldots Y_{k+2n-2} = \mathbf{T}$ then
$\mathbf{prefix}(\mathbf{Gar_R}(X_{k+2n-1})) = \mathbf{prefix}(\mathbf{Gar_R}(X_\ell))$ for all $\ell \geqslant k + 2n - 1$.



Random walk in dimer groups

Random walk

1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ in $\{\mathbf{S}_1^{\pm 1}, \ldots, \mathbf{S}_n^{\pm 1}\}$.
2. Random process $(X_k)_{k \geqslant 0}$ defined by:
$$X_0 = \mathbf{1} \text{ and } X_{k+1} = X_k Y_k.$$

Random walk in dimer groups

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Theorem [Folklore]

Convergence of the words

	$\mathbf{Gar_L}(X_k)_{k \geqslant 0}$	$\mathbf{Gar_R}(X_k)_{k \geqslant 0}$
prefix-	✓	✓
suffix-	✗	✗

Random walk in dimer groups

Random walk

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Theorem [Folklore]

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	$\mathbf{Gar_L}(X_k)_{k \geqslant 0}$	$\mathbf{Gar_R}(X_k)_{k \geqslant 0}$
prefix-	✓	✓
suffix-	✗	✗

$\Rightarrow$ The proof of **suffix-divergence** still works!

Random walk in irreducible trace groups

Random walk

1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ in $\{\mathbf{S}_1^{\pm 1}, \ldots, \mathbf{S}_n^{\pm 1}\}$.
2. Random process $(X_k)_{k \geqslant 0}$ defined by:

$$X_0 = \mathbf{1} \text{ and } X_{k+1} = X_k Y_k.$$

Theorem [Folklore]

Convergence of the words

	$\mathbf{Gar_L}(X_k)_{k \geqslant 0}$	$\mathbf{Gar_R}(X_k)_{k \geqslant 0}$
prefix-	✓	✓
suffix-	✗	✗

$\Rightarrow$ The proof of **suffix-divergence** still works!

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar}_{\mathbf{L}}(X_k)_{k \geqslant 0}$

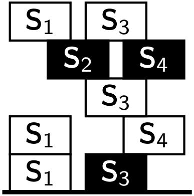
Key ingredient: $\mathbf{C}_i(x) = \#\{\text{occurrences of } \mathbf{S}_i^{\pm 1} \text{ or } \mathbf{S}_{i+1}^{\pm 1} \text{ in } \mathbf{Gar}_{\mathbf{L}}(x)\}$.

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar_L}(X_k)_{k \geqslant 0}$

Key ingredient: $\mathbf{C}_i(x) = \#\{\text{occurrences of } \mathbf{S}_i^{\pm 1} \text{ or } \mathbf{S}_{i+1}^{\pm 1} \text{ in } \mathbf{Gar_L}(x)\}$.

Lemma: $\min_i \mathbf{C}_i(X_k) \to +\infty$



Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar}_{\mathbf{L}}(X_k)_{k \geqslant 0}$

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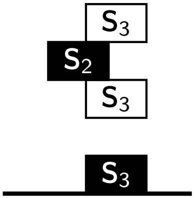

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar}_{\mathbf{L}}(X_k)_{k \geqslant 0}$

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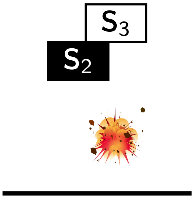
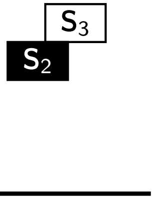


Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar_L}(X_k)_{k \geqslant 0}$

Key ingredient: $\mathbf{C}_i(x) = \#\{\text{occurrences of } \mathbf{S}_i^{\pm 1} \text{ or } \mathbf{S}_{i+1}^{\pm 1} \text{ in } \mathbf{Gar_L}(x)\}$.

Lemma: $\min_i \mathbf{C}_i(X_k) \to +\infty$ (escape rate in free groups)



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Prefix-convergence of $\mathbf{Gar_L}(X_k)_{k \geqslant 0}$

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Lemma 2: for all $x \in \mathcal{M}_n$, if $\mathbf{C}_i(x) \geqslant 2$, then

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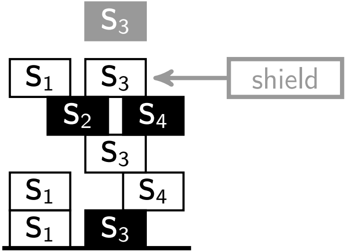

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar}_{\mathbf{L}}(X_k)_{k \geqslant 0}$

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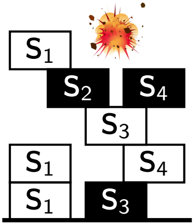

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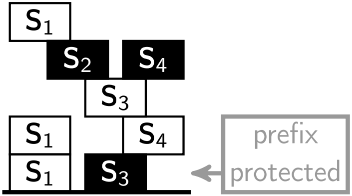

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

Lemma: $\mathbb{P}[\forall k \geqslant 1, \mathbf{prefix}(\mathbf{Gar_L}(X_k)) = \{\mathbf{S}_1\}] > 0$

Random walk in dimer groups

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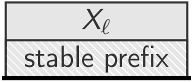

After $\ell \geqslant K$ steps,
with proba > 0

Random walk in dimer groups

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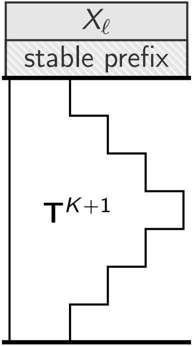

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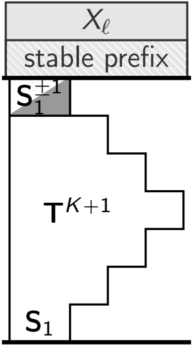

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Random walk in dimer groups

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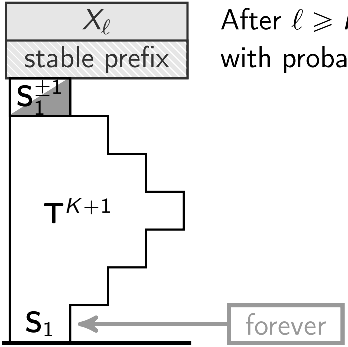

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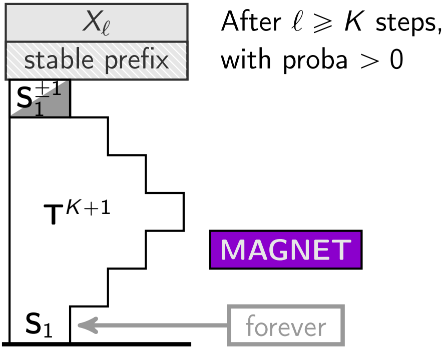

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

Lemma: $\mathbb{P}[\forall k \geqslant 1, \mathbf{prefix}(\mathbf{Gar_R}(X_k)) = \{\mathbf{S}_1\}] > 0$

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Random walk in dimer groups

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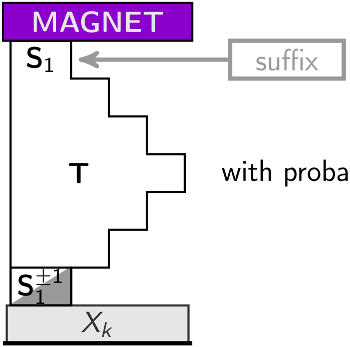

Random walk in dimer groups

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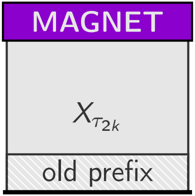
Consequence: $\mathbf{prefix}(\mathbf{Gar}_{\mathbf{R}}(X_k))_{k \geqslant 0}$ converges



$\tau_0, \tau_1, \ldots, \tau_k$:
stopping times

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

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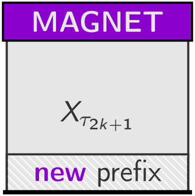

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Random walk in dimer groups

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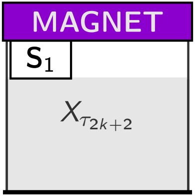

$\tau_0, \tau_1, \ldots, \tau_k$: stopping times

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar}_\mathbf{R}(X_k)_{k \geqslant 0}$

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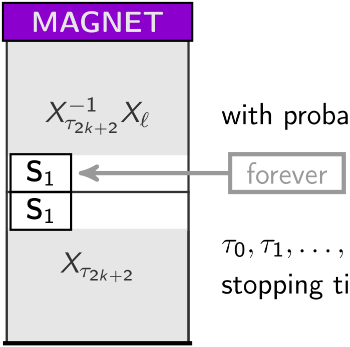

with proba > 0

$\tau_0, \tau_1, \ldots, \tau_k$: stopping times

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar}_{\mathbf{R}}(X_k)_{k \geqslant 0}$

Lemma: $\mathbb{P}[\forall k \geqslant 1, \mathbf{prefix}(\mathbf{Gar}_{\mathbf{R}}(X_k)) = \{\mathbf{S}_1\}] > 0$

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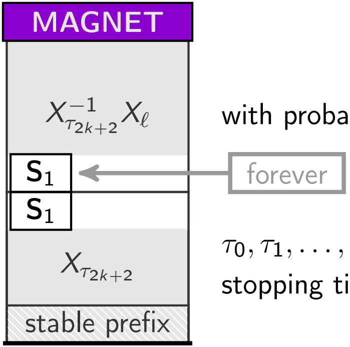

with proba > 0

$\tau_0, \tau_1, \ldots, \tau_k$: stopping times

Random walk in dimer groups

Prefix-convergence of $\mathbf{Gar}_{\mathbf{R}}(X_k)_{k \geqslant 0}$

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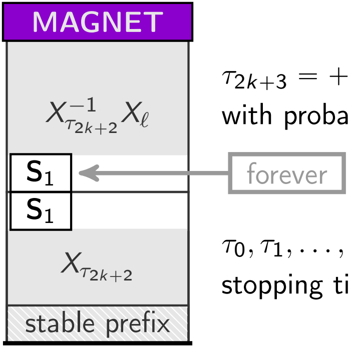
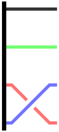


$\tau_{2k+3} = +\infty$
with proba > 0

$\tau_0, \tau_1, \ldots, \tau_k$:
stopping times

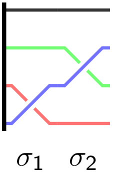
Contents

Do you like braiding your hair clockwise?



σ_1

Do you like braiding your hair clockwise?



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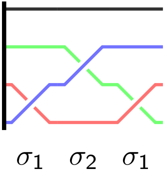

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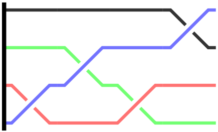

$\sigma_1 \quad \sigma_2 \quad \sigma_1 \quad \sigma_3$

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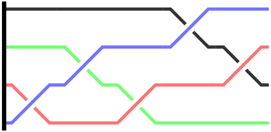

$\sigma_1 \quad \sigma_2 \quad \sigma_1 \quad \sigma_3 \quad \sigma_2$

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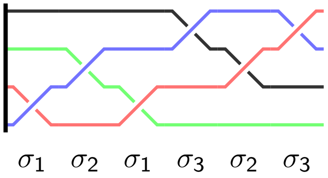
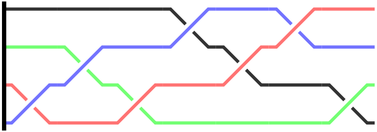


Do you like braiding your hair clockwise?



$\sigma_1 \quad \sigma_2 \quad \sigma_1 \quad \sigma_3 \quad \sigma_2 \quad \sigma_3 \quad \sigma_1$

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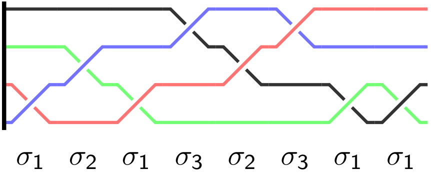

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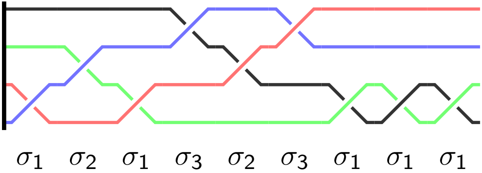

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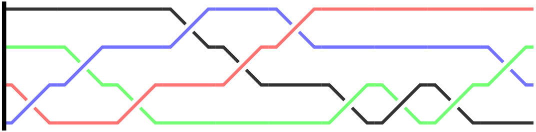

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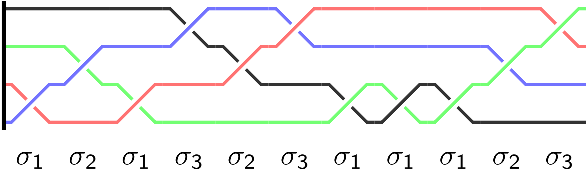

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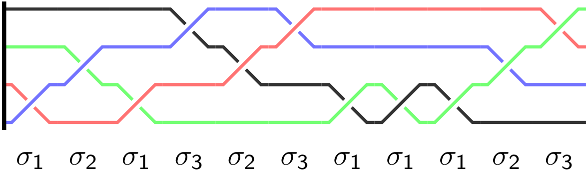

Artin braid diagrams

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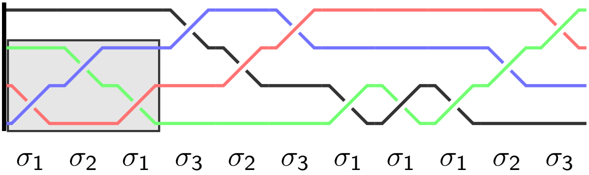

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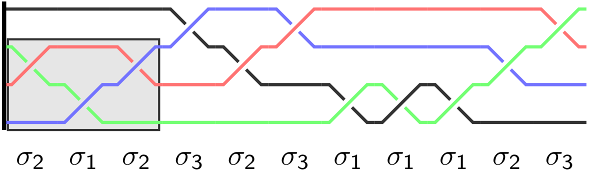

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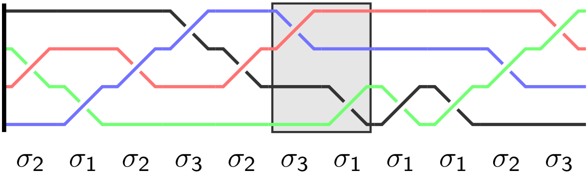

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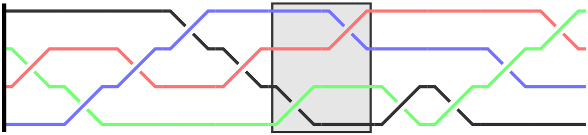

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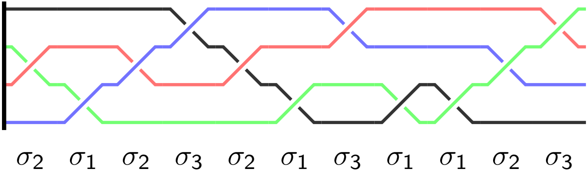

Artin braid diagrams

Braid monoid

$\mathcal{B}_n^+ = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, |i-j| \geqslant 2 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i \rangle^+$

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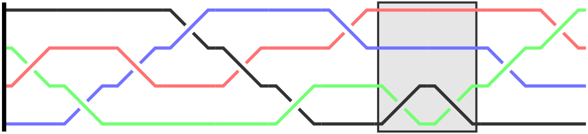

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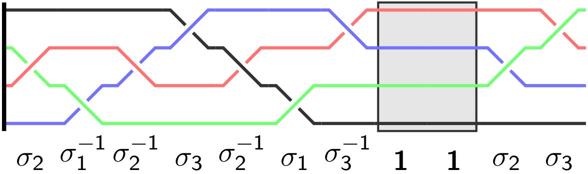

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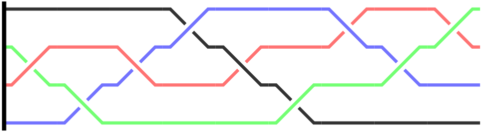

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Simple braids & Garside normal forms

Theorem [Garside 1969]

$\mathcal{B}_n^+$ is a lattice for both orders $\leqslant$ and $\geqslant$.

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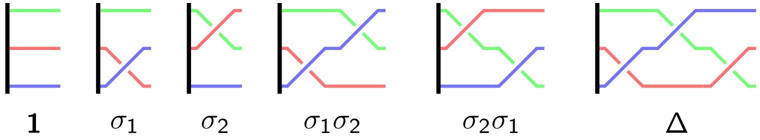

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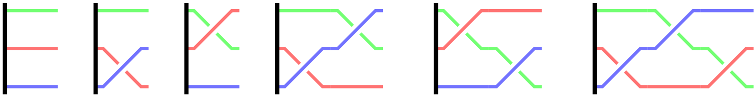

1 σ_1 σ_2 $\sigma_1\sigma_2$ $\sigma_2\sigma_1$ Δ

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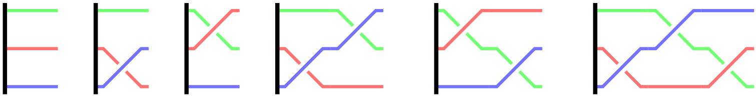
Simple braids?

Positive braids where strands never cross twice:

- in bijection with $\mathfrak{S}_n$;
- closed under $\leqslant$- and $\geqslant$-divisibility.



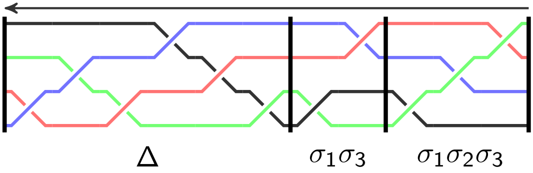
$\mathbf{1}$ σ_1 σ_2 $\sigma_1\sigma_2$ $\sigma_2\sigma_1$ Δ

Simple braids & Garside normal forms

Left Garside normal form

Minimal factorisation of the form $\beta = \beta_1 \beta_2 \cdots \beta_k$ such that

$$\forall i \leqslant k, \beta_i = \mathbf{GCD}_{\leqslant}(\Delta_n, \beta_i \beta_{i+1} \cdots \beta_k)$$



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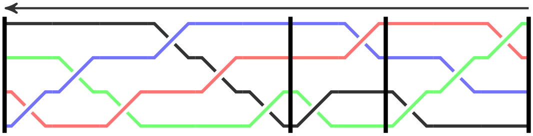

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$\mathbf{R}(\beta) = \{i : \beta \geqslant \sigma_i\}$

	Δ	$\sigma_1\sigma_3$	$\sigma_1\sigma_2\sigma_3$
L :	$\{1,2,3\}$	$\{1,3\}$	$\{1\}$
R :	$\{1,2,3\}$	$\{1,3\}$	$\{3\}$

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Right Garside normal form

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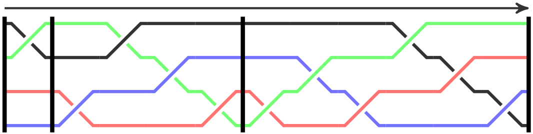

	σ_3	$\sigma_1\sigma_3\sigma_2\sigma_1$	Δ
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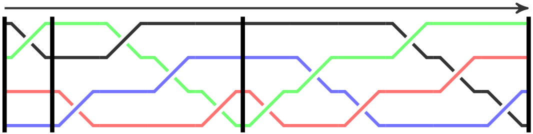

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Random walk in braid monoids

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1. Select i.i.d. generators $(Y_k)_{k \geqslant 0}$ in $\{\sigma_1, \ldots, \sigma_n\}$.

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Theorem [Folklore]

Convergence of the words

	$\mathbf{Gar_L}(X_k)_{k \geqslant 0}$	$\mathbf{Gar_R}(X_k)_{k \geqslant 0}$
prefix-	✓	
suffix-	✗	✓

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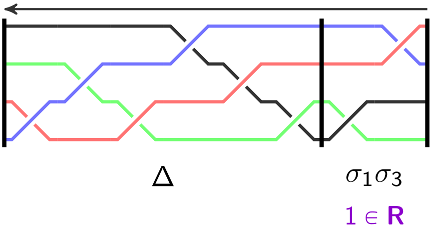
Random walk in braid monoids

Suffix-divergence of $\mathbf{Gar_L}(X_k)_{k \geqslant 0}$

Lemma: if $Y_k Y_{k+1} = \sigma_i \sigma_{i\pm1}$ for some $i \in \mathbf{R}(\mathbf{suffix}(\mathbf{Gar_L}(X_k)))$ then



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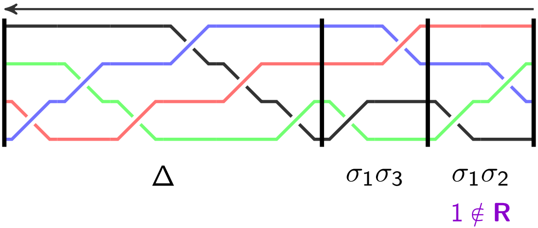

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Random walk in braid monoids

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$\Rightarrow$ Also proves that $\mathbf{Gar_L}(X_k)_{k \geqslant 0}$ prefix-converges

Random walk in braid monoids

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

Key ingredient: blocking braid $\mathbf{B} = \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6$ such that

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Random walk in braid monoids

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$\Rightarrow$ There exist blocking braids if n is even:

β_1

Random walk in braid monoids

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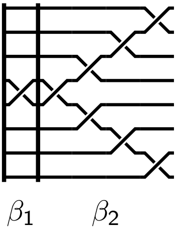

Random walk in braid monoids

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

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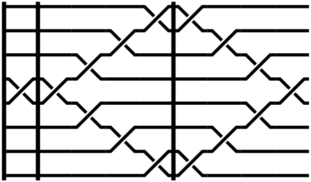

β_1 β_2

Random walk in braid monoids

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

Key ingredient: blocking braid $\mathbf{B} = \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6$ such that

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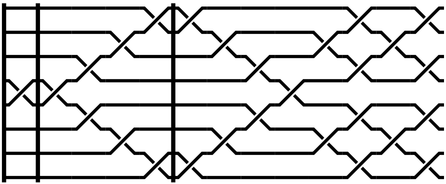

β_1 β_2 β_3

Random walk in braid monoids

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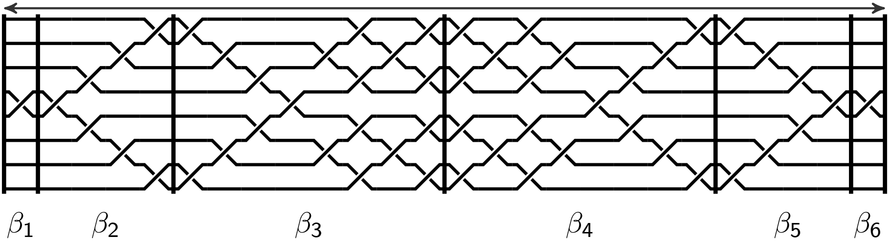

Random walk in braid monoids

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k \geqslant 0}$

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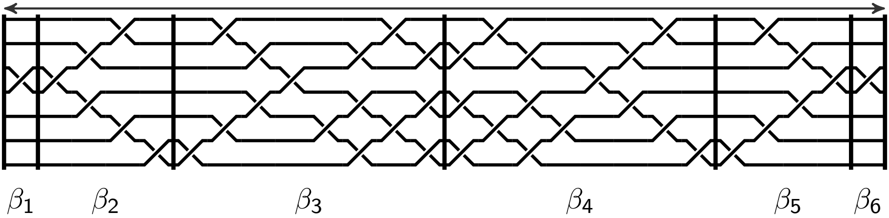

Random walk in braid monoids

Prefix-convergence of $\mathbf{Gar}_\mathbf{R}(X_k)_{k \geqslant 0}$ – continued

Lemma #1: for all $\alpha, \beta \in B_n^+$, if $\mathbf{Gar}_\mathbf{R}(\alpha\mathbf{B}) = \mathbf{Gar}_\mathbf{R}(\alpha) \cdot \mathbf{Gar}_\mathbf{R}(\mathbf{B})$ and ($\alpha\mathbf{B}\beta$ is Δ-free or $\beta \in \{\sigma_1, \ldots, \sigma_n\}$), then

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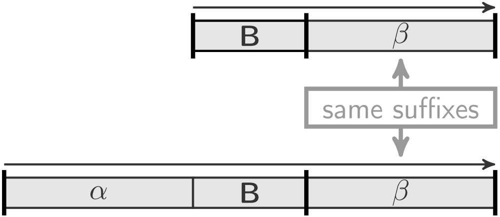

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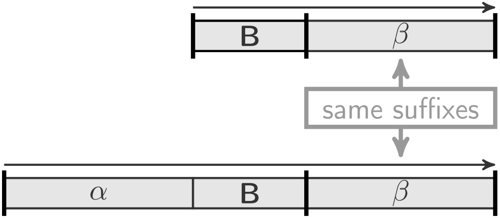

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Uses:
- Lemma #1
- Playing with magnets
- Tricky inductions

Random walk in braid monoids

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- Lemma #1
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- Ugly calculations

Prefix-convergence of $\mathbf{Gar_R}(X_k)_{k\geqslant 0}$ – continued

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Random walk in braid monoids

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$\Rightarrow$ Mission accomplished! ☺

Contents

What is our limit object?

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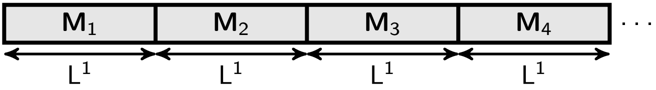
1. Limit of an infinite-state Markov chain with L^1 factors;



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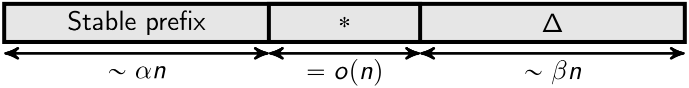

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Seeking non-trivial limits

Delete Δs

- **Unsatisfactory result**: $\mathbf{prefix}(\mathbf{Gar}_{\mathbf{L}}(X_k))_{k \geqslant 0}$ and $\mathbf{suffix}(\mathbf{Gar}_{\mathbf{R}}(X_k))_{k \geqslant 0}$ **must** converge (a.s. towards Δ)

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Convergence of the words

	$\mathbf{Gar_L}(X_k)_{k \geqslant 0}$	$\mathbf{Gar_R}(X_k)_{k \geqslant 0}$
prefix-	✓	✓
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Random walk in irreducible trace monoids

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Vincent Jugé & Jean Mairesse Stabilization of random braids

This looks like déjà vu...

Random walk in irreducible trace groups

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Vincent Jugé & Jean Mairesse Stabilization of random braids

From braid monoids to other monoids

Key ingredients

1. **Length-preserving**, **left-right symmetric** relations
2. **Two-way Garside family** $\mathcal{F}$: closed under $\leqslant$, $\geqslant$, $\mathbf{LCM}_{\leqslant}$ and $\mathbf{LCM}_{\geqslant}$
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Extensions to other monoids with Garside families

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Extensions to other monoids with Garside families

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From braid monoids to other monoids

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1. **Length-preserving**, **left-right symmetric** relations
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Extensions to other monoids with Garside families

- **Finite** family in dual braid monoids (of every spherical type)
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- **Infinite** family in all Artin-Tits monoids (without elements σ_i^2)
- **Finite**, **one-way Garside family** in all Artin–Tits monoids

From (dual) braid monoids to groups

Which Garside normal form do we choose?

From (dual) braid monoids to groups

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$\Rightarrow$ Similar or identical results in non-degenerate cases.

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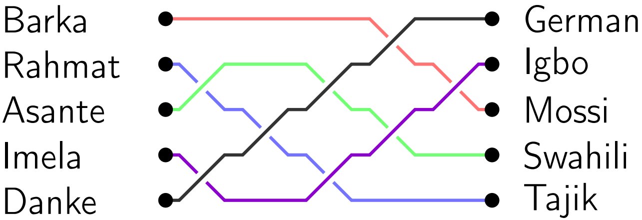
Thank you very much for your attention!



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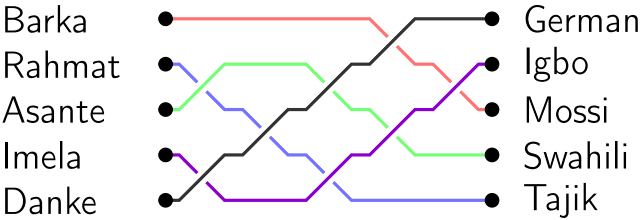

Questions?