The diagonal of the Stasheff polytope

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To Murray Gerstenhaber and Jim Stasheff

Summary. We construct an A-infinity structure on the tensor product of two A-infinity algebras by using the simplicial decomposition of the Stasheff polytope. The key point is the construction of an operad AA-infinity based on the simplicial Stasheff polytope. The operad AA-infinity admits a coassociative diagonal and the operad A-infinity is a retract by deformation of it. We compare these constructions with analogous constructions due to Saneblidze-Umble and Markl-Shnider based on the Boardman-Vogt cubical decomposition of the Stasheff polytope.

Key words: Stasheff polytope, associahedron, operad, bar construction, cobar construction, A-infinity algebra, AA-infinity algebra, diagonal.

Introduction

An associative algebra up to homotopy, or A_∞-algebra, is a chain complex (A, d_A) equipped with an n-ary operation μ_n for each $n \geq 2$ verifying $\mu \circ \mu = 0$. See [13], or, for instance, [3]. Here we put

$$\mu := d_A + \mu_2 + \mu_3 + \cdots \ : T(A) \to T(A),$$

where μ_n has been extended to the tensor module $T(A)$ by derivation. In particular μ_2 is not associative, but only associative up to homotopy in the following sense:

$$\mu_2 \circ (\mu_2 \otimes \mathrm{id}) - \mu_2 \circ (\mathrm{id} \otimes \mu_2) = d_A \circ \mu_3 + \mu_3 \circ d_{A^{\otimes 3}} \ .$$

Putting an A_∞-algebra structure on the tensor product of two A_∞-algebras is a long standing problem, cf. for instance [10, 2]. Recently a solution has been constructed by Saneblidze and Umble, cf. [11, 12], by constructing a diagonal

$A_\infty \to A_\infty \otimes A_\infty$ on the operad A_∞ which governs the A_∞-algebras. Recall that, over a field, the operad A_∞ is the minimal model of the operad As governing the associative algebras. The differential graded module $(A_\infty)_n$ of the n-ary operations is the chain complex of the Stasheff polytope. In [9] Markl and Shnider give a conceptual construction of the Saneblidze-Umble diagonal by using the Boardman-Vogt model of As. This model is the bar-cobar construction on As, denoted ΩBAs, in the operadic framework. It turns out that there exists a coassociative diagonal on ΩBAs. This diagonal, together with the quasi-isomorphisms $q : A_\infty \to \Omega BAs$ and $p : \Omega BAs \to A_\infty$ permit them to construct a diagonal on A_∞ by composition:

$$A_\infty \xrightarrow{q} \Omega BAs \to \Omega BAs \otimes \Omega BAs \xrightarrow{p\otimes p} A_\infty \otimes A_\infty \ .$$

The aim of this paper is to give an alternative solution to the diagonal problem by relying on the *simplicial decomposition of the Stasheff polytope* described in [6]. It leads to a new model AA_∞ of the operad As, whose dg module $(AA_\infty)_n$ is the chain complex of a simplicial decomposition of the Stasheff polytope. Because of its simplicial nature, the operad AA_∞ has a coassociative diagonal (Alexander-Whitney map) and therefore we get a new diagonal on A_∞ by composition:

$$A_\infty \xrightarrow{q'} AA_\infty \to AA_\infty \otimes AA_\infty \xrightarrow{p'\otimes p'} A_\infty \otimes A_\infty \ .$$

Since A_∞ is the minimal model of As the quasi-isomorphisms q and q' are well-defined. However for p and p' one has choices. The choice for p taken in [9] has a nice geometric interpretation as a factorization through the cube. We describe a choice for p' which factorizes through the simplex. It is related to the shortest path in the Tamari poset structure of the planar binary trees.

Moreover we provide an explicit comparison map between the two models ΩBAs and AA_∞ by using the simplicialization of the cubical decomposition of the Stasheff polytope.

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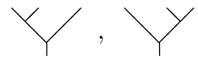
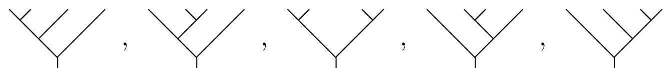
1 Stasheff polytope (associahedron)

We recall briefly the construction of the Stasheff polytope, also called associahedron, and its simplicial realization, which is the key tool of this paper. All chain complexes in this paper are made of free modules over a commutative ring $\mathbb{K}$ (which can be $\mathbb{Z}$ or a field).

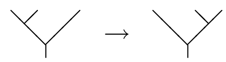
1.1 Planar binary trees

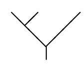
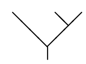
We denote by PBT_n the set of *planar binary trees* having n leaves:

$PBT_1 := \{|\}, PBT_2 := \{$  $\}, PBT_3 := \{$  $\}$,

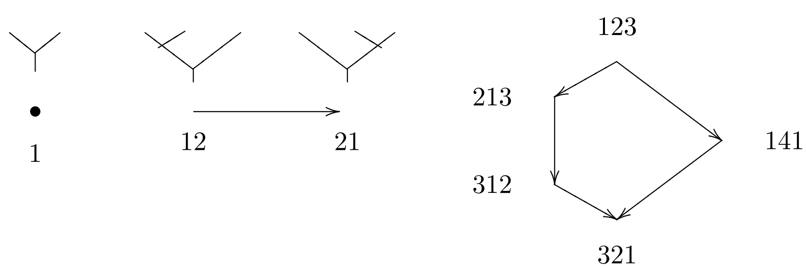
$PBT_4 := \{$  $\}$.

So $t \in PBT_n$ has one root, n leaves, $(n-1)$ internal vertices, $(n-2)$ internal edges. Each vertex is binary (two inputs, one output). The number of elements in PBT_{n+1} is known to be the *Catalan number* $c_n = \frac{(2n)!}{n!\,(n+1)!}$. There is a partial order on PBT_n called the *Tamari order* and defined as follows. On PBT_3 it is given by



More generally, if t and s are two planar binary trees with the same number of leaves, there is a covering relation $t \to s$ if and only if s can be obtained from t by replacing a local pattern like  by  . In other words s is obtained from t by moving a leaf or an internal edge from left to right over a fork.

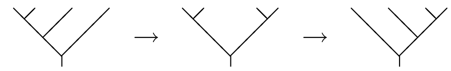
Examples:



where the elements of PBT_4 (listed above) are denoted $123, 213, 141, 312, 321$, respectively (coordinates in $\mathbb{R}^4$, cf. [5]).

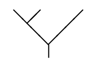
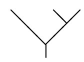
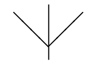
The Tamari poset admits an initial element: the right comb, and a terminal element: the left comb. There is a shortest path from the initial element to the terminal element. It is made of the trees which are the grafting of some right comb with a left comb. In PBT_n there are $n-1$ of them. This sequence of planar binary trees will play a significant role in the comparison of different cell realizations of the Stasheff polytope.

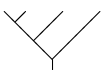
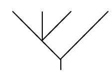
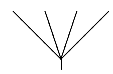
Example: the shortest path in PBT_4:



1.2 Planar trees

We now consider the planar trees for which an internal vertex has one root and k leaves, where k can be any integer greater than or equal to 2. We denote by PT_n the set of planar trees with n leaves:

$PT_1 := \{|\}, PT_2 := \{$  $\}, PT_3 := \{$  ,  ,  $\}$,

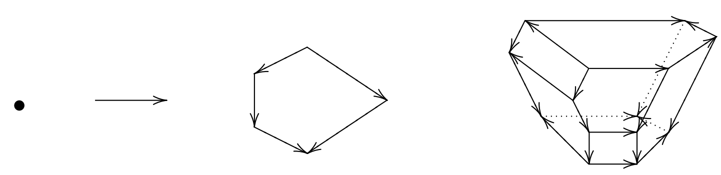
$PT_4 := \{$  $, \ldots,$  $, \ldots,$  $\}$.

Each set PT_n is graded according to the number of internal vertices, i.e. $PT_n = \bigcup_{n \geq k} PT_{n,k}$ where $PT_{n,k}$ is the set of planar trees with n leaves and k internal vertices. For instance $PT_{n,1}$ contains only one element which we call the n-*corolla* (the last element in the above sets). It is clear that $PT_{n,n-1} = PBT_n$.

1.3 The Stasheff polytope, alias associahedron

The *associahedron* is a cellular complex $\mathcal{K}^n$ of dimension n, first constructed by Jim Stasheff [13], which can be realized as a convex polytope whose extremal vertices are in one-to-one correspondence with the planar binary trees in PBT_{n+2}, cf. [5] for details. The edges of the polytope are indexed by the covering relations of the Tamari poset.

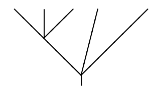
Examples:



$\mathcal{K}^0$ $\mathcal{K}^1$ $\mathcal{K}^2$ $\mathcal{K}^3$

Its k-cells are in one-to-one correspondence with the planar trees in $PT_{n+2,n+1-k}$. For instance the 0-cells are indexed by the planar binary trees, and the top cell is indexed by the corolla.

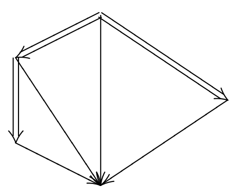
It will prove helpful to adopt the notation $\mathcal{K}^t$ to denote the cell in $\mathcal{K}^n$ indexed by $t \in PT_{n+2}$. For instance, if t is the corolla, then $\mathcal{K}^t = \mathcal{K}^n$. As a space $\mathcal{K}^t$ is the product of p associahedrons (or associahedra, as you like), where p

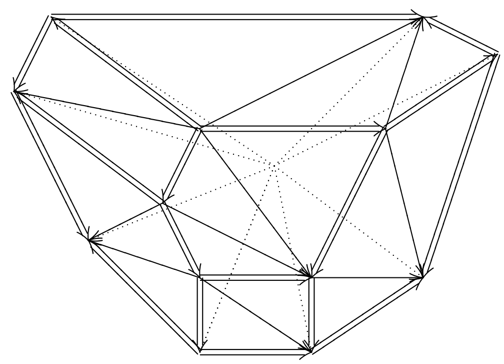
is the number of internal vertices of t. For instance, if $t =$  , then $\mathcal{K}^t = \mathcal{K}^1 \times \mathcal{K}^1$.

1.4 Chain complex of the simplicial associahedron

In [6] we constructed a simplicial set $\mathcal{K}^n_{\text{simp}}$ whose geometric realization gives a simplicial decomposition of the associahedron. In other words the associahedron $\mathcal{K}^n$ is viewed as a union of n-simplices (there are $(n+1)^{n-1}$ of them). For $n = 1$, we have $\mathcal{K}^1_{\text{simp}} = \mathcal{K}^1$.

Examples: $\mathcal{K}^2_{\text{simp}}$ and $\mathcal{K}^3_{\text{simp}}$





This simplicial decomposition is constructed inductively as follows. We fatten the simplicial set $\mathcal{K}^{n-1}_{\text{simp}}$ into a new simplicial set fat$\mathcal{K}^{n-1}_{\text{simp}}$, cf. [6]. Then $\mathcal{K}^n_{\text{simp}}$ is defined as the cone over fat$\mathcal{K}^{n-1}_{\text{simp}}$ (as in the original construction of Stasheff [13]). In the pictures the fatten space as been indicated by double arrows. Since, in the process of fattenization, the new cells are products of smaller dimensional associahedrons we get the following main property.

Proposition 1.5 *The simplicial decomposition of a face $\mathcal{K}^{i_1} \times \cdots \times \mathcal{K}^{i_k}$ of $\mathcal{K}^n$ is the product of the simplicializations of each component $\mathcal{K}^{i_j}$.*

Proof. It is immediate from the inductive procedure which constructs $\mathcal{K}^n$ out of $\mathcal{K}^{n-1}$. □

Considered as a cellular complex, still denoted $\mathcal{K}^n_{\text{simp}}$, the simplicialized associahedron gives rise to a chain complex denoted $C_*(\mathcal{K}^n_{\text{simp}})$. This chain complex is the normalized chain complex of the simplicial set. It is the quotient of the chain complex associated to the simplicial set, divided out by the degeneracies (cf. for instance [7] Chapter VIII). A basis of $C_0(\mathcal{K}^n_{\text{simp}})$ is given by PBT_{n+2} and a basis of $C_n(\mathcal{K}^n_{\text{simp}})$ is given by the $(n+1)^{n-1}$ top simplices (in bijection with the parking functions, cf. [6]). It is zero higher up.

2 The operad AA_∞

We construct the operad AA_∞ and compare it with the operad A_∞ governing the associative algebras up to homotopy.

2.1 Differential graded non-symmetric operad [8]

By definition a *differential graded non-symmetric operad*, dgns operad for short, is a family of chain complexes $\mathcal{P}_n = (\mathcal{P}_n, d)$ equipped with chain complex morphisms

$$\gamma_{i_1\cdots i_n} : \mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \cdots \otimes \mathcal{P}_{i_n} \to \mathcal{P}_{i_1+\cdots+i_n},$$

which satisfy the following associativity property. Let $\mathcal{P}$ be the endofunctor of the category of chain complexes over $\mathbb{K}$ defined by $\mathcal{P}(V) := \bigoplus_n \mathcal{P}_n \otimes V^{\otimes n}$. The maps $\gamma_{i_1\cdots i_n}$ give rise to a transformation of functors $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$. This transformation of functors γ is supposed to be associative. Moreover we suppose that $\mathcal{P}_0 = 0, \mathcal{P}_1 = \mathbb{K}$ (trivial chain complex concentrated in degree 0). The transformation of functors $\mathrm{Id} \to \mathcal{P}$ determined by $\mathcal{P}_1$ is supposed to be a unit for γ. So we can denote by id the generator of $\mathcal{P}_1$. Since $\mathcal{P}_n$ is a graded module, $\mathcal{P}$ is bigraded. The integer n is called the "arity" in order to differentiate it from the degree of the chain complex.

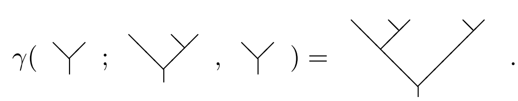
2.2 The fundamental example A_∞

The operad A_∞ is a dgns operad constructed as follows:

$$A_{\infty,n} := C_*(\mathcal{K}^{n-2}) \text{ (chain complex of the cellular space } \mathcal{K}^{n-2}).$$

Let us observe that $C_k(\mathcal{K}^{n-2}) = \mathbb{K}[PT_{n,n-1-k}]$.

Let us denote by $As^{¡}$ the family of one dimensional modules $(As^{¡}_n)_{n\geq 1}$ generated by the corollas (unique top cells). It is easy to check that there is a natural identification of graded (by arity) modules $A_\infty = \mathcal{T}(As^{¡})$, where $\mathcal{T}(As^{¡})$ is the free operad over $As^{¡}$. This identification is given by grafting on the leaves as follows. Given trees $t, t_1, \ldots, t_n$ where t has n leaves, the tree $\gamma(t; t_1, \ldots, t_n)$ is obtained by identifying the ith leaf of t with the root of t_i. For instance:



Moreover, under this identification, the composition map γ is a chain map, therefore A_∞ is a dgns operad.

This construction is a particular example of the so-called "cobar construction" Ω, i.e. $A_\infty = \Omega As^{¡}$ where $As^{¡}$ is considered the cooperad governing the coassociative coalgebras (cf. [8]).

For any chain complex A there is a well-defined dgns operad $\mathrm{End}(A)$ given by $\mathrm{End}(A)_n = \mathrm{Hom}(A^{\otimes n}, A)$. An A_∞-algebra is nothing but a morphism of operads $A_\infty \to \mathrm{End}(A)$. The image of the corolla is the n-ary operation μ_n alluded to in the introduction.

2.3 Hadamard product of operads, the diagonal problem

Given two operads $\mathcal{P}$ and $\mathcal{Q}$, their Hadamard product, also called tensor product, is the operad $\mathcal{P}\otimes\mathcal{Q}$ defined as $(\mathcal{P}\otimes\mathcal{Q})_n := \mathcal{P}_n\otimes\mathcal{Q}_n$. The composition map is simply the tensor product of the two composition maps.

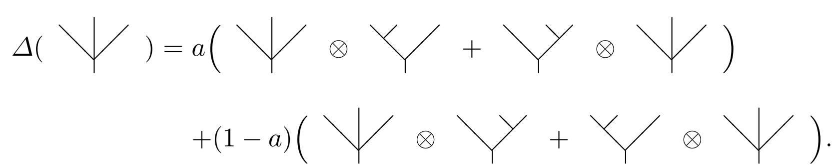
It is a long-standing problem to decide if, given two A_∞-algebras A and B, there is a natural A_∞-structure on their tensor product $A \otimes B$ which extends the natural dg nonassociative algebra structure, cf. [10, 2]. It amounts to construct a diagonal on A_∞, i.e. an operad morphism $\Delta : A_\infty \to A_\infty \otimes A_\infty$, since, by composition, we get an A_∞-structure on $A \otimes B$:

$$A_\infty \to A_\infty \otimes A_\infty \to \mathrm{End}(A) \otimes \mathrm{End}(B) \to \mathrm{End}(A \otimes B)\ .$$

Let us recall that the classical associative structure on the tensor product of two associative algebras can be interpreted operadically as follows. There is a diagonal on the operad As given by

$$As_n \to As_n \otimes As_n, \quad \mu_n \mapsto \mu_n \otimes \mu_n\ .$$

Since we want the diagonal Δ to be compatible with the diagonal on As, there is no choice in arity 2, and we have $\Delta(\mu_2) = \mu_2 \otimes \mu_2$. Observe that these two elements are in degree 0. In arity 3, since μ_3 is of degree 1 and $\mu_3 \otimes \mu_3$ of degree 2, this last element cannot be the answer. In fact there is already a choice (parameter a):



By some tour de force Samson Saneblidze and Ron Umble constructed such a diagonal on A_∞ in [11]. Their construction was re-interpreted in [9] by Markl and Shnider through the Boardman-Vogt construction (see section 3 below for a brief account of their work). We will use the simplicialization of the associahedron described in [6] to give another solution to the diagonal problem.

2.4 Construction of the operad AA_∞

We define the dgns operad AA_∞ as follows. The chain complex $AA_{\infty,n}$ is the chain complex of the simplicialization of the associahedron considered as a cellular complex (cf. 1.4):

$$AA_{\infty,n} = C_*(\mathcal{K}^{n-2}_{\mathrm{simp}}) \ .$$

In low dimension we take $AA_{\infty,0} = 0, AA_{\infty,1} = \mathbb{K}\,\mathrm{id}$. So a basis of $AA_{\infty,n}$ is made of the cells (nondegenerate simplices) of $\mathcal{K}^{n-2}_{\mathrm{simp}}$. Let us now construct the composition map

$$\gamma = \gamma^{AA_\infty} : AA_{\infty,n} \otimes AA_{\infty,i_1} \otimes \cdots \otimes AA_{\infty,i_n} \to AA_{\infty,i_1+\cdots+i_n}.$$

We denote by Δ^k the standard k-simplex. Let $\iota : \Delta^k \rightarrowtail \mathcal{K}^{n-2}_{\mathrm{simp}}$ be a cell, i.e. a linear generator of $C_*(\mathcal{K}^{n-2}_{\mathrm{simp}})$. Given such cells

$$\iota_0 \in AA_{\infty,n},\ \iota_1 \in AA_{\infty,i_1},\ \ldots,\ \iota_n \in AA_{\infty,i_n}$$

we construct their image $\gamma(\iota_0; \iota_1, \ldots, \iota_n) \in AA_{\infty,m}$, where $m := i_1 + \cdots + i_n$ as follows. We denote by k_i the dimension of the cell ι_i.

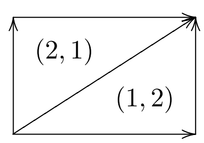
Let t_n be the n-corolla in PT_n and let $s := \gamma(t_n; t_{i_1}, \ldots, t_{i_n}) \in PT_m$ be the grafting of the trees $t_{i_1}, \ldots, t_{i_n}$ on the leaves of t_n. As noted before this is the composition in the operad A_∞. The tree s indexes a cell $\mathcal{K}^s$ of the space $\mathcal{K}^{m-2}$, which is combinatorially homeomorphic to $\mathcal{K}^{n-2} \times \mathcal{K}^{i_1-2} \times \cdots \times \mathcal{K}^{i_n-2}$. In other words it determines a map

$$s_* : \mathcal{K}^{n-2} \times \mathcal{K}^{i_1-2} \times \cdots \times \mathcal{K}^{i_n-2} = \mathcal{K}^s \rightarrowtail \mathcal{K}^{m-2}.$$

The product of the inclusions $\iota_j, j = 0, \ldots, n$, defines a map

$$\iota_0 \times \iota_1 \times \cdots \times \iota_n : \Delta^{k_0} \times \Delta^{k_1} \times \cdots \times \Delta^{k_n} \rightarrowtail \mathcal{K}^{n-2} \times \mathcal{K}^{i_1-2} \times \cdots \times \mathcal{K}^{i_n-2}.$$

Let us recall that a product of standard simplices can be decomposed into the union of standard simplices. These pieces are indexed by the multi-shuffles α. Example: $\Delta^1 \times \Delta^1 = \Delta^2 \cup \Delta^2$:



So, for any multi-shuffle α there is a map

$$f_\alpha : \Delta^l \to \Delta^{k_0} \times \Delta^{k_1} \times \cdots \times \Delta^{k_n},$$

where $l = k_0 + \cdots + k_n$. By composition of maps we get

$$s_* \circ (\iota_0 \times \cdots \times \iota_n) \circ f_\alpha : \Delta^l \to \mathcal{K}^{m-2}$$

which is a linear generator of $C_l(\mathcal{K}^{m-2}_{\mathrm{simp}})$ by construction of the triangulation of the associahedron, cf. [5]. By definition $\gamma(\iota_0; \iota_1, \ldots, \iota_n)$ is the algebraic sum of the cells $s_* \circ (\iota_0 \times \cdots \times \iota_n) \circ f_\alpha$ over the multi-shuffles.

Proposition 2.5 *The graded chain complex AA_∞ and γ constructed above define a dgns operad, denoted AA_∞. The operad AA_∞ is a model of the operad As.*

Proof. We need to prove associativity for γ. It is an immediate consequence of the associativity for the composition of trees (operadic structure of A_∞) and the associativity property for the decomposition of the product of simplices into simplices.

Since the associahedron is contractible, taking the homology gives a graded linear map $C_*(\mathcal{K}^{n-2}_{\text{simp}}) \to \mathbb{K}\,\mu_n$, where μ_n is in degree 0. This map obviously induces an isomorphism on homology. These maps assemble into a dgns operad morphism $AA_\infty \to As$ which is quasi-isomorphism. Hence AA_∞ is a resolution of As, that is a model of As in the category of dgns operads. □

2.6 Remark

In order to construct the operad AA_∞ we could also construct, first, a simplicial ns operad $n \mapsto \mathcal{K}^{n-2}_{\text{simp}}$ (simplicial set) for $n \geq 2$ and $\{\text{id}\}$ (trivial simplicial set) for $n = 1$. Second, we use the Eilenberg-Zilber map and the Alexander-Whitney map to induce an operadic structure on the normalized chain complex.

Proposition 2.7 *The operad AA_∞ admits a coassociative diagonal.*

Proof. This diagonal $\Delta : AA_\infty \to AA_\infty \otimes AA_\infty$ is determined by its value in arity n for all n, that is a chain complex morphism

$$C_*(\mathcal{K}^{n-2}_{\text{simp}}) \to C_*(\mathcal{K}^{n-2}_{\text{simp}}) \otimes C_*(\mathcal{K}^{n-2}_{\text{simp}}).$$

This morphism is defined as the composite

$$C_*(\mathcal{K}^{n-2}_{\text{simp}}) \xrightarrow{\Delta_*} C_*(\mathcal{K}^{n-2}_{\text{simp}} \times \mathcal{K}^{n-2}_{\text{simp}}) \xrightarrow{AW} C_*(\mathcal{K}^{n-2}_{\text{simp}}) \otimes C_*(\mathcal{K}^{n-2}_{\text{simp}}),$$

where Δ_* is induced by the diagonal on the simplicial set, and where AW is the Alexander-Whitney map. Let us recall from [7], Chapter VIII, the construction of the AW map. Denote by $d_0, \ldots, d_n$ the face operators of the simplicial set. If x is a simplex of dimension n, then we define $d_{max}(x) := d_n(x)$. So, for instance $(d_{max})^2(x) = d_{n-1}d_n(x)$. By definition the AW map is given by

$$(x, y) \mapsto \sum_{i=0}^{n} \big((d_0)^i(x), (d_{max})^{n-i}(y)\big).$$

It is straightforward to check that this diagonal is compatible with the operad structure.

The coassociativity property follows from the coassociativity property of the Alexander-Whitney map. □

2.8 Comparing A_∞ to AA_∞

Since $\mathcal{K}^{n-2}_{\text{simp}}$ is a simplicialization of $\mathcal{K}^{n-2}$, there is a chain complex map

$$q' : C_*(\mathcal{K}^{n-2}) \to C_*(\mathcal{K}^{n-2}_{\text{simp}}),$$

where a cell of $\mathcal{K}^{n-2}$ is sent to the algebraic sum of the simplices it is made of.

Proposition 2.9 *The map $q' : A_\infty \to AA_\infty$ induced by the maps $q' : C_*(\mathcal{K}^n) \to C_*(\mathcal{K}^n_{\text{simp}})$ is a quasi-isomorphism of dgns operads.*

Proof. It is sufficient to prove that the maps q' on the chain complexes are compatible with the operadic composition:

$$q'(\gamma^{As}(t; t_1, \dots, t_n)) = \gamma^{AA_\infty}(q'(t); q'(t_1), \dots, q'(t_n)).$$

This equality follows from the definition of γ^{AA_∞} and Proposition 1.5. □

Moreover we have commutative diagrams:

$$\begin{array}{ccc} C_*(\mathcal{K}^{n-2}) & \xrightarrow{q'} & C_*(\mathcal{K}^{n-2}_{\text{simp}}) \\ & {\scriptstyle H_*}\searrow \quad \swarrow{\scriptstyle H_*} & \\ & \mathbb{K}\,\mu_n & \end{array} \qquad \begin{array}{ccc} A_\infty & \xrightarrow{q'} & AA_\infty \\ & \searrow \quad \swarrow & \\ & As & \end{array}$$

2.10 Comparing AA_∞ to A_∞

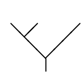
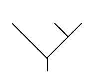
Let us construct an inverse quasi-isomorphism $p' : AA_\infty \to A_\infty$. We construct a quasi-isomorphism $p' : C_*(\mathcal{K}^n_{\text{simp}}) \to C_*(\mathcal{K}^n)$ as a composite

$$C_*(\mathcal{K}^n_{\text{simp}}) \xrightarrow{p'_1} C_*(\Delta^n) \xrightarrow{p'_2} C_*(\mathcal{K}^n)$$

where Δ^n is the standard n-simplex and the first map is induced by a simplicial map.

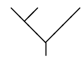
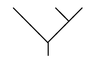
2.11 The simplicial map $p'_1 : \mathcal{K}^n_{\text{simp}} \to \Delta^n$

Since a simplex in $\mathcal{K}^n_{\text{simp}}$ is completely determined by its vertices, it suffices to define p'_1 on the vertices. Recall that the 0-cells of $\mathcal{K}^n_{\text{simp}}$ are indexed by the planar binary trees, and that the 0-cells of Δ^n are indexed by the integers $\underline{0}, \dots, \underline{n}$ (with poset structure given by the standard order). Let $t \in PBT_{n+2}$ and let $\omega(t)$ be the number of leaves on the right side of the root of t. For instance:

$\omega($  $) = 1, \quad \omega($  $) = 2$.

In $\mathcal{K}^{n-2}_{\text{simp}}$ we single out the $(n-2)$-simplex whose vertices are on the shortest path from the right comb to the left comb, cf. 1.1.

Lemma 2.12 *The map $p_1' : \mathcal{K}^n_{\text{simp}} \to \Delta^n$ induced by $t \mapsto \underline{\omega(t)-1}$ on the 0-cells is a simplicial map. It maps bijectively the singled out simplex to its image.*

Proof. It suffices to check that this is a poset map. For any covering relation $t \to s$ some local pattern  is changed into . If the root is not involved, then $\omega(s) = \omega(t)$ and we are done. If the root is involved, then $\omega(s) = \omega(t) + 1$ and we are done too.

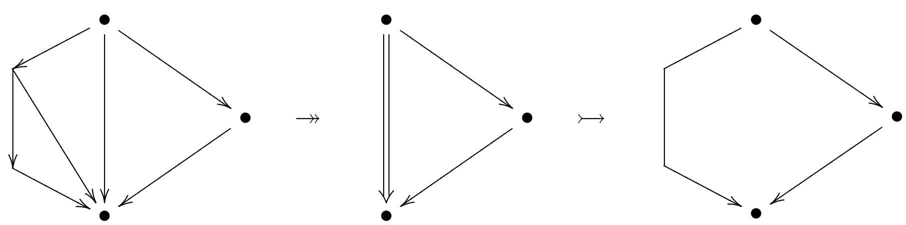
The vertices of the singled out simplex are indexed by trees t, which are the grafting of a rightcomb with a left comb. If the left comb has i leaves, then $\omega(t) = i$ and the image of t is precisely $\underline{i-1}$. Hence the simplex made out of these trees is mapped bijectively to Δ^n. □

2.13 The chain complex map $p_2' : C_*(\Delta^n) \to C_*(\mathcal{K}^n)$

There is a unique chain complex map $C_*(\Delta^n) \to C_*(\mathcal{K}^n)$ which, geometrically, sends the vertices of Δ^n to the vertices of the shortest path. In particular the top-cell of Δ^n is sent to the top-cell of $\mathcal{K}^n$.

Composing p_1' and p_2' we obtain the map $p' : C_*(\mathcal{K}^n_{\text{simp}}) \to C_*(\mathcal{K}^n)$ for all n.

Example $n = 2$:



The double line indicates that an entire square is mapped to this interval.

The maps p' assemble into a morphism $p' : AA_\infty \to A_\infty$ of graded chain complexes. It is obviously a quasi-inverse of q'.

Theorem 2.14 *The map $p' : AA_\infty \to A_\infty$ is a morphism of dgns operads. The composite*

$$A_\infty \xrightarrow{q'} AA_\infty \xrightarrow{\Delta} AA_\infty \otimes AA_\infty \xrightarrow{p' \otimes p'} A_\infty \otimes A_\infty \ .$$

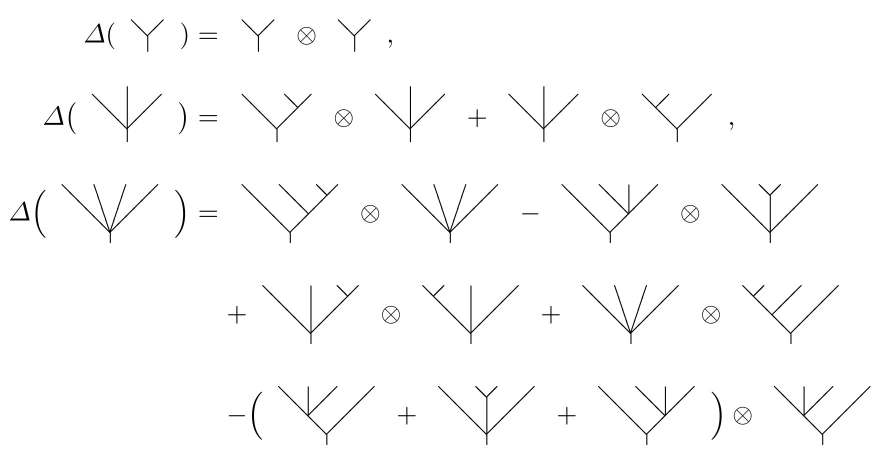
is a diagonal for the operad A_∞.

Proof. Recall from section 2 that the operad structure of AA_∞ is esssentially defined by the operad structure of A_∞. Except for the cells of the singled out simplex the image of the cells of AA_∞ are 0. Since the top-cell of the singled out simplex is the top-cell of the associahedron, the compatibility of the operadic structure follows.

From the properties of q', Δ and p', it follows that this composition is compatible with the diagonal on As, therefore this composite permits us to put an A_∞-structure on $A \otimes B$ (A and B being A_∞-algebras) such that the product μ_2 is as expected. □

2.15 The first formulas

Let us give the explicit form of $\Delta(\mu_n)$ for $n = 2, 3, 4$:



2.16 Remarks

Though the diagonal of AA_∞ that we constructed is coassociative, the diagonal of As is not because of the behavior of p'. In fact it has been shown in [9] that there does not exist any coassociative diagonal on A_∞. However the diagonal on A_∞ is coassociative up to homotopy.

Proposition 2.17 *If A is an associative algebra and B an A_∞ algebra, then the A_∞-structure on $A \otimes B$ is given by*

$$\mu_n(a_1 \otimes b_1, \ldots, a_n \otimes b_n) = a_1 \cdots a_n \otimes \mu_n(b_1, \ldots, b_n).$$

Proof. In the formula for Δ we have $\mu_n = 0$ for all $n \geq 3$, that is, any tree with a k-valent vertex for $k \geq 3$ is 0 on the left side. Hence the only term which is left is *comb* ⊗ *corolla*, whence the assertion. □

3 Comparing the operads AA_∞ and ΩBAs

We first give a brief account of [9, 11] where a diagonal of the operad A_∞ is constructed by using a coassociative diagonal on ΩBAs. Then we compare the two operads AA_∞ and ΩBAs.

3.1 Cubical decomposition of the associahedron [1]

The associahedron can be decomposed into cubes as follows.

For each tree $t \in PBT_{n+2}$ we take a copy of the cube I^n (where $I = [0,1]$ is the interval) which we denote by I^n_t. Then the associahedron $\mathcal{K}^n$ is the quotient

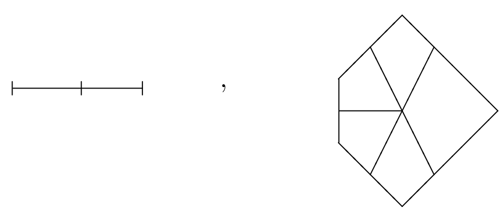
$$\mathcal{K}^n := \bigsqcup_t I^n_t / \sim$$

where the equivalence relation is as follows. We think of an element $\tau = (t; \lambda_1, \ldots, \lambda_n) \in I^n_t$ as a tree of type t where the λ_i's are the lengths of the internal edges. If some of the λ_i's are 0, then the geometric tree determined by τ is not binary anymore (since some of its internal edges have been shrinked to a point). We denote the new tree by $\bar{\tau}$. For instance, if none of the λ_i's is zero, then $\bar{\tau} = t$; if all the λ_i's are zero, then the tree $\bar{\tau}$ is the corolla (only one vertex). The equivalence relation $\tau \sim \tau'$ is defined by the following two conditions:

- $\bar{\tau} = \bar{\tau}'$,
- the lengths of the nonzero-length edges of τ are the same as those of τ'.

Hence $\mathcal{K}^n$ is obtained as a cubical realization denoted $\mathcal{K}^n_{\text{cub}}$.

Examples:



$\mathcal{K}^1$, $\mathcal{K}^2$

3.2 Markl-Shnider version of Saneblidze-Umble diagonal [9, 11]

In [1] Boardman and Vogt showed that the bar-cobar construction on the operad As is a dgns operad ΩBAs whose chain complex in arity n can be identified with the chain complex of the cubical decomposition of the associahedron:

$$(\Omega BAs)_n = C_*(\mathcal{K}^{n-2}_{\text{cub}}) \ .$$

In [9] (where $\mathcal{K}^{n-2}_{\text{cub}}$ is denoted W_n and $\mathcal{K}^{n-2}$ is denoted K_n) Markl and Shnider use this result to construct a coassociative diagonal on the operad ΩBAs. There is an obvious quasi-isomorphism $q : A_\infty \to \Omega BAs$. They construct an inverse quasi-isomorphism $p : \Omega BAs \to A_\infty$ by giving explicit algebraic formulas. At the chain level the map $p : C_*(\mathcal{K}^n_{\text{cub}}) \to C_*(\mathcal{K}^n)$ is the composite of two maps

$$C_*(\mathcal{K}^{n-2}_{\text{cub}}) \xrightarrow{p_1} C_*(I^{n-2}) \xrightarrow{p_2} C_*(\mathcal{K}^{n-2}) \ .$$

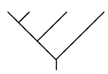
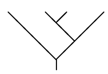
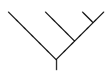
In the next section we give a geometric interpretation of these maps following a cubical description of the associahedron given in the Appendix. These maps p assemble to give the morphism of operads $p : \Omega BAs \to A_\infty$.

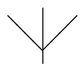
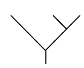
Markl and Shnider claim that the composite

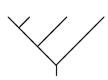
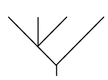
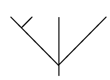
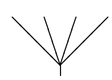
$$A_\infty \xrightarrow{q} \Omega BAs \to \Omega BAs \otimes \Omega BAs \xrightarrow{p\otimes p} A_\infty \otimes A_\infty$$

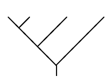
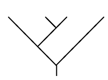
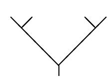
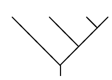
is the Saneblidze-Umble diagonal.

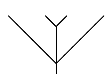
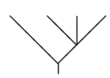
3.3 The geometric maps p_1 and p_2

The map $p_1 : C_*(\mathcal{K}^n_{\text{cub}}) \to C_*(I^n)$ is induced by a cellular (in fact cubical) map which is completely determined by the image of the vertices of $\mathcal{K}^n$. Let t be such a vertex. In the cubical description of $\mathcal{K}^n$ given in the Appendix, it has coordinates $(\alpha_1, \ldots, \alpha_n)$. For instance the trees  ,  ,  have coordinates $(0,0), (0, \frac{1}{2}), (1,1)$ respectively. The tree t is sent under p_1 to the vertex with coordinates $(\bar{\alpha_1}, \ldots, \bar{\alpha_n})$ of I^n, where $\bar{\alpha_j} = 0$ if $\alpha_j = 0$ and $\bar{\alpha_j} = 1$ if $\alpha_j \neq 0$. Under this map the cube of $\mathcal{K}^n_{\text{cub}}$ indexed by the left comb is mapped bijectively to I^n. All the other ones are shrinked to smaller cubes.

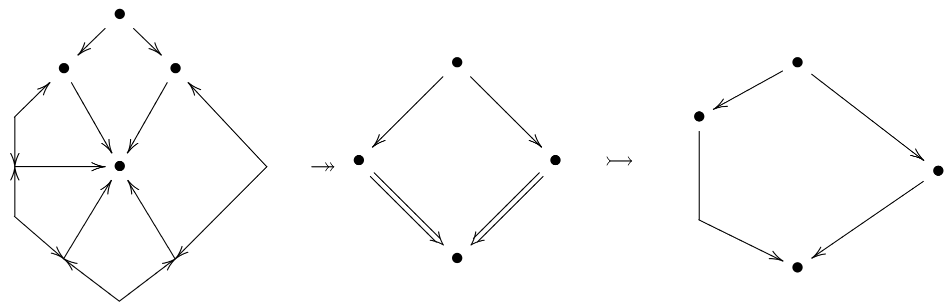
The map $p_2 : C_*(I^n) \to C_*(\mathcal{K}^n)$ is induced by a cellular map which sends the vertices of the cube to some vertices of the associahedron under the following rule. We consider the cube of $\mathcal{K}^n_{\text{cub}}$ indexed by the right comb. Each of its vertices is indexed by a planar tree. For instance one of them is the right comb and one of them is the corolla. For each such tree t we define $\widetilde{t}$ as follows: starting with t we replace each vertex by a left comb. Therefore we get a planar binary tree, which we denote by $\widetilde{t}$. For instance, if the arity of the vertex is 2, then we do nothing. If the arity of the vertex is 3, then, locally,  is replaced by  . Examples:

$t =$  ,  ,  ,  ,

$\widetilde{t} =$  ,  ,  ,  .

The chain map $p_2 : C_*(I^n) \twoheadrightarrow C_*(\mathcal{K}^n)$ is precisely obtained by identifying the cube to the decorated cube. For instance the image of the cell $\{1\} \times I$ is the sum  +  (compare with [9] Example 9).

These constructions determine the chain maps $p : C_*(\mathcal{K}^{n-2}_{\mathrm{cub}}) \to C_*(\mathcal{K}^{n-2})$. Example:



3.4 Comparison of the operads AA_∞ and ΩBAs

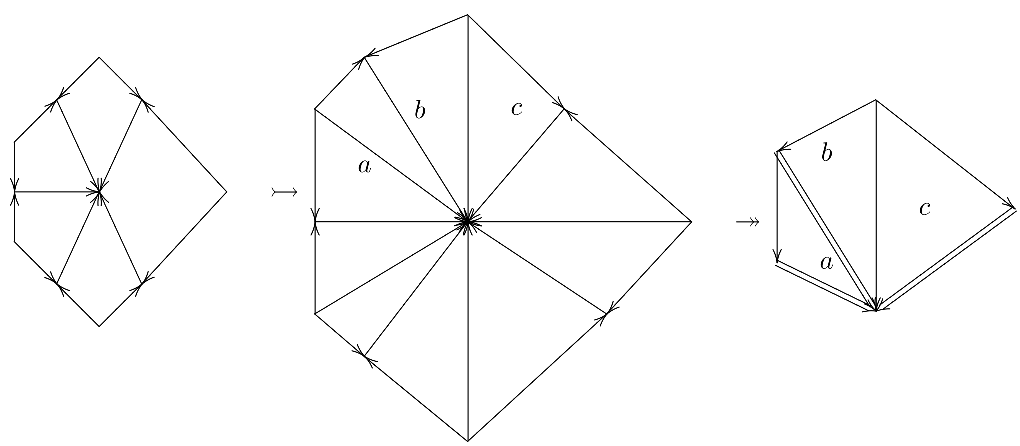
From the geometric nature of AA_∞ and ΩBAs we can deduce an explicit quasi-isomorphism of operads

$$\Omega BAs \to AA_\infty$$

as follows. A cube admits a simplicial decomposition. Hence we can simplicialize the cubical decomposition of $\mathcal{K}^{n-2}_{\mathrm{cub}}$ to obtain a new cellular complex $\mathcal{K}^n_{\mathrm{cub,simp}}$. There are explicit quasi-isomorphisms

$$C_*(\mathcal{K}^n_{\mathrm{cub}}) \rightarrowtail C_*(\mathcal{K}^n_{\mathrm{cub,simp}}) \twoheadrightarrow C_*(\mathcal{K}^n_{\mathrm{simp}}) \ .$$

We leave it to the reader to figure out the explicit formulas from the 2-dimensional case:



Hint: identify the top-simplices of $\mathcal{K}^n_{\mathrm{cub,simp}}$ which are mapped bijectively onto their image (like a, b, c above).

Observe that the intermediate spaces $\mathcal{K}^n_{\mathrm{cub,simp}}$ give rise to a new dgns operad along the same lines as before. Let us denote it by AAA_∞. The operad morphisms

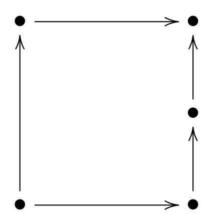
$$\Omega BAs \to AAA_\infty \to AA_\infty$$

are quasi-isomorphisms and are compatible with the diagonals (Hopf operad morphisms).

4 Appendix: Drawing a Stasheff polytope on a cube

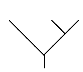
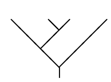
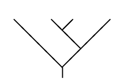
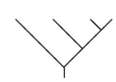
This is an account of some effort to construct the Stasheff polytope that I did in 2002 while visiting Northwestern University. During this visit I had the opportunity to meet Samson Saneblidze and Ron Umble, who were drawing the same kind of figures for different reasons (explained above). It makes the link between Markl and Shnider algebraic description of the map p, the pictures appearing in Saneblidze and Umble paper, and some algebraic properties of the planar binary trees.

There is a way of constructing an associahedron structure on a cube as follows. For $n = 0$ and $n = 1$ there is nothing to do since $\mathcal{K}^0$ and $\mathcal{K}^1$ are the cubes I^0 and I^1 respectively. For $n = 2$ we simply add one point in the middle of an edge to obtain a pentagon:

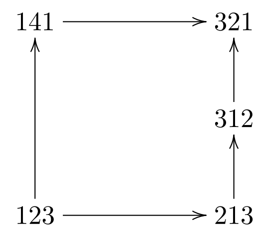


Inductively we draw $\mathcal{K}^n$ on I^n out of the drawing of $\mathcal{K}^{n-1}$ on I^{n-1} as follows. Any tree $t \in PBT_{n+1}$ gives rise to an ordered sequence of trees $(t_1, \ldots, t_k)$ in PBT_{n+2} as follows. We consider the edges which are on the right side of t, including the root. The tree t_1 is formed by adding a leaf which starts from the middle of the root and goes rightward (see [4] p. XXX). The tree t_2 is formed by adding a leaf which starts from the middle of the next edge and goes rightward. And so forth. Obviously k is the number of vertices lying on the right side of t plus one (so it is always greater than or equal to 2).

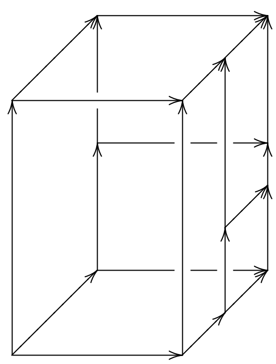
Example:

if $t =$  , then $t_1 =$  , $t_2 =$  , $t_3 =$  .

In $I^n = I^{n-1} \times I$ we label the point $\{t\} \times \{0\}$ by t_1, the point $\{t\} \times \{1\}$ by t_k, and we introduce (in order) the points $t_2, \ldots, t_{k-1}$ on the edge $\{t\} \times I$. For $n = 2$ we obtain (with the coding introduced in section 1.1):

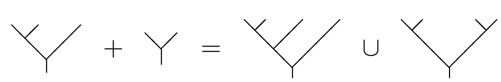


For $n = 3$ we obtain the following picture:



(It is a good exercise to draw the tree at each vertex). Compare with [11], p. 3). The case $n = 4$ can be found on my home-page. It is important to observe that the order induced on the vertices by the canonical orientation of the cube coincides precisely with the Tamari poset structure.

Surprisingly, this way of viewing the associahedron is related to an algebraic structure on the set of planar binary trees $PBT = \bigcup_{n\geq 1} PBT_n$, related to dendriform algebras. Indeed there is a non-commutative monoid structure on the set of homogeneous nonempty subsets of PBT constructed in [4]. It comes from the associative structure of the free dendriform algebra on one generator. This monoid structure is denoted by $+$, the neutral element is the tree $|$. If $t \in PBT_p$ and $s \in PBT_q$, then $s + t$ is a subset of PBT_{p+q-1}. It is proved in [4] that the trees which lie on the edge $\{t\} \times I \subset I^n$ are precisely the trees of $t +$  . For instance:



and

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