Bloom filters

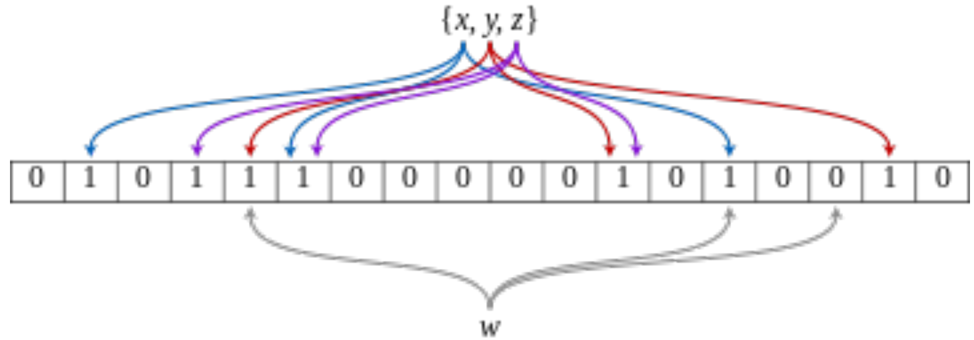
Approximate membership data structures

Bloom filters: generalities

- Bloom (1970)
- generalizes the bitmap representation of sets
- *approximate membership data structure*: supports INSERT and LOOKUP
- LOOKUP only checks for the presence, no satellite data
- produces false positives (with low probability)
- cannot iterate over the elements of the set
- DELETE is not supported (in the basic variant)
- *very* space efficient, keys themselves are ***not*** stored
- *Example*: forbidden passwords

Bloom filter: how it works

- U : universe of possible elements
- K : subset of elements, $|K| = n$
- m : size of allocated **bit array**



- define d hash functions $h_1, \dots, h_d\colon U \to \{0, \dots, m-1\}$

- INSERT(k): set $h_i(k) = 1$ for all i
- LOOKUP(k): check $h_i(k) = 1$ for all i

- false positives but no false negatives

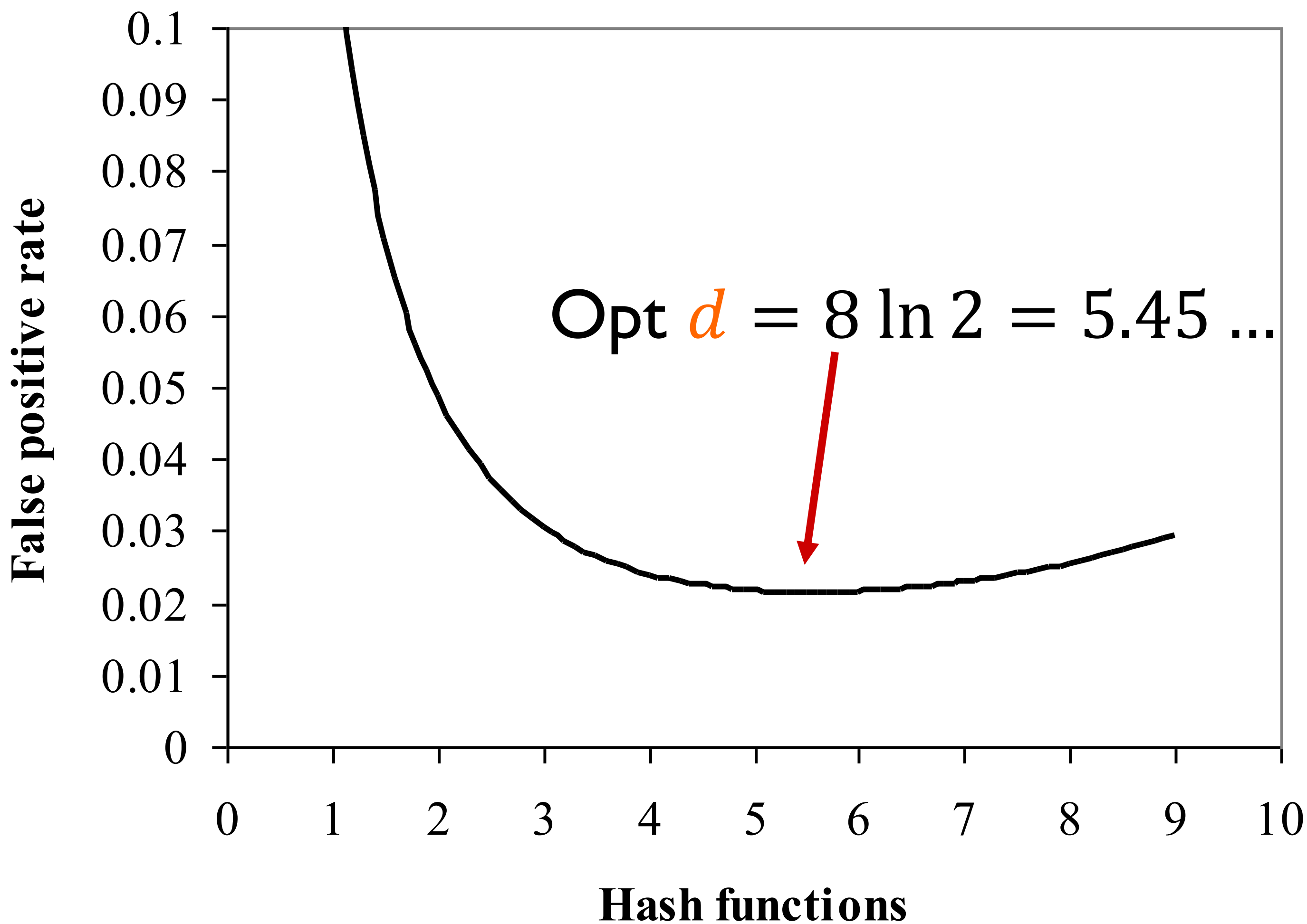
Bloom filters: analysis

- P[specific bit of filter is 0] $= (1 - 1/m)^{dn} \approx e^{-dn/m} \equiv p$
- P[false positive] $= (1 - p)^d = (1 - e^{-dn/m})^d$
- Optimal number d of hash functions: $d = \ln 2 \cdot \frac{m}{n} \approx 0.693 \cdot \frac{m}{n}$
- Therefore, for the optimal number of hash functions,

$$\text{P[false positive]} = 2^{-\ln 2 \cdot \frac{m}{n}} \approx 0.6185^{\frac{m}{n}}$$

- E.g. with 10 bits per element, P[false positive] is less than 1%
- To insure the FP rate ε: $m = \log_2 e \cdot n \cdot \log_2 \frac{1}{\varepsilon} \approx 1.44 \cdot n \cdot \log_2 \frac{1}{\varepsilon}$

Dependence on the nb of hash functs



$m/n = 8$

n elements
m bits
d hash functions

Lower bound on the size of approximate membership data structures (AMD)

- Bloom filter takes $1.44 \cdot \log\frac{1}{\varepsilon}$ bits per key, is this optimal?

- How many AMDs are there to store all sets of size n drawn from universe U with FPP ε?
- Each AMD specifies a set of size $\varepsilon|U|$ (assuming $|U|$ large) containing a set of size n
- Any set of size n should be covered, and the number of such sets is $\geq \binom{|U|}{n} / \binom{\varepsilon|U|}{n} \approx \left(\frac{1}{\varepsilon}\right)^n$ (cf Erdős&Spencer 74, Rödl 85)
- $\Rightarrow$ each FPP must take $\geq n \cdot \log\frac{1}{\varepsilon}$ bits

Bloom filter: properties/operations

- For the optimal number of hash function, about a half of the bits is 1 [*immedate from the formula*]
- The Bloom filter for the union is the OR of the Bloom filters
- Is similar true for the intersection? [*explain*]
- If a Bloom filter is sparse, it is easy to halve its size

Bloom filters: applications

- Bloom filters are very easy to implement
- Used e.g. for
  - spell-checkers (in early UNIX-systems)
  - unsuitable passwords, "approximate" unsuitable passwords (Manber&Wu 1994)
  - online applications (traffic monitoring, …)
  - distributed databases
  - malicious sites in Google Chrome
  - read articles in publishing systems (Medium)
  - Google Bigtable, Apache HBase, Bitcoin, bioinformatics, …
- Sometimes (when the set of possible queries is limited) it is possible to store the set of false positives in a separate data structure

Cuckoo filters

filters via Cuckoo hashing

Filters via MPHF

- Given a set $K \subset U$, build an MPHF $h: K \longrightarrow [1..n]$
- Given ε, pick a hash function f mapping keys of K into *fingerprints* of $\log\frac{1}{\varepsilon}$ bits
- Build an array F of fingerprints: $F[h(k)] = f(k)$

- $P[f(x) = f(y)] = \frac{1}{2^{\log\frac{1}{\varepsilon}}} = \varepsilon$ (false positive proba)
- Space: $\overbrace{n \cdot \log\frac{1}{\varepsilon}}^{F} + \text{<size of MPFR>}$
- *lower bound*: size of MPFR $\geq 1.44n$
- K must be static, does not support insertions/deletions

Cuckoo filter: ideas

- Use Cuckoo hash table (e.g. (2,4)-table) instead of MPHF

- *Problem*: How to move a fingerprint? i.e. how to know its alternative bucket?

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$$h_1 : K \to 2^{\log |T|},\ \ h_2 : 2^{\log \frac{1}{\varepsilon}} \to 2^{\log |T|}$$

location 1: $h_1(k)$

location 2: $h_1(k) \oplus h_2(f(k))$

- Alternative location of a fingerprint α at location i is

$$i \oplus h_2(\alpha)$$

Remarks

- Two locations of a key are not fully independent. E.g. two keys sharing the same bucket and the same fingerprint have the same alternative location. (⇒ store *multisets* in b-element buckets)

- *Practical*: Cuckoo vs. Bloom: for small false positive rate ($< 3\%$) and $b = 4$, Cuckoo filter achieves the same performance as Bloom with smaller space

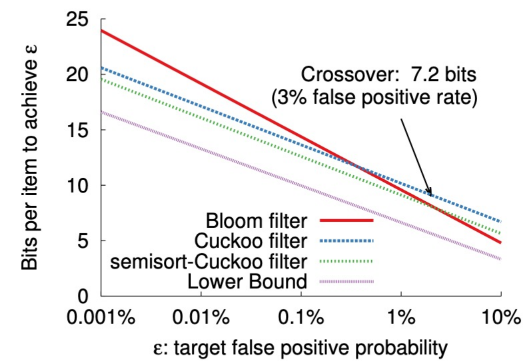

Figure 4: False positive rate vs. space cost per element. For low false positive rates ($<3\%$), cuckoo filters require fewer bits per element than the space-optimized Bloom filters. The load factors to calculate space cost of cuckoo filters are obtained empirically.

[Fan et al. Cuckoo filter: practically better than Bloom, CoNEXT 2014]

Count-Min sketch (aka Spectral Bloom filter)

Storing count information

How to support deletions in Bloom filters?

How to support deletions in Bloom filters?

- Counting Bloom filter: Bloom filter that, instead of 0 and 1, stores (small) counters

  - INSERT(k): $B[h_i(k)] \leftarrow B[h_i(k)] + 1$ for all i
  - LOOKUP(k): check $B[h_i(k)] > 0$ for all i
  - DELETE(k): $B[h_i(k)] \leftarrow B[h_i(k)] - 1$ for all i

- Also works for multi-sets
- Analysis shows that

$$P\left[\max \text{counter} \geq 16\right] < 1.37m \cdot 10^{-15}$$

i.e. 4 bits/counter suffices for practical purposes

[Fan et al. IEEE/ACM Trans. on Networking, 2000]

Count-Min sketch

- What if we want to estimate the multiplicities (number of occurrences) of elements of a multi-set stored in a counting Bloom filter?
- Streaming framework
- *Example:* $m = 8,\ C[0..7]$

	a	b	c	d	e	f
h_1	0	2	3	7	4	5
h_2	4	5	1	2	1	7

$b\ a\ d\ a\ e\ f\ c\ a\ \ldots$

$\#a?\ \#e?\ \#f?\quad \#c?\ \#e?\ \#f?$

Count-Min: operations

- UPDATE(k): $C[h_i(k)] \leftarrow C[h_i(k)] + 1$ for all i
- $\hat{f}(k) = \min_i h_i(k)$

Count-Min sketch: analysis

- *Theorem*: if $d = \log_2 \frac{1}{\delta}$ and $m = \frac{ed}{\varepsilon}$, then

$$P[\hat{f}(k) \geq f(k) + \varepsilon n] \leq \delta,$$

where d is the number of hash functions, $f(k)$ is the true count of k and n is the total number of elements in the stream

Proof: $C[h_i(k)] = f(k) + X_i(k)$

$$E[X_i(k)] = \frac{1}{m}\sum_{l \neq k} f(l) + \sum_{j \neq i} \frac{1}{m} \sum_{k} f(k) \leq \frac{d}{m} n = \frac{\varepsilon}{e} n$$

$$P[X_i(k) > \varepsilon n] < \frac{1}{e}$$

Markov inequality
$P[X \geq a] \leq \frac{E[X]}{a}$

Since $\hat{f}(k) = f(k) + \min_i X_i(k)$, we have

$$P\left[\hat{f}(k) - f(k) \geq \varepsilon n\right] \leq e^{-d} = \delta$$

Count-Min: properties

- Total space $m = \frac{e}{\epsilon} \log \frac{1}{\delta}$
- *Example ("Heavy Hitters")*: Assume we want to output all elements that occur $n/50$ of times. Set $\varepsilon = 1/100$, i.e. $m = 271 \cdot \log \frac{1}{\delta}$. Then we will output all desired elements, but also some elements occurring less, but not less than $n/100$, with $p = 1 - \delta$.
- Bound in Theorem is in terms of n
- CM-sketch also applies to increments > 1
- Decrements ("deletions") are also supported provided that counters remain non-negatives
- Count-Min [Cormode&Muthukrishnan 2005] sketch vs Spectral Bloom filters [Cohen&Matias 2003]

What if n is not known in advance?

- *Idea*: Maintain a min-heap of current frequent items, update after each element
- After processing each element x, estimate $\hat{f}(x)$
- If $\hat{f}(x) \geq \varepsilon n$ (n current stream size), insert x to the heap with value $\hat{f}(x)$ (or update if it was already there)
- If the smallest value of the heap (computed in $O(1)$) is $< \varepsilon m$, delete it from the heap
- At the end, output all elements of the heap