Symbolic dynamics

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Contents

1 Introduction

Symbolic dynamics is part of dynamical systems theory. It studies discrete dynamical systems called shift spaces and their relations under appropriately defined morphisms, in particular isomorphisms called conjugacies. A special emphasis has been put on the classification of shift spaces up to conjugacy or flow equivalence.

There is a considerable overlap between symbolic dynamics and automata theory. Actually, one of the basic objects of symbolic dynamics, the sofic systems, are essentially the same as finite automata. In addition, the morphisms of shift spaces are a particular case of rational transductions, that is functions defined by finite automata with output. The difference is that symbolic dynamics considers mostly infinite words and that all states of the automata are initial and final. Also, the morphisms are particular transductions which are given by local maps.

This chapter presents some of the links between automata theory and symbolic dynamics. The emphasis is on two particular points. The first one is the interplay between some particular classes of automata, such as local automata and results on embeddings of shifts of finite type. The second one is the connection between syntactic semigroups and the classification of sofic shifts up to conjugacy.

The chapter is organized as follows. In Section 2, we introduce the basic notions of symbolic dynamics: shift spaces, conjugacy and flow equivalence. We state without proof two important results: the Decomposition Theorem and the Classification Theorem.

In Section 3, we introduce automata in relation to sofic shifts. In Section 4, we define two kinds of minimal automata for shift spaces: the Krieger automaton and the Fischer automaton. We also relate these automata with the syntactic semigroup of a shift space.

In Section 5, we state and prove an analogue due to Nasu of the Decomposition Theorem and of the Classification Theorem.

In Section 6 we consider two special families of automata: local automata and automata with finite delay. We show that they are related to shifts of finite type and of almost finite type, respectively. We prove an embedding theorem (Theorem 6.4) which is a counterpart for automata of a result known as Nasu's masking lemma.

In Section 7 we study syntactic invariants of sofic shifts. We introduce the syntactic graph of an automaton. We show that that the syntactic graph of an automaton is invariant under conjugacy (Theorem 7.4) and also under flow equivalence. We finally state some results concerning the shift spaces corresponding to some pseudovarieties of ordered semigroups.

We follow the notation of the book of Doug Lind and Brian Marcus [19]. In general, we have not not reproduced the proofs of the results which can be found there. We thank Mike Boyle and Alfredo Costa for their help.

2 Shift spaces

This section contains basic definitions concerning symbolic dynamics.

The first subsection gives the definition of shift spaces, and the important case of edge shifts.

The next subsection and thus also under (Section 2.2) introduces conjugacy, and the basic notion of state splitting and merging. It contains the statement of two important theorems, the Decomposition Theorem (Theorem 2.12) and the Classification Theorem (Theorem 2.14).

The last subsection (Section 2.3) introduces flow equivalence, and states Frank's characterization of flow equivalent edge shifts (Theorem 2.16).

2.1 Shift spaces

Let A be a finite alphabet. We denote by A^* the set of words on A and by A^+ the set of nonempty words. A word v is a *factor* of a word t if $t = uvw$ for some words u, w.

We denote by $A^{\mathbb{Z}}$ the set of biinfinite sequences of symbols from A. This set is a topological space in the product topology of the discrete topology on A. The *shift transformation* on $A^{\mathbb{Z}}$ is the map σ_A from $A^{\mathbb{Z}}$ onto itself defined by $y = \sigma_A(x)$ if $y_n = x_{n+1}$ for $n \in \mathbb{Z}$. A set $X \subset A^{\mathbb{Z}}$ is *shift invariant* if $\sigma(X) = X$. A *shift space* on the alphabet A is a shift-invariant subset of $A^{\mathbb{Z}}$ which is closed in the topology. The set $A^{\mathbb{Z}}$ itself is a shift space called the *full shift*.

For a set $W \subset A^*$ of words (whose elements are called the *forbidden factors*), we denote by $X^{(W)}$ the set of $x \in A^{\mathbb{Z}}$ such that no $w \in W$ is a factor of x.

Proposition 2.1. *The shift spaces on the alphabet A are the sets $X^{(W)}$, for $W \subset A^*$.*

A shift space X is of *finite type* if there is a finite set $W \subset A^*$ such that $X = X^{(W)}$.

Example 2.1. Let $A = \{a, b\}$, and let $W = \{bb\}$. The shift $X^{(W)}$ is composed of the sequences without two consecutive b's. It is a shift of finite type, called the *golden mean shift*.

Recall that a set $W \subset A^*$ is said to be *recognizable* if it can be recognized by a finite automaton or, equivalently, defined by a regular expression. A shift space X is said to be *sofic* if there is a recognizable set W such that $X = X^{(W)}$. Since a finite set is recognizable, any shift of finite type is sofic.

Example 2.2. Let $A = \{a, b\}$, and let $W = a(bb)^*ba$. The shift $X^{(W)}$ is composed of the sequences where two consecutive occurrences of the symbol a are separated by an even number of b's. It is a sofic shift called the *even shift*. It is not a shift of finite type. Indeed, assume that $X = X^{(V)}$ for a finite set $V \subset A^*$. Let n be the maximal length of the words of V. A biinfinite repetition of the word ab^n has the same blocks of length at most n as a biinfinite repetition of the word ab^{n+1}. However, one is in X if and only if the other is not in X, a contradiction.

Example 2.3. Let $A = \{a, b\}$ and let $W = \{ba^n b^m a \mid n, m \geqslant 1, n \neq m\}$. The shift $X^{(W)}$ is composed of infinite sequences of the form $\ldots a^{n_i} b^{n_i} a^{n_{i+1}} b^{n_{i+1}} \ldots$. The set W is not recognizable and it can be shown that X is not sofic.

Edge shifts. In this chapter, a *graph* $G = (Q, \mathcal{E})$ is a pair composed of a finite set Q of *vertices* (or *states*), and a finite set $\mathcal{E}$ of *edges*. The graph is equipped with two maps $i, t : \mathcal{E} \to Q$ which associate, to an edge e, its *initial* and *terminal* vertex[1]. We say that e starts in $i(e)$ and ends in $t(e)$. Sometimes, $i(e)$ is called the *source* and $t(e)$ is called the *target* of e.

We also say that e is an incoming edge for $t(e)$, and an outgoing edge for $i(e)$. Two edges $e, e' \in \mathcal{E}$ are *consecutive* if $t(e) = i(e')$.

For $p, q \in Q$, we denote by $\mathcal{E}_p^q$ the set of edges of a graph $G = (Q, \mathcal{E})$ starting in state p and ending in state q. The *adjacency matrix* of a graph $G = (Q, \mathcal{E})$ is the $Q \times Q$-matrix $M(G)$ with elements in $\mathbb{N}$ defined by

$$M(G)_{pq} = \mathrm{Card}(\mathcal{E}_p^q)\,.$$

A (finite or biinfinite) *path* is a (finite or biinfinite) sequence of consecutive edges. The *edge shift* on the graph G is the set of biinfinite paths in G. It is denoted by X_G and is a shift of finite type on the alphabet of edges. Indeed, it can be defined by taking the set of non-consecutive edges for the set of forbidden factors. The converse does not hold, since the golden mean shift is not an edge shift. However, we shall see below (Proposition 2.5) that every shift of finite type is conjugate to an edge shift.

A graph is *essential* if every state has at least one incoming and one outgoing edge. This implies that every edge is on a biinfinite path. The *essential part* of a graph G is the subgraph obtained by restricting to the set of vertices and edges which are on a biinfinite path.

2.2 Conjugacy

Morphisms. Let X be a shift space on an alphabet A, and let Y be a shift space on an alphabet B.

A *morphism* φ from X into Y is a continuous map from X into Y which commutes with the shift. This means that $\varphi \circ \sigma_A = \sigma_B \circ \varphi$.

Let k be a positive integer. A k-*block* of X is a factor of length k of an element of X. We denote by $\mathcal{B}(X)$ the set of all blocks of X and by $\mathcal{B}_k(X)$ the set of k-blocks of X. A function $f : \mathcal{B}_k(X) \to B$ is called a k-*block substitution*. Let now m, n be fixed nonnegative integers with $k = m + 1 + n$. Then the function f defines a map φ called *sliding block map* with *memory* m and *anticipation* n as follows. The image of $x \in X$ is the element $y = \varphi(x) \in B^{\mathbb{Z}}$ given by

$$y_i = f(x_{i-m} \cdots x_i \cdots x_{i+n})\,.$$

We denote $\varphi = f_\infty^{[m,n]}$. It is a sliding block map from X into Y if y is in Y for all x in X. We also say that φ is a k-block map from X into Y. The simplest case occurs when $m = n = 0$. In this case, φ is a 1-block map.

The following result is Theorem 6.2.9 in [19].

Theorem 2.2 (Curtis–Lyndon–Hedlund). *A map from a shift space X into a shift space Y is a morphism if and only if it is a sliding block map.*

[1] We avoid the use of the terms 'initial state' or 'terminal state' of an edge to avoid confusion with the initial or terminal states of an automaton

Conjugacies of shifts. A morphism from a shift X onto a shift Y is called a *conjugacy* if it is one-to-one from X onto Y. Note that in this case, using standard topological arguments, one shows that the inverse mapping is also a morphism, and thus a conjugacy.

We define the n-th *higher block shift* $X^{[n]}$ of a shift X over the alphabet A as follows. The alphabet of $X^{[n]}$ is the set $B = \mathcal{B}_n(X)$ of blocks of length n of X.

Proposition 2.3. *The shifts X and $X^{[n]}$ for $n \geqslant 1$ are conjugate.*

Proof. Let $f : \mathcal{B}_n(X) \to B$ be the n-block substitution which maps the factor $x_1 \cdots x_n$ to itself, viewed as a symbol of the alphabet B. By construction, the shift $X^{[n]}$ is the image of X by the map $f_\infty^{[n-1,0]}$. This map is a conjugacy since it is bijective, and its inverse is the 1-block map g_∞ corresponding to the 1-block map which associates to the symbol $x_1 \cdots x_n$ of B the symbol x_n of A. □

Let $G = (Q, \mathcal{E})$ be a graph. For an integer $n \geqslant 1$, denote by $G^{[n]}$ the following graph called the n-th *higher edge graph* of G. For $n = 1$, one has $G^{[1]} = G$. For $n > 1$, the set of states of $G^{[n]}$ is the set of paths of length $n-1$ in G. The edges of $G^{[n]}$ are the paths of length n of G. The start state of an edge $(e_1, e_2, \ldots, e_n)$ is $(e_1, e_2, \ldots, e_{n-1})$ and its end state is $(e_2, e_3, \ldots, e_n)$.

The following result shows that the higher block shifts of an edge shift are again edge shifts.

Proposition 2.4. *Let G be a graph. For $n \geqslant 1$, one has $X_G^{[n]} = X_{G^{[n]}}$.*

A shift of finite type need not be an edge shift. For example the golden mean shift of Example 2.1 is not an edge shift. However, any shift of finite type comes from an edge shift in the following sense.

Proposition 2.5. *Every shift of finite type is conjugate to an edge shift.*

Proof. We show that for every shift of finite type X there is an integer n such that $X^{[n]}$ is an edge shift. Let $W \subset A^*$ be a finite set of words such that $X = X^{(W)}$, and let n be the maximal length of the words of W. If $n = 0$, X is the full shift. Thus we assume $n \geqslant 1$. Define a graph G whose vertices are the blocks of length $n-1$ of X, and whose edges are the block of length n of X. For $w \in \mathcal{B}_n(X)$, the initial (resp. terminal) vertex of w is the prefix (resp. suffix) of length $n-1$ of w.

We show that $X_G = X^{[n]}$. An element of $X^{[n]}$ is always an infinite path in G. To show the other inclusion, consider an infinite path y in G. It is the sequence of n-blocks of an element x of $A^{\mathbb{Z}}$ which does not contain any block on W. Since $X = X^{(W)}$, we get that x is in X. Consequently, y is in $X^{[n]}$. This proves the equality. □

Proposition 2.6. *A shift space that is conjugate to a shift of finite type is itself of finite type.*

Proof. Let $\varphi : X \to Y$ be a conjugacy from a shift of finite type X onto a shift space Y. By Proposition 2.5, we may assume that $X = X_G$ for some graph G. Changing G into some higher edge graph, we may assume that φ is 1-block. We may consider

G as a graph labeled by φ. Suppose that φ^{-1} has memory m and anticipation n. Set $\varphi^{-1} = f_\infty^{[m,n]}$. Let W be the set of words of length $m+n+2$ which are not the label of a path in G. We show that $Y = X^{(W)}$, which implies that Y is of finite type. Indeed, the inclusion $Y \subset X^{(W)}$ is clear. Conversely, consider y in $X^{(W)}$. For each $i \in \mathbb{Z}$, set $x_i = f(y_{i-m} \cdots y_i \cdots y_{i+n})$. Since $y_{i-m} \cdots y_i \cdots y_{i+n} y_{i+n+1}$ is the label of a path in G, the edges x_i and x_{i+1} are consecutive. Thus $x = (x_i)_{i \in \mathbb{Z}}$ is in X and $y = \varphi(x)$ is in Y. □

Conjugacy invariants. No effective characterization of conjugate shift spaces is known, even for shifts of finite type. There are however several quantities that are known to be invariant under conjugacy.

The *entropy* of a shift space X is defined by

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log s_n \,,$$

where $s_n = \mathrm{Card}(\mathcal{B}_n(X))$. The limit exists because the sequence s_n is sub-additive (see [19] Lemma 4.1.7). Note that since $\mathrm{Card}(\mathcal{B}_n(X)) \leqslant \mathrm{Card}(A)^n$, we have $h(X) \leqslant \log \mathrm{Card}(A)$. If X is nonempty, then $0 \leqslant h(X)$.

The following statement shows that the entropy is invariant under conjugacy (see [19] Corollary 4.1.10).

Theorem 2.7. *If X, Y are conjugate shift spaces, then $h(X) = h(Y)$.*

Example 2.4. Let X be the golden mean shift of Example 2.1. Then a block of length $n+1$ is either a block of length $n-1$ followed by ab or a block of length n followed by a. Thus $s_{n+1} = s_n + s_{n-1}$. As a classical result, $h(X) = \log \lambda$ where $\lambda = (1+\sqrt{5})/2$ is the golden mean.

An element x of a shift space X over the alphabet A has *period* n if $\sigma_A^n(x) = x$. If $\varphi : X \to Y$ is a conjugacy, then an element x of X has period n if and only if $\varphi(x)$ has period n.

The *zeta function* of a shift space X is the power series

$$\zeta_X(z) = \exp \sum_{n \geqslant 0} \frac{p_n}{n} z^n \,,$$

where p_n is the number of elements x of X of period n.

It follows from the definition that the sequence $(p_n)_{n \in \mathbb{N}}$ is invariant under conjugacy, and thus the zeta function of a shift space is invariant under conjugacy.

Several other conjugacy invariants are known. One of them is the Bowen-Franks group of a matrix which defines an invariant of the associated shift space. This will be defined below.

Example 2.5. Let $X = A^{\mathbb{Z}}$. Then $\zeta_X(z) = \frac{1}{1-kz}$, where $k = \mathrm{Card}(A)$. Indeed, one has $p_n = k^n$, since an element x of $A^{\mathbb{Z}}$ has period n if and only if it is a biinfinite repetition of a word of length n over A.

State splitting. Let $G = (Q, \mathcal{E})$ and $H = (R, \mathcal{F})$ be graphs. A pair (h, k) of surjective maps $k : R \to Q$ and $h : \mathcal{F} \to \mathcal{E}$ is called a *graph morphism* from H onto G if the two diagrams in Figure 1 are commutative.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "h"] \arrow[d, "i"] & \mathcal{E} \arrow[d, "i"] \\ R \arrow[r, "k"] & Q \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F} \arrow[r, "h"] \arrow[d, "t"] & \mathcal{E} \arrow[d, "t"] \\ R \arrow[r, "k"] & Q \end{tikzcd}$$

Figure 1. Graph morphism.

A graph morphism (h, k) from H onto G is an *in-merge* from H onto G if for each $p, q \in Q$ there is a partition $(\mathcal{E}_p^q(t))_{t \in k^{-1}(q)}$ of the set $\mathcal{E}_p^q$ such that for each $r \in k^{-1}(p)$ and $t \in k^{-1}(q)$, the map h is a bijection from $\mathcal{F}_r^t$ onto $\mathcal{E}_p^q(t)$. If this holds, then G is called an *in-merge* of H, and H is an *in-split* of G.[2]

Thus an in-split H is obtained from a graph G as follows: each state $q \in Q$ is split into copies which are the states of H in the set $k^{-1}(q)$. Each of these states t receives a copy of $\mathcal{E}_p^q(t)$ starting in r and ending in t for each r in $k^{-1}(p)$.

Each r in $k^{-1}(p)$ has the same number of edges going out of r and coming in s, for any $s \in R$.

Moreover, for any $p, q \in Q$ and $e \in \mathcal{E}_p^q$, all edges in $h^{-1}(e)$ have the same terminal vertex, namely the state t such that $e \in \mathcal{E}_p^q(t)$.

Example 2.6. Let G and H be the graphs represented on Figure 2. Here $Q = \{1, 2\}$ and $R = \{3, 4, 5\}$. The graph H is an in-split of the graph G. The graph morphism (h, k)

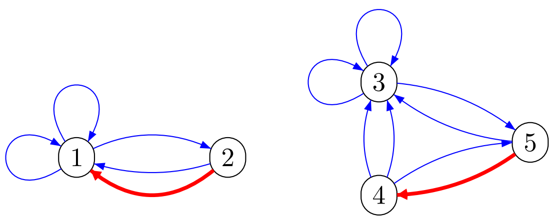

Figure 2. An in-split from G (on the left) onto H (on the right).

is defined by $k(3) = k(4) = 1$ and $k(5) = 2$. Thus the state 1 of G is split into two states 3 and 4 of H, and the map h is associated to the partition obtained as follows: the edges from 2 to 1 are partitioned into two classes, indexed by 3 and 4 respectively, and containing each one edge from 2 to 1.

The following result is well-known (see [19]). It shows that if H is an in-split of a graph G, then X_G and X_H are conjugate.

Proposition 2.8 ([19, Theorem 2.4.10]). *If (h, k) is an in-merge of a graph H onto a graph G, then h_∞ is a 1-block conjugacy from X_H onto X_G and its inverse is 2-block.*

[2] In this chapter, a *partition* of a set X is a family $(X_i)_{i \in I}$ of pairwise disjoint, possibly empty subsets of X, indexed by a set I, such that X is the union of the sets X_i for $i \in I$.

The map h_∞ from X_H to X_G is called an *edge in-merging map* and its inverse an *edge in-splitting map*.

A *column division matrix* over two sets R, Q is an $R \times Q$-matrix D with elements in $\{0, 1\}$ such that each column has at least one 1 and each row has exactly one 1. Thus, the columns of such a matrix represent a partition of R into $\mathrm{Card}(Q)$ sets.

The following result is Theorem 2.4.14 of [19].

Proposition 2.9. *Let G and H be essential graphs. The graph H is an in-split of the graph G if and only if there is an $R \times Q$-column division matrix D and a $Q \times R$-matrix E with nonnegative integer entries such that*

$$M(G) = ED, \quad M(H) = DE. \tag{2.1}$$

Example 2.7. For the graphs G, H of Example 2.6, one has $M(G) = DE$ and $M(H) = ED$ with

$$E = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Observe that a particular case of a column division matrix is a permutation matrix. The corresponding in-split (or merge) is a renaming of the states of a graph.

The notion of an *out-merge* is defined symmetrically. A graph morphism (h, k) from H onto G is an *out-merge* from H onto G if for each $p, q \in Q$ there is a partition $(\mathcal{E}_p^q(r))_{r \in k^{-1}(p)}$ of the set $\mathcal{E}_p^q$ such that for each $r \in k^{-1}(p)$ and $t \in k^{-1}(q)$, the map h is a bijection from $\mathcal{F}_r^t$ onto $\mathcal{E}_p^q(r)$. If this holds, then G is called an *out-merge* of H, and H is an *out-split* of G.

Proposition 2.8 also has a symmetrical version. Thus if (h, k) is an out-merge from G onto H, then h_∞ is a 1-block conjugacy from X_H onto X_G whose inverse is 2-block. The conjugacy h_∞ is called an *edge out-merging map* and its inverse an *edge out-splitting map*.

Symmetrically, a *row division matrix* is a matrix with elements in the set $\{0, 1\}$ such that each column has at least one 1 and each row has exactly one 1.

The following statement is symmetrical to Proposition 2.9.

Proposition 2.10. *Let G and H be essential graphs. The graph H is an out-split of the graph G if and only if there is a row division matrix D and a matrix E with nonnegative integer entries such that*

$$M(G) = DE, \quad M(H) = ED. \tag{2.2}$$

Example 2.8. Let G and H be the graphs represented on Figure 3. Here $Q = \{1, 2\}$ and $R = \{3, 4, 5\}$. The graph H is an out-split of the graph G. The graph morphism (h, k) is defined by $k(3) = k(4) = 1$ and $k(5) = 2$. The map h is associated with the partition indicated by the colors. One has $M(G) = ED$ and $M(H) = DE$ with

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}.$$

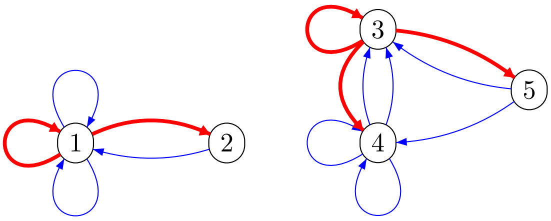

Figure 3. The graphs G and H.

We use the term *split* to mean either an in-split or an out-split. The same convention holds for a *merge*.

Proposition 2.11. *For $n \geqslant 2$, the graph $G^{[n-1]}$ is an in-merge of the graph $G^{[n]}$.*

Proof. Consider for $n \geqslant 2$ the equivalence on the states of $G^{[n]}$ which relates two paths of length $n-1$ which differ only by the first edge. It is clear that this equivalence is such that two equivalent elements have the same output. Thus $G^{[n-1]}$ is an in-merge of $G^{[n]}$. □

The Decomposition Theorem. The following result is known as the *Decomposition Theorem* (Theorem 7.1.2 in [19]).

Theorem 2.12. *Every conjugacy from an edge shift onto another is the composition of a sequence of edge splitting maps followed by a sequence of edge merging maps.*

The statement of Theorem 2.12 given in [19] is less precise, since it does not specify the order of splitting and merging maps.

The proof relies on the following statement (Lemma 7.1.3 in [19]).

Lemma 2.13. *Let G, H be graphs and let $\varphi : X_G \to X_H$ be a 1-block conjugacy whose inverse has memory $m \geqslant 1$ and anticipation $n \geqslant 0$. There are in-splittings $\overline{G}, \overline{H}$ of the graphs G, H and a 1-block conjugacy with memory $m-1$ and anticipation n $\overline{\varphi} : X_{\overline{G}} \to X_{\overline{H}}$ such that the following diagram commutes.*

$$\begin{tikzcd}
X_G \arrow[r] \arrow[d, "\varphi"] & X_{\overline{G}} \arrow[d, "\overline{\varphi}"] \\
X_H \arrow[r] & X_{\overline{H}}
\end{tikzcd}$$

The horizontal edges in the above diagram represent the edge in-splitting maps from X_G to $X_{\overline{G}}$ and from X_H to $X_{\overline{H}}$ respectively.

The Classification Theorem. Two nonnegative integral square matrices M, N are *elementary equivalent* if there exists a pair R, S of nonnegative integral matrices such that

$$M = RS\,, \quad N = SR\,.$$

Thus if a graph H is a split of a graph G, then, by Proposition 2.9, the matrices $M(G)$ and $M(H)$ are elementary equivalent. The matrices M and N are *strong shift equivalent* if there is a sequence $(M_0, M_1, \ldots, M_n)$ of nonnegative integral matrices such that M_i and M_{i+1} are elementary equivalent for $0 \leqslant i < n$ with $M_0 = M$ and $M_n = N$.

The following theorem is Williams' Classification Theorem (Theorem 7.2.7 in [19]).

Theorem 2.14. *Let G and H be two graphs. The edge shifts X_G and X_H are conjugate if and only if the matrices $M(G)$ and $M(H)$ are strong shift equivalent.*

Note that one direction of this theorem is contained in the Decomposition Theorem. Indeed, if X_G and X_H are conjugate, there is a sequence of edge splitting and edge merging maps from X_G to X_H. And if G is a split or a merge of H, then $M(G)$ and $M(H)$ are elementary equivalent, whence the result in one direction follows. Note also that, in spite of the easy definition of strong shift equivalence, it is not even known whether there exists a decision procedure for determining when two nonnegative integral matrices are strong shift equivalent.

2.3 Flow equivalence

In this section, we give basic definitions and properties concerning flow equivalence of shift spaces. The notion comes from the notion of equivalence of continuous flows, see Section 13.6 of [19]. A characterization of flow equivalence for shift spaces (which we will take below as our definition of flow equivalence for shift spaces) is due to Parry and Sullivan [23]. It is noticeable that the flow equivalence of irreducible shifts of finite type has an effective characterization, by Franks' Theorem (Theorem 2.16).

Let A be an alphabet and a be a letter in A. Let ω be a letter which does not belong to A. Set $B = A \cup \omega$. The *symbol expansion* of a set $W \subset A^+$ relative to a is the image of W by the semigroup morphism $\varphi : A^+ \to B^+$ such that $\varphi(a) = a\omega$ and $\varphi(b) = b$ for all $b \in A \setminus a$. Recall that a *semigroup morphism* $f : A^+ \to B^+$ is a map satisfying $f(xy) = f(x)f(y)$ for all words x, y. It should not be confused with the morphisms of shift spaces defined earlier. The semigroup morphism φ is also called a symbol expansion. Let X be a shift space on the alphabet A. The *symbol expansion* of X relative to a is the least shift space X' on the alphabet $B = A \cup \omega$ which contains the symbol expansion of $\mathcal{B}(X)$. Note that if φ is a symbol expansion, it defines a bijection from $\mathcal{B}(X)$ onto $\mathcal{B}(X')$. The inverse of a symbol expansion is called a *symbol contraction*.

Two shift spaces X, Y are said to be *flow equivalent* if there is a sequence $X_0, \ldots, X_n$ of shift spaces such that $X_0 = X$, $Y_n = Y$ and for $0 \leqslant i \leqslant n-1$, either X_{i+1} is the image of X_i by a conjugacy, a symbol expansion or a symbol contraction.

Example 2.9. Let $A = \{a, b\}$. The symbol expansion of the full shift $A^{\mathbb{Z}}$ relative to b is conjugate to the golden mean shift. Thus the full shift on two symbols and the golden mean shift are flow equivalent.

For edge shifts, symbol expansion can be replaced by another operation. Let G be a graph and let p be a vertex of G. The *graph expansion* of G relative to p is the graph G' obtained by replacing p by an edge from a new vertex p' to p to and replacing all edges coming in p by edges coming in p' (see Figure 4). The inverse of a graph expansion is called a *graph contraction*. Note that graph expansion (relative to vertex 1) changes the

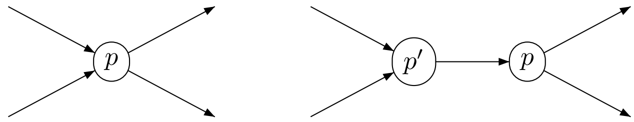

Figure 4. Graph expansion

adjacency matrix of a graph as indicated below.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & a_{11} & a_{12} & \dots & a_{1n} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & & \\ 0 & a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Proposition 2.15. *The flow equivalence relation on edge shifts is generated by conjugacies and graph expansions.*

Proof. Let $G = (Q, E)$ be a graph and let p be a vertex of G. The graph expansion of G relative to p can be obtained by a symbol expansion of each of the edges coming into p followed by a conjugacy which merges all the new symbols into one new symbol. Conversely, let e be an edge of G. The symbol expansion of X_G relative to e can be obtained by a input split which makes e the only edge going into its end vertex q followed by a graph expansion relative to q. $\square$

The *Bowen-Franks group* of a square $n \times n$-matrix M with integer elements is the Abelian group

$$BF(M) = \mathbb{Z}^n / \mathbb{Z}^n(I - M)$$

where $\mathbb{Z}^n(I - M)$ is the image of $\mathbb{Z}^n$ under the matrix $I - M$ acting on the right. In other terms, $\mathbb{Z}^n(I - M)$ is the Abelian group generated by the rows of the matrix $I - M$. This notion is due to Bowen and Franks [5], who have shown that it is an invariant for flow equivalence.

The following result is due to Franks [14]. We say that a graph is *trivial* if it is reduced to one cycle.

Theorem 2.16. *Let G, G' be two strongly connected nontrivial graphs and let M, M' be their adjacency matrices. The edge shifts $X_G, X_{G'}$ are flow equivalent if and only if $\det(I - M) = \det(I - M')$ and the groups $BF(M)$, $BF(M')$ are isomorphic.*

In the case trivial graphs, the theorem is false. Indeed, any two edge shifts on strongly connected trivial graphs are flow equivalent and are not flow equivalent to any edge shift on a nontrivial irreducible graph. For any trivial graph G with adjacency matrix M, one has $\det(I - M) = 0$ and $BF(M) \sim \mathbb{Z}$. However there are nontrivial strongly connected graphs such that $\det(I - M) = 0$ and $BF(M) \sim \mathbb{Z}$.

The case of arbitrary shifts of finite type has been solved by Huang (see [6, 8]). A similar characterization for sofic shifts is not known (see [7]).

Example 2.10. Let

$$M = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}, \quad M' = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}.$$

One has $\det(I - M) = \det(I - M') = -4$. Moreover $BF(M) \sim \mathbb{Z}/4\mathbb{Z}$. Indeed, the rows of the matrix $I - M$ are $\begin{bmatrix} -3 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \end{bmatrix}$. They generate the same group as $\begin{bmatrix} 4 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \end{bmatrix}$. Thus $BF(M) \sim \mathbb{Z}/4\mathbb{Z}$. In the same way, $BF(M') \sim \mathbb{Z}/4\mathbb{Z}$. Thus, according to Theorem 2.16, the edge shifts X_G and $X_{G'}$ are flow equivalent.

Actually X_G and $X_{G'}$ are both flow equivalent to the full shift on 5 symbols.

3 Automata

In this section, we start with the definition and notation for automata recognizing shifts, and we show that sofic shifts are precisely the shifts recognized by finite automata (Proposition 3.3).

We introduce the notion of labeled conjugacy; it is a conjugacy preserving the labeling. We extend the Decomposition Theorem and the Classification Theorem to labeled conjugacies (Theorems 3.8 and 3.9).

3.1 Automata and sofic shifts

The automata considered in this section are finite automata. We do not mention the initial and final states in the notation when all states are both initial and final. Thus, an automaton is denoted by $\mathcal{A} = (Q, E)$ where Q is the finite set of *states* and $E \subset Q \times A \times Q$ is the set of *edges*. The edge (p, a, q) has initial state p, label a and terminal state q. The underlying graph of $\mathcal{A}$ is the same as $\mathcal{A}$ except that the labels of the edges are not used.

An automaton is *essential* if its underlying graph is essential. The *essential part* of an automaton is its restriction to the essential part of its underlying graph.

We denote by $X_{\mathcal{A}}$ the set of biinfinite paths in $\mathcal{A}$. It is the edge shift of the underlying graph of $\mathcal{A}$. Note that since the automaton is supposed finite, the shift space $X_{\mathcal{A}}$ is on a finite alphabet, as required for a shift space. We denote by $L_{\mathcal{A}}$ the set of labels of biinfinite paths in $\mathcal{A}$. We denote by $\lambda_{\mathcal{A}}$ the 1-block map from $X_{\mathcal{A}}$ into the full shift $A^{\mathbb{Z}}$ which assigns to a path its label. Thus $L_{\mathcal{A}} = \lambda_{\mathcal{A}}(X_{\mathcal{A}})$. If this holds, we say that $X_{\mathcal{A}}$ is the shift space *recognized* by $\mathcal{A}$.

The following propositions describe how this notion of recognition is related to that for finite words. In the context of finite words, we denote by $\mathcal{A} = (Q, I, E, T)$ an automaton with distinguished subsets I (resp. T) of initial (resp. terminal) states. A word w is *recognized* by $\mathcal{A}$ if there is a path from a state in I to a state in T labeled w. Recall that a set is recognizable if it is the set of words recognized by a finite automaton. An automaton $\mathcal{A} = (Q, I, T)$ is *trim* if, for every state p in Q, there is a path from a state in I to p and a path from p to a state in T.

Proposition 3.1. *Let $W \subset A^*$ be a recognizable set and let $\mathcal{A} = (Q, I, T)$ be a trim finite automaton recognizing the set $A^* \setminus A^*WA^*$. Then $L_{\mathcal{A}} = X^{(W)}$.*

Proof. The label of a biinfinite path in the automaton $\mathcal{A}$ does not contain a factor w in W. Otherwise, there is a finite path $p \xrightarrow{w} q$ which is a segment of this infinite path. The path $p \xrightarrow{w} q$ can be extended to a path $i \xrightarrow{u} p \xrightarrow{w} q \xrightarrow{v} t$ for some $i \in I, t \in T$, and uwv is accepted by $\mathcal{A}$, which is a contradiction.

Next, consider a biinfinite word $x = (x_i)_{i\in\mathbb{Z}}$ in $X^{(W)}$. For every $n \geqslant 0$, there is a path π_n in the automaton $\mathcal{A}$ labeled $w_n = x_{-n} \cdots x_0 \cdots x_n$ because the word w_n has no factor in W. By compactness (König's lemma) there is an infinite path in $\mathcal{A}$ labeled x. Thus x is in $L_{\mathcal{A}}$. $\square$

The following proposition states in some sense the converse.

Proposition 3.2. *Let X be a sofic shift over A, and let $\mathcal{A} = (Q, I, T)$ be a trim finite automaton recognizing the set $\mathcal{B}(X)$ of blocks of X. Then $L_{\mathcal{A}} = X$.*

Proof. Set $W = A^* \setminus \mathcal{B}(X)$. Then one easily checks that $X = X^{(W)}$. Next, $\mathcal{A}$ recognizes $A^* \setminus A^*WA^*$. By Proposition 3.1, one has $L_{\mathcal{A}} = X$. $\square$

Proposition 3.3. *A shift X over A is sofic if and only if there is a finite automaton $\mathcal{A}$ such that $X = L_{\mathcal{A}}$.*

Proof. The forward implication results from Proposition 3.1. Conversely, assume that $X = L_{\mathcal{A}}$ for some finite automaton $\mathcal{A}$. Let W be the set of finite words which are not labels of paths in $\mathcal{A}$. Clearly $X \subset X^{(W)}$. Conversely, if $x \in X^{(W)}$, then all its factors are labels of paths in $\mathcal{A}$. Again by compactness, x itself is the label of a biinfinite path in $\mathcal{A}$. $\square$

Example 3.1. The golden mean shift of Example 2.1 is recognized by the automaton of Figure 5 on the left while the even shift of Example 2.2 is recognized by the automaton of Figure 5 on the right.

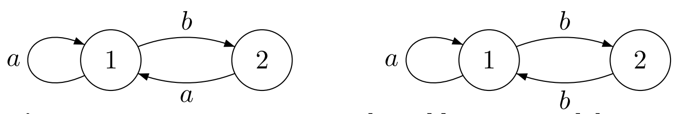

Figure 5. Automata recognizing the golden mean and the even shift

The *adjacency matrix* of the automaton $\mathcal{A} = (Q, E)$ is the $Q \times Q$-matrix $M(\mathcal{A})$ with elements in $\mathbb{N}\langle A\rangle$ defined by

$$(M(\mathcal{A})_{pq}, a) = \begin{cases} 1 & \text{if } (p, a, q) \in E\,, \\ 0 & \text{otherwise.} \end{cases}$$

We write M for $M(\mathcal{A})$ when the automaton is understood. The entries in the matrix M^n, for $n \geqslant 0$, have an easy combinatorial interpretation: for each word w of length n, the coefficient $(M^n_{p,q}, w)$ is the number of distinct paths from p to q carrying the label w.

A matrix M is called *alphabetic* over the alphabet A if its elements are homogeneous polynomials of degree 1 over A with nonnegative coefficients. Adjacency matrices are special cases of alphabetic matrices. Indeed, its elements are homogeneous polynomials of degree 1 with coefficients 0 or 1.

3.2 Labeled conjugacy

Let $\mathcal{A}$ and $\mathcal{B}$ be two automata on the alphabet A. A *labeled conjugacy* from $X_{\mathcal{A}}$ onto $X_{\mathcal{B}}$ is a conjugacy φ such that $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}}\varphi$, that is such that the following diagram is commutative. We say that $\mathcal{A}$ and $\mathcal{B}$ are *conjugate* if there exists a labeled conjugacy

$$\begin{tikzcd} X_{\mathcal{A}} \arrow[rr, "\varphi"] \arrow[dr, "\lambda_{\mathcal{A}}"'] & & X_{\mathcal{B}} \arrow[dl, "\lambda_{\mathcal{B}}"] \\ & A^{\mathbb{Z}} & \end{tikzcd}$$

from $X_{\mathcal{A}}$ to $X_{\mathcal{B}}$. The aim of this paragraph is to give two characterizations of labeled conjugacy.

Labeled split and merge. Let $\mathcal{A} = (Q, E)$ and $\mathcal{B} = (R, F)$ be two automata. Let G, H be the underlying graphs of $\mathcal{A}$ and $\mathcal{B}$ respectively.

A *labeled in-merge* from $\mathcal{B}$ onto $\mathcal{A}$ is an in-merge (h, k) from H onto G such that for each $f \in F$ the labels of f and $h(f)$ are equal. We say that $\mathcal{B}$ is a *labeled in-split* of $\mathcal{A}$, or that $\mathcal{A}$ is a *labeled in-merge* of $\mathcal{B}$.

The following statement is the analogue of Proposition 2.8 for automata.

Proposition 3.4. *If (h, k) is a labeled in-merge from the automaton $\mathcal{B}$ onto the automaton $\mathcal{A}$, then the map h_∞ is a labeled conjugacy from $X_{\mathcal{B}}$ onto $X_{\mathcal{A}}$.*

Proof. Let (h, k) be a labeled in-merge from $\mathcal{B}$ onto $\mathcal{A}$. By Proposition 2.8, the map h_∞ is a 1-block conjugacy from $X_{\mathcal{B}}$ onto $X_{\mathcal{A}}$. Since the labels of f and $h(f)$ are equal for each edge f of $\mathcal{B}$, this map is a labeled conjugacy. $\square$

The next statement is the analogue of Proposition 2.9 for automata.

Proposition 3.5. *An automaton $\mathcal{B} = (R, F)$ is a labeled in-split of the automaton $\mathcal{A} = (Q, E)$ if and only if there is an $R \times Q$-column division matrix D and an alphabetic $Q \times R$-matrix N such that*

$$M(\mathcal{A}) = ND, \quad M(\mathcal{B}) = DN. \tag{3.1}$$

Proof. Suppose first that D and N are as described in the statement, and define a map $k : R \to Q$ by $k(r) = q$ if $D_{rq} = 1$. We define $h : F \to E$ as follows. Consider an edge $(r, a, s) \in F$. Set $p = k(r)$ and $q = k(s)$. Since $M(\mathcal{B}) = DN$, we have $(N_{ps}, a) = 1$. Since $M(\mathcal{A}) = ND$, this implies that $(M(\mathcal{A})_{pq}, a) = 1$ or, equivalently, that $(p, a, q) \in E$. We set $h(r, a, s) = (p, a, q)$. Then (h, k) is a labeled in-merge. Indeed h is associated with the partitions defined by

$$E_p^q(t) = \{(p, a, q) \in E \mid (N_{pt}, a) = 1 \text{ and } k(t) = q\}.$$

Suppose conversely that (h, k) is a labeled in-merge from $\mathcal{B}$ onto $\mathcal{A}$. Let D be the $R \times Q$-column division matrix defined by

$$D_{rq} = \begin{cases} 1 & \text{if } k(r) = q \\ 0 & \text{otherwise} \end{cases}$$

For $p \in Q$ and $t \in R$, we define N_{rt} as follows. Set $q = k(t)$. By definition of an in-merge, there is a partition $(E_p^q(t))_{t \in k^{-1}(q)}$ of E_p^q such that h is a bijection from F_r^t onto $E_p^q(t)$. For $a \in A$, set

$$(N_{pt}, a) = \begin{cases} 1 & \text{if } (p, a, q) \in E_p^q(t) \\ 0 & \text{otherwise} \end{cases}$$

Then $M(\mathcal{A}) = ND$ and $M(\mathcal{B}) = DN$. □

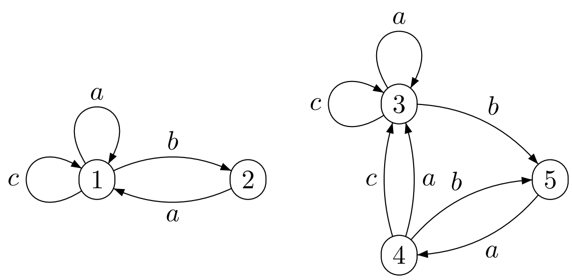

Figure 6. An in-split from $\mathcal{A}$ to $\mathcal{B}$.

Example 3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be the automata represented on Figure 6. Here $Q = \{1, 2\}$ and $R = \{3, 4, 5\}$. One has $M(\mathcal{A}) = ND$ and $M(\mathcal{B}) = DN$ with

$$N = \begin{bmatrix} a + c & 0 & b \\ 0 & a & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A *labeled out-merge* from $\mathcal{B}$ onto $\mathcal{A}$ is an out-merge (h, k) from H onto G such that for each $f \in F$ the labels of f and $h(f)$ are equal.

We say that $\mathcal{B}$ is a *labeled out-split* of $\mathcal{A}$, or that $\mathcal{A}$ is a *labeled in-merge* of $\mathcal{B}$.

Thus if $\mathcal{B}$ is a labeled out-split of $\mathcal{A}$, there is a labeled conjugacy from $X_{\mathcal{B}}$ onto $X_{\mathcal{A}}$.

Proposition 3.6. *The automaton $\mathcal{B} = (R, F)$ is a labeled out-split of the automaton $\mathcal{A} = (Q, E)$ if and only if there is a $Q \times R$-row division matrix D and an alphabetic $R \times Q$-matrix N such that*

$$M(\mathcal{A}) = DN\,, \quad M(\mathcal{B}) = ND\,. \tag{3.2}$$

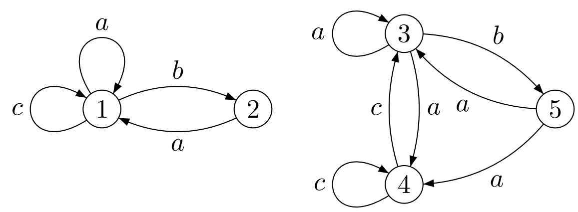

Figure 7. An out-split from $\mathcal{A}$ to $\mathcal{B}$.

Example 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be the automata represented on Figure 7. Here $Q = \{1, 2\}$ and $R = \{3, 4, 5\}$. One has $M(\mathcal{A}) = ND$ and $M(\mathcal{B}) = DN$ with

$$N = \begin{bmatrix} a & b \\ c & 0 \\ a & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\mathcal{A} = (Q, E)$ be an automaton. For a pair of integers $m, n \geqslant 0$, denote by $\mathcal{A}^{[m,n]}$ the following automaton called the (m, n)-th *extension* of $\mathcal{A}$. The underlying graph of $\mathcal{A}^{[m,n]}$ is the higher edge graph $G^{[k]}$ for $k = m + n + 1$. The label of an edge

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_m} p_m \xrightarrow{a_{m+1}} p_{m+1} \xrightarrow{a_{m+2}} \cdots \xrightarrow{a_{m+n}} p_{m+n} \xrightarrow{a_{m+n+1}} p_{m+n+1}$$

is the letter a_{m+1}. Observe that $\mathcal{A}^{[0,0]} = \mathcal{A}$. By this construction, each graph $G^{[k]}$ produces k extensions according to the choice of the labeling.

Proposition 3.7. *For $m \geqslant 1, n \geqslant 0$, the automaton $\mathcal{A}^{[m-1,n]}$ is a labeled in-merge of the automaton $\mathcal{A}^{[m,n]}$ and for $m \geqslant 0, n \geqslant 1$, the automaton $\mathcal{A}^{[m,n-1]}$ is a labeled out-merge of the automaton $\mathcal{A}^{[m,n]}$.*

Proof. Suppose that $m \geqslant 1, n \geqslant 0$. Let k be the map from the paths of length $m + n$ in $\mathcal{A}$ onto the paths of length $m + n - 1$ which erases the first edge of the path. Let h be the map from the set of edges of $\mathcal{A}^{[m,n]}$ to the set of edges of $\mathcal{A}^{[m-1,n]}$ defined by $h(\pi, a, \rho) = (k(\pi), a, k(\rho))$. Then (h, k) is a labeled in-merge from $\mathcal{A}^{[m,n]}$ onto

$\mathcal{A}^{[m-1,n]}$. The proof that, for $m \geqslant 0, n \geqslant 1$, the automaton $\mathcal{A}^{[m,n-1]}$ is an out-merge of the automaton $\mathcal{A}^{[m,n]}$ is symmetrical. □

The following result is the analogue, for automata, of the Decomposition Theorem.

Theorem 3.8. *Every conjugacy of automata is a composition of labeled splits and merges.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two conjugate automata. Let φ be a labeled conjugacy from $\mathcal{A}$ onto $\mathcal{B}$. Let G_0 and H_0 be the underlying graphs of $\mathcal{A}$ and $\mathcal{B}$, respectively. By the Decomposition Theorem 2.12, there are sequences $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_m)$ of graphs with $G_n = H_m$ and such that G_{i+1} is a split of G_i for $0 \leqslant i < n$ and H_{j+1} is a split of H_j for $0 \leqslant j < m$. Moreover, φ is the composition of the sequence of edge splitting maps from G_i onto G_{i+1} followed by the sequence of edge merging maps from H_{j+1} onto H_j. Let (h_i, k_i), for $1 \leqslant i \leqslant n$, be a merge from G_i onto G_{i-1} and (u_j, v_j), for $1 \leqslant j \leqslant m$ be a merge from H_j onto H_{j-1}. Then we may define labels on the edges of $G_1, \ldots, G_n$ in such a way that G_i becomes the underlying graph of an automaton $\mathcal{A}_i$ and (h_i, k_i) is a labeled merge from $\mathcal{A}_i$ onto $\mathcal{A}_{i-1}$. In the same way, we may define labels on the edges of H_j in such a way that H_j becomes the underlying graph of an automaton $\mathcal{B}_j$ and (u_j, v_j) is a labeled merge from $\mathcal{B}_j$ onto $\mathcal{B}_{j-1}$.

$$G_0 \xleftarrow{(h_1,k_1)} G_1 \cdots \xleftarrow{(h_n,k_n)} G_n = H_m \xrightarrow{(u_m,v_m)} \cdots H_1 \xrightarrow{(u_1,v_1)} H_0 .$$

Let $h = h_1 \cdots h_n$ and $u = u_1 u_2 \cdots u_m$. Since $\varphi = u_\infty h_\infty^{-1}$, and φ is a labeled conjugacy, we have $\lambda_{\mathcal{A}} h_\infty = \lambda_{\mathcal{B}} u_\infty$. This shows that the automata $\mathcal{A}_n$ and $\mathcal{B}_m$ are equal. Thus there is a sequence of labeled splitting maps followed by a sequence of labeled merging maps which is a equal to φ. □

Let M and M' be two alphabetic square matrices over the same alphabet A. We say that M and M' are *elementary equivalent* if there exists a nonnegative integral matrix D and an alphabetic matrix N such that

$$M = DN\,, \quad M' = ND \quad \text{or vice-versa.}$$

By Proposition 3.5, if $\mathcal{B}$ is an in-split of $\mathcal{A}$, then $M(\mathcal{B})$ and $M(\mathcal{A})$ are elementary equivalent. We say that M, M' are *strong shift equivalent* if there is a sequence $(M_0, M_1, \ldots, M_n)$ such that M_i and M_{i+1} are elementary equivalent for $0 \leqslant i < n$ with $M_0 = M$ and $M_n = M'$. The following result is the version, for automata, of the Classification Theorem.

Theorem 3.9. *Two automata are conjugate if and only if their adjacency matrices are strong shift equivalent.*

Note that when D is a column division matrix, the statement results from Propositions 3.4 and 2.9. The following statement proves the theorem in one direction.

Proposition 3.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be two automata. If $M(\mathcal{A})$ is elementary equivalent to $M(\mathcal{B})$, then $\mathcal{A}$ and $\mathcal{B}$ are conjugate.*

Proof. Let $\mathcal{A} = (Q, E)$ and $\mathcal{B} = (R, F)$. Let D be an $R \times Q$ nonnegative integral matrix and let N be an alphabetic $Q \times R$ matrix such that

$$M(\mathcal{A}) = ND, \quad M(\mathcal{B}) = DN.$$

Consider the map f from the set of paths of length 2 in $\mathcal{A}$ into F defined as follows (see Figure 8 on the left). Let $p \xrightarrow{a} q \xrightarrow{b} r$ be a path of length 2 in $\mathcal{A}$. Since $(M(\mathcal{A})_{pq}, a) = 1$ and $M(\mathcal{A}) = ND$ there is a unique $t \in R$ such that $(N_{pt}, a) = D_{tq} = 1$. In the same way, since $(M(\mathcal{A})_{qr}, b) = 1$, there is a unique $u \in R$ such that $(N_{qu}, b) = D_{ur} = 1$. Since $M(\mathcal{B}) = DN$, we have $(M(\mathcal{B})_{tu}, b) = D_{tq} = (N_{qu}, b) = 1$ and thus (t, u, b) is an edge of $\mathcal{B}$. We set

$$f(p \xrightarrow{a} q \xrightarrow{b} r) = t \xrightarrow{b} u$$

Similarly, we may define a map g from the set of paths of length 2 in $\mathcal{B}$ into E by

$$\begin{tikzcd} {} \arrow[dd, "f"'] \\ \\ {} \end{tikzcd} \qquad \begin{tikzcd} p \arrow[r, "a"] \arrow[dr, "N"'] & q \arrow[r, "b"] \arrow[dr, "N"] & r \\ & t \arrow[u, "D"] \arrow[r, "b"] & u \arrow[u, "D"'] \end{tikzcd} \qquad \begin{tikzcd} p \arrow[r, "a"] \arrow[dr, "N"] & q \arrow[dr, "N"] & \\ s \arrow[u, "D"] \arrow[r, "a"] & t \arrow[u, "D"] \arrow[r, "b"] & u \end{tikzcd} \qquad \begin{tikzcd} {} \\ \\ {} \arrow[uu, "g"'] \end{tikzcd}$$

Figure 8. The maps f and g.

$$g(s \xrightarrow{a} t \xrightarrow{b} u) = p \xrightarrow{a} q$$

if $D_{sp} = (N_{pt}, a) = D_{tq} = 1$. Let $\varphi = f_\infty^{[1,0]}$ and $\gamma = g_\infty^{[0,1]}$ (see Figure 8 on the right). We verify that

$$\varphi\gamma = \mathrm{Id}_F, \quad \gamma\varphi = \mathrm{Id}_E$$

where Id_E and Id_F are the identities on $E^{\mathbb{Z}}$ and $F^{\mathbb{Z}}$. Let indeed π be a path in $X_{\mathcal{A}}$ and let $\rho = \varphi(\pi)$. Set $\pi_i = (p_i, a_i, p_{i+1})$ and $\rho_i = (r_i, b_i, r_{i+1})$ (see Figure 9). Then, by definition of φ, we have for all $i \in \mathbb{Z}$, $b_i = a_i$ and $(N_{p_i r_{i+1}}, a_i) = D_{r_i p_i} = 1$. Let $\sigma = \gamma(\rho)$ and $\sigma = (s_i, c_i, s_{i+1})$. By definition of γ, we have $c_i = b_i$ and $D_{r_i s_i} = (N_{s_i r_{i+1}}, b_i) = 1$. Thus we have simultaneously $D_{r_i p_i} = (N_{p_i r_{i+1}}, a_i) = 1$ and $D_{r_i s_i} = (N_{s_i r_{i+1}}, a_i) = 1$. Since $M(\mathcal{A}) = DN$, this forces $p_i = s_i$. Thus $\sigma = \pi$ and this shows that $\gamma\varphi = \mathrm{Id}_E$. The fact that $\varphi\gamma = \mathrm{Id}_F$ is proved in the same way.

$$\begin{tikzcd} {} \arrow[r] & p_{i-1} \arrow[r, "a_{i-1}"] \arrow[dr, "N"] & p_i \arrow[r, "a_i"] \arrow[dr, "N"] & p_{i+1} \arrow[r] & {} \\ {} \arrow[r] & r_{i-1} \arrow[u, "D"] \arrow[r, "a_{i-1}"'] & r_i \arrow[u, "D"] \arrow[r, "a_i"'] & r_{i+1} \arrow[u, "D"] \arrow[r] & {} \end{tikzcd}$$

Figure 9. Conjugacy of automata.

$\square$

Proof of Theorem 3.9. In one direction, the above statement is a direct consequence of the Decomposition Theorem 2.12. Indeed, if $\mathcal{A}$ and $\mathcal{B}$ are conjugate, there is a sequence

$\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ of automata such that $\mathcal{A}_i$ is a split or a merge of $\mathcal{A}_{i+1}$ for $0 \leqslant i < n$ with $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_n = \mathcal{B}$. The other direction follows from Proposition 3.10. $\square$

4 Minimal automata

In this section, we define two notions of minimal automaton for sofic shifts: the Krieger automaton and the Fischer automaton. The first is defined for any sofic shift, and the second for irreducible ones.

The main result is that the Fischer automaton has the minimal number of states among all deterministic automata recognizing a given sofic shift (Proposition 4.6).

We then define the syntactic semigroup of a sofic shift, as an ordered semigroup. We show that this semigroup is isomorphic to the transition semigroup of the Krieger automaton and, for irreducible shifts, to the transition semigroup of the Fischer automaton (Proposition 4.8).

Minimal automata of sets of finite words. Recall that an automaton $\mathcal{A} = (Q, E)$ recognizes a shift X if $X = L_{\mathcal{A}}$. There should be no confusion with the notion of acceptance for sets of finite words in the usual sense: if $\mathcal{A}$ has an initial state i and a set of terminal states T, the set of finite words recognized by $\mathcal{A}$ is the set of labels of finite paths from i to a terminal state t in T. In this chapter[3], an automaton is called *deterministic* if, for each state p and each letter a, there is at most one edge starting in p and carrying the label a. We write, as usual, $p \cdot u$ for the unique end state, provided it exists, of a path starting in p and labeled u. For a set W of A^*, there exists a unique deterministic minimal automaton (this time with a unique initial state) recognizing W. Its states are the nonempty sets $u^{-1}W$ for $u \in A^*$, called the *right contexts* of u, and the edges are the triples $(u^{-1}W, a, (ua)^{-1}W)$, for $a \in A$ (see the chapter of J.-É. Pin).

Let $\mathcal{A} = (Q, E)$ be a finite automaton. For a state $p \in Q$, we denote by $L_p(\mathcal{A})$ or simply L_p the set of labels of finite paths starting from p. The automaton $\mathcal{A}$ is said to be *reduced* if $p \neq q$ implies $L_p \neq L_q$.

A word w is *synchronizing* for a deterministic automaton $\mathcal{A}$ if the set of paths labeled w is nonempty and all paths labeled w end in the same state. An automaton is *synchronized* if there is a synchronizing word. The following result holds because all states are terminal.

Proposition 4.1. *A reduced deterministic automaton is synchronized.*

Proof. Let $\mathcal{A} = (Q, E)$ be a reduced deterministic automaton. Given any word x, we denote by $Q \cdot X$ the set $Q \cdot x = \{q \cdot x \mid q \in Q\}$.

Let x be a word such that $Q \cdot x$ has minimal nonzero cardinality. Let p, q be two elements of the set $Q \cdot x$. If u is a word such that $p \cdot u$ is nonempty, then $q \cdot u$ is also nonempty since otherwise $Q \cdot xu$ would be of nonzero cardinality less than $Q \cdot x$. This implies that $L_p = L_q$ and thus $p = q$ since $\mathcal{A}$ is reduced. Thus x is synchronizing. $\square$

[3]This contrasts the more traditional definition which assumes in addition that there is a unique initial state.

4.1 Krieger automata and Fischer automata

Krieger automata. We denote by $A^{-\mathbb{N}}$ the set of left infinite words $x = \cdots x_{-1}x_0$. For $y = \cdots y_{-1}y_0 \in A^{-\mathbb{N}}$ and $z = z_0 z_1 \cdots \in A^{\mathbb{N}}$, we denote by $y \cdot z = (w_i)_{i \in \mathbb{Z}}$ the biinfinite word defined by $w_i = y_{i+1}$ for $i < 0$ and $w_i = z_i$ for $i \geqslant 0$. Let X be a shift space. For $y \in A^{-\mathbb{N}}$, the set of *right contexts* of y is the set $C_X(y) = \{z \in A^{\mathbb{N}} \mid y \cdot z \in X\}$. For $u \in A^+$, we denote $u^\omega = uu\cdots$ and $u^{-\omega} = \cdots uu$.

The *Krieger automaton* of a shift space X is the deterministic automaton whose states are the nonempty sets of the form $C_X(y)$ for $y \in A^{-\mathbb{N}}$, and whose edges are the triples (p, a, q) where $p = C_X(y)$ for some left infinite word, $a \in A$ and $q = C_X(ya)$.

The definition of the Krieger automaton uses infinite words. One could use instead of the sets $C_X(y)$ for $y \in A^{-\mathbb{N}}$, the sets

$$D_X(y) = \{u \in A^* \mid \exists z \in A^{\mathbb{N}} : yuz \in X\}.$$

Indeed $C_X(y) = C_X(y')$ if and only if $D_X(y) = D_X(y')$. However, one cannot dispense completely with infinite words (see Proposition 4.2).

Example 4.1. Let $A = \{a, b\}$, and let $X = X^{(ba)}$. The Krieger automaton of X is represented in Figure 10. The states are the sets $1 = C_X(a^{-\omega}) = a^\omega \cup a^*b^\omega$ and $2 = C_X(a^{-\omega}b) = b^\omega$.

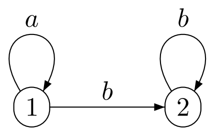

Figure 10. The Krieger automaton of $X^{(ba)}$.

Proposition 4.2. *The Krieger automaton of a shift space X is reduced and recognizes X. It is finite if and only if X is sofic.*

Proof. Let $\mathcal{A} = (Q, E)$ be the Krieger automaton of X. Let $p, q \in Q$ and let $y, z \in A^{-\mathbb{N}}$ be such that $p = C_X(y)$, $q = C_X(z)$. If $L_p = L_q$, then the labels of infinite paths starting from p and q are the same. Thus $p = q$. This shows that $\mathcal{A}$ is reduced. If $\mathcal{A}$ finite, then X is sofic by Proposition 3.3. Conversely, if X is sofic, let $\mathcal{A}$ be a finite automaton recognizing X. The set of right contexts of a left infinite word y only depends on the set of states p such that there is a path in the automaton $\mathcal{A}$ labeled y ending in state p. Thus the family of sets of right contexts is finite. □

We say that a deterministic automaton $\mathcal{A} = (Q, E)$ over the alphabet A is a *subautomaton* of a deterministic automaton $\mathcal{A}' = (Q', E')$ if $Q \subset Q'$ and if for each edge $(p, a, q) \in E$ such that $p \in Q$ one has $q \in Q$ and $(p, a, q) \in E'$.

The following proposition appears in [22] and in [11] where an algorithm to compute the states of the minimal automaton which are in the Krieger automaton is described.

Proposition 4.3. *The Krieger automaton of a sofic shift X is, up to an isomorphism, a subautomaton of the minimal automaton of the set of blocks of X.*

Proof. Let X be a sofic shift. Let $y \in A^{-\mathbb{N}}$ and set $y = \cdots y_{-1}y_0$ with $y_i \in A$ for $i \leqslant 0$. Set $u_i = y_{-i} \cdots y_0$ and $U_i = u_i^{-1}\mathcal{B}(X)$. Since $\mathcal{B}(X)$ is regular, the chain

$$\ldots \subset U_i \subset \ldots \subset U_1 \subset U_0$$

is stationary. Thus there is an integer $n \geqslant 0$ such that $U_{n+i} = U_n$ for all $i \geqslant 0$. We define $s(y) = U_n$.

We show that the map $C_X(y) \mapsto s(y)$ is well-defined and injective. Suppose first that $C_X(y) = C_X(y')$ for some $y, y' \in A^{-\mathbb{N}}$. Let $u \in A^*$ be such that $y_{-m} \cdots y_0 u \in \mathcal{B}(X)$ for all $m \geqslant n$. By compactness, there exists a $z \in A^{\mathbb{N}}$ such that $yuz \in X$. Then $y' \cdot uz \in X$ implies $u \in s(y')$. Symmetrically $u \in s(y')$ implies $u \in s(y)$. This shows that the map is well-defined.

To show that it is injective, consider $y, y' \in A^{-N}$ such that $s(y) = s(y')$. Let $z \in C_X(y)$. For each integer $m \geqslant 0$, we have $z_0 \cdots z_m \in s(y)$ and thus $z_0 \cdots z_m \in s(y')$. Since X is closed, this implies that $y' \cdot z \in X$ and thus $z \in C_X(y')$. The converse implication is proved in the same way. $\square$

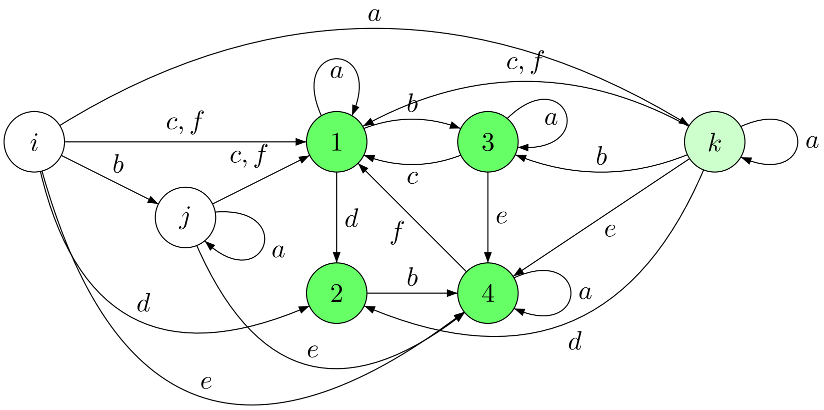

Figure 11. An example of Krieger automaton.

Example 4.2. Consider the automaton $\mathcal{A}$ on 7 states $i, j, k, 1, 2, 3, 4$ given in Figure 11. It can obtained, starting with the subautomaton over the states $1, 2, 3, 4$, using the subset construction computing the accessible nonempty sets of states, starting from the state $i = \{1, 2, 3, 4\}$ with $j = \{3, 4\}$, $k = \{1, 3, 4\}$.

The subautomaton with dark shaded states $1, 2, 3, 4$ is strongly connected and recognizes an irreducible sofic shift denoted by X. The whole automaton is the minimal automaton (with initial state i) of the set $\mathcal{B}(X)$. We identify each residual of $\mathcal{B}(X)$ with one of the 7 states of $\mathcal{A}$. Thus, for example, $i = \mathcal{B}(X)$ and $k = a^{-1}\mathcal{B}(X)$. The Krieger automaton of X is the automaton on the five shaded states. Indeed, with the map s defined in the proof of Proposition 4.3, there is no left infinite word y such that $s(y) = i$ or $s(y) = j$. On the contrary, since $i \cdot a = k$ and $k \cdot a = k$, one has $s(a^{-\omega}) = k$.

Fischer automata of irreducible shift spaces. A shift space $X \subset A^{\mathbb{Z}}$ is called *irreducible* if for any $u, v \in \mathcal{B}(X)$ there exists a $w \in \mathcal{B}(X)$ such that $uwv \in \mathcal{B}(X)$.

An automaton is said to be strongly connected if its underlying graph is strongly connected. Clearly a shift recognized by a strongly connected automaton is irreducible.

A strongly connected component of an automaton $\mathcal{A}$ is *minimal* if all successors of vertices of the component are themselves in the component. One may verify that a minimal strongly connected component is the same as a strongly connected subautomaton.

The following result is due to Fischer [13] (see also [19, Section 3]). It implies in particular that an irreducible sofic shift can be recognized by a strongly connected automaton.

Proposition 4.4. *The Krieger automaton of an irreducible sofic shift X is synchronized and has a unique minimal strongly connected component.*

Proof. Let $\mathcal{A} = (Q, E)$ be the Krieger automaton of X. By Proposition 4.2, $\mathcal{A}$ is reduced and by Proposition 4.1, it follows that it is synchronized.

Let x be a synchronizing word. Let R be the set of states reachable from the state $q = Q \cdot x$. The set R is a minimal strongly connected component of $\mathcal{A}$. Indeed, for any $r \in R$ there is a path $q \xrightarrow{y} r$. Since X is irreducible there is a word z such that $yzx \in \mathcal{B}(X)$. Since $q \cdot yzx = q$, r belongs to the same strongly connected component as q. Next, if p belongs to a minimal strongly connected component S of $\mathcal{A}$, since X is irreducible, there is a word y such that $p \cdot yx$ is not empty. Thus q is in S, which implies $S = R$. Thus R is the only minimal strongly component of $\mathcal{A}$. □

Example 4.3. Let X be the even shift. The Krieger and Fischer automata of X are represented on Figure 12. The word a is synchronizing.

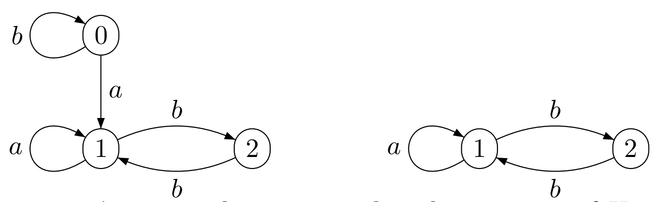

Figure 12. The Krieger and Fischer automata of X.

Example 4.4. The Fischer automaton of the irreducible shift of Example 4.2 is the subautomaton on states $1, 2, 3, 4$ represented with dark shaded states in Figure 11.

Let X be an irreducible sofic shift X. The minimal strongly connected component of the Krieger automaton of X is called its *Fischer automaton*.

Proposition 4.5. *The Fischer automaton of an irreducible sofic shift X recognizes X.*

Proof. The Fischer automaton $\mathcal{F}$ of X is a subautomaton of the Krieger automaton of X which in turn is a subautomaton of the minimal automaton $\mathcal{A}$ of the set $\mathcal{B}(X)$. Let i be the initial state of $\mathcal{A}$. Since $\mathcal{A}$ is trim, there is a word w such that $i \cdot w$ is a state of $\mathcal{F}$. Let v be any block of X. Since X is irreducible, there is a word u such that wuv is a block of X. This shows that v is a label of a path in $\mathcal{F}$. Thus every block of X is a label of a path in $\mathcal{F}$ and conversely. In view of Proposition 3.2, the automaton $\mathcal{F}$ recognizes X. □

Let $\mathcal{A} = (Q, E)$ and $B = (R, F)$ be two deterministic automata. A *reduction* from $\mathcal{A}$ onto $\mathcal{B}$ is a map h from Q onto R such that for any letter $a \in A$, one has $(p, a, q) \in E$ if and only if $(h(p), a, h(q)) \in F$. Thus any labeled in or out-merge is a reduction. However the converse is not true since a reduction is not, in general, a conjugacy.

For any automaton $\mathcal{A} = (Q, E)$, there is reduction from $\mathcal{A}$ onto a reduced automaton $\mathcal{B}$. It is obtained by identifying the pairs of states $p, q \in Q$ such that $L_p = L_q$.

The following statement is Corollary 3.3.20 of [19].

Proposition 4.6. *Let X be an irreducible shift space. For any strongly connected deterministic automaton $\mathcal{A}$ recognizing X there is a reduction from $\mathcal{A}$ onto the Fischer automaton of X.*

Proof. Let $\mathcal{A} = (Q, E)$ be a strongly connected automaton recognizing X. Let $\mathcal{B} = (R, F)$ be the reduced automaton obtained from $\mathcal{A}$ identifying the pairs $p, q \in Q$ such that $L_p = L_q$. By Proposition 4.1, $\mathcal{B}$ is synchronized.

We now show that $\mathcal{B}$ can be identified with the Fischer automaton of X. Let w be a synchronizing word for $\mathcal{B}$. Set $s = Q \cdot w$. Let r be a state such that $r \cdot w = s$. and let $y \in A^{-\mathbb{N}}$ be the label of a left infinite path ending in the state s. For any state t in R, let u be a word such that $s \cdot u = t$. The set $C_X(ywu)$ depends only on the state t, and not on the word u such that $s \cdot u = t$. Indeed, for each right infinite word z, one has $ywuz$ in X if and only if there is a path labeled z starting at t. This holds because w is synchronizing.

Thus the map $t \mapsto C_X(ywu)$ is well-defined and defines a reduction from $\mathcal{B}$ onto the Fischer automaton of X. □

This statement shows that the Fischer automaton of an irreducible shift X is minimal in the sense that it has the minimal number of states among all deterministic strongly connected automata recognizing X.

The statement also gives the following practical method to compute the Fischer automaton of an irreducible shift. We start with a strongly connected deterministic automaton recognizing X and merge the pairs of states p, q such that $L_p = L_q$. By the above result, the resulting automaton is the Fischer automaton of X.

4.2 Syntactic semigroup

Recall that a *preorder* on a set is a relation which is reflexive and transitive. The equivalence associated to a preorder is the equivalence relation defined by $u \equiv v$ if and only if $u \leqslant v$ and $v \leqslant u$.

Let S be a semigroup. A preorder on S is said to be *stable* if $s \leqslant s'$ implies $us \leqslant us'$ and $su \leqslant s'u$ for all $s, s', u \in S$. An *ordered semigroup* S is a semigroup equipped with a stable preorder. Any semigroup can be considered as an ordered semigroup equipped with the equality order.

A *congruence* in an ordered semigroup S is the equivalence associated to a stable preorder which is coarser than the preorder of S. The quotient of an ordered semigroup by a congruence is the ordered semigroup formed by the classes of the congruence.

The *set of contexts* of a word u with respect to a set $W \subset A^+$ is the set $\Gamma_W(u)$ of pairs of words defined by $\Gamma_W(u) = \{(\ell, r) \in A^* \times A^* \mid \ell ur \in W\}$. The preorder on A^+ defined by $u \leqslant_W v$ if $\Gamma_W(u) \subset \Gamma_W(v)$ is stable and thus defines a congruence of the semigroup A^+ equipped with the equality order called the *syntactic congruence*. The *syntactic semigroup* of a set $W \subset A^*$ is the quotient of the semigroup A^+ by the syntactic congruence.

Let $\mathcal{A} = (Q, E)$ be a deterministic automaton on the alphabet A. Recall that for $p \in Q$ and $u \in A^+$, there is at most one path π labeled u starting in p. We set $p \cdot u = q$ if q is the end of π and $p \cdot u = \emptyset$ if π does not exist. The preorder defined on A^+ by $u \leqslant_{\mathcal{A}} v$ if $p \cdot u \subset p \cdot v$ for all $p \in Q$ is stable. The quotient of A^+ by the congruence associated to this preorder is the *transition semigroup* of $\mathcal{A}$.

The following property is standard, see the chapter of J.-É Pin.

Proposition 4.7. *The syntactic semigroup of a set $W \subset A^+$ is isomorphic to the transition semigroup of the minimal automaton of W.*

The *syntactic semigroup* of a shift space X is by definition the syntactic semigroup of $\mathcal{B}(X)$.

Proposition 4.8. *Let X be a sofic shift and let S be its syntactic semigroup. The transition semigroup of the Krieger automaton of X is isomorphic to S. Moreover, if X is irreducible, then it is isomorphic to the transition semigroup of its Fischer automaton.*

Proof. Let $\mathcal{A}$ be the minimal automaton of $\mathcal{B}(X)$, and let $\mathcal{K}$ be the Krieger automaton of X. We have to show that for any $u, v \in A^+$, one has $u \leqslant_{\mathcal{A}} v$ if and only if $u \leqslant_{\mathcal{K}} v$. Since, by Proposition 4.3, $\mathcal{K}$ is isomorphic to a subautomaton of $\mathcal{A}$, the direct implication is clear. Indeed, if p is a state of $\mathcal{K}$, then $L_p(\mathcal{K})$ is equal to the set $L_p(\mathcal{A})$. Consequently, if $u \leqslant_{\mathcal{A}} v$ then $u \leqslant_{\mathcal{K}} v$. Conversely, suppose that $u \leqslant_{\mathcal{K}} v$. We prove that $u \leqslant_{\mathcal{B}(X)} v$. For this, let $(\ell, r) \in \Gamma_{\mathcal{B}(X)}(u)$. Then $\ell ur \in \mathcal{B}(X)$. Then $y \cdot \ell urz \in X$ for some $y \in A^{-\mathbb{N}}$ and $z \in A^{\mathbb{N}}$. But since $C_X(y\ell u) \subset C_X(y\ell v)$, this implies $rz \in C_X(y\ell v)$ and thus $\ell vr \in \mathcal{B}(X)$. Thus $u \leqslant_{\mathcal{B}(X)} v$ which implies $u \leqslant_{\mathcal{A}} v$.

Next, suppose that X is irreducible. We have to show that $u \leqslant_{\mathcal{A}} v$ if and only if $u \leqslant_{\mathcal{F}(X)} v$. Since $\mathcal{F}(X)$ is a subautomaton of $\mathcal{K}(X)$ and $\mathcal{K}(X)$ is a subautomaton of $\mathcal{A}$, the direct implication is clear. Conversely, assume that $u \leqslant_{\mathcal{F}(X)} v$. Suppose that $\ell ur \in \mathcal{B}(X)$. Let i be the initial state of $\mathcal{A}$ and let w be such that $i \cdot w$ is a state of $\mathcal{F}(X)$. Since X is irreducible, there is a word s such that $ws\ell ur \in \mathcal{B}(X)$. But then $i \cdot ws\ell ur \neq \emptyset$ implies $i \cdot ws\ell vr \neq \emptyset$. Thus $\ell vr \in \mathcal{B}(X)$. This shows that $u \leqslant_{\mathcal{B}(X)} v$ and thus $u \leqslant_{\mathcal{A}} v$. □

5 Symbolic conjugacy

This section is concerned with a new notion of conjugacy between automata called symbolic conjugacy. It extends the notion of labeled conjugacy and captures the fact that the automata may be over different alphabets. The table below summarizes the various notions.

object type	isomorphism	elementary transformation
shift spaces	conjugacy	split/merge
edge shifts	conjugacy	edge split/merge
integer matrices	strong shift equivalence	elementary equivalence
automata (same alphabet)	labeled conjugacy	labeled split/merge
automata	symbolic conjugacy	split/merge
alphabetic matrices	symbolic strong shift	elementary symbolic

There are two main results in this section. Theorem 5.7 due to Nasu is a version of the Classification Theorem for sofic shifts. It implies in particular that conjugate sofic shifts have symbolic conjugate Krieger or Fisher automata.The proof uses the notion of bipartite automaton, which corresponds to the symbolic elementary equivalence of adjacency matrices. Theorem 5.8 is due to Hamachi and Nasu: it characterizes symbolic conjugate automata by means of their adjacency matrices.

In this section, we will use for convenience automata in which several edges with the same source and target can have the same label. Formally, such an automaton is a pair $\mathcal{A} = (G, \lambda)$ of a graph $G = (Q, \mathcal{E})$ and a map assigning to each edge $e \in \mathcal{E}$ of a label $\lambda(e) \in A$. The adjacency matrix of $\mathcal{A}$ is the $Q \times Q$-matrix $M(\mathcal{A})$ with elements in $\mathbb{N}\langle A\rangle$ defined by

$$(M(\mathcal{A})_{pq}, a) = \mathrm{Card}\{e \in \mathcal{E} \mid \lambda(e) = a\}. \tag{5.1}$$

Note that $M(\mathcal{A})$ is alphabetic but may have arbitrary nonnegative coefficients. The advantage of this version of automata is that for any alphabetic $Q \times Q$-matrix M there is an automaton $\mathcal{A}$ such that $M(\mathcal{A}) = M$.

We still denote by $X_\mathcal{A}$ the edge shift X_G and by $L_\mathcal{A}$ the set of labels of infinite paths in G.

Symbolic conjugate automata. Let $\mathcal{A}, \mathcal{B}$ be two automata. A *symbolic conjugacy* from $\mathcal{A}$ onto $\mathcal{B}$ is a pair (φ, ψ) of conjugacies $\varphi : X_\mathcal{A} \to X_\mathcal{B}$ and $\psi : L_\mathcal{A} \to L_\mathcal{B}$ such that the following diagram is commutative.

$$\begin{tikzcd} X_\mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\lambda_\mathcal{A}"] & X_\mathcal{B} \arrow[d, "\lambda_\mathcal{B}"] \\ L_\mathcal{A} \arrow[r, "\psi"] & L_\mathcal{B} \end{tikzcd}$$

5.1 Splitting and merging maps

Let A, B be two alphabets and let $f : A \to B$ be a map from A onto B. Let X be a shift space on the alphabet A. We consider the set of words $A' = \{f(a_1)a_2 \mid a_1a_2 \in \mathcal{B}_2(X)\}$ as a new alphabet. Let $g : \mathcal{B}_2(X) \to A'$ be the 2-block substitution defined by $g(a_1a_2) = f(a_1)a_2$.

The *in-splitting map* defined on X and relative to f or to g is the sliding block map $g_\infty^{1,0}$ corresponding to g. It is a conjugacy from X onto its image by $X' = g_\infty^{1,0}(X)$ since its inverse is 1-block. The shift space X', is called the *in-splitting* of X, relative to f or g. The inverse of an in-splitting map is called an *in-merging map*.

In addition, any renaming of the alphabet of a shift space is also considered to be an in-splitting map (and an in-merging map).

Example 5.1. Let $A = B$ and let f be the identity on A. The out-splitting of a shift X relative to f is the second higher block shift of X.

The following proposition relates splitting maps to edge splittings as defined in Section 2.2.

Proposition 5.1. *An in-splitting map on an edge shift is an edge in-splitting map, and conversely.*

Proof. Let first $G = (Q, \mathcal{E})$ be a graph, and let $f : \mathcal{E} \to I$ be a map from $\mathcal{E}$ onto a set I. Set $\mathcal{E}' = \{f(e_1)e_2 \mid e_1e_2 \in \mathcal{B}_2(X_G)\}$. Let $g : \mathcal{B}_2(X_G) \to \mathcal{E}'$ be the 2-block substitution defined by $g(e_1e_2) = f(e_1)e_2$. Let $G' = (Q', \mathcal{E}')$ be the graph on the set of states $Q' = I \times Q$ defined for $e' = f(e_1)e_2$ by $i(e') = (f(e_1), i(e_2))$ and $t(e') = (f(e_2), t(e_2))$. Define $h : \mathcal{E}' \to \mathcal{E}$ and $k : Q' \to Q$ by $h(f(e_1)e_2) = e_2$ for $e_1e_2 \in \mathcal{B}_2(X_G)$ and $k(i, q) = q$ for $(i, q) \in I \times Q$. Then the pair (h, k) is an in-merge from G' onto G and h_∞ is the inverse of $g_\infty^{1,0}$. Indeed, one may verify that (h, k) is a graph morphism from G' onto G. Next it is an in-merge because for each $p, q \in Q$, the partition $(\mathcal{E}_p^q(t))_{t \in k^{-1}(q)}$ of $\mathcal{E}_p^q$ is defined by $\mathcal{E}_p^q(i, q) = E_p^q \cap f^{-1}(i)$.

Conversely, set $G = (Q, \mathcal{E})$ and $G' = (Q', \mathcal{E}')$. Let (h, k) be an in-merge from G' onto G. Consider the map $f : \mathcal{E} \to Q'$ defined by $f(e) = r$ if r is the common end of the edges in $h^{-1}(e)$. The map α from $\mathcal{E}'$ to $Q' \times \mathcal{E}$ defined by $\alpha(i) = (r, h(i))$ where r is the origin of i is a bijection by definition of an in-merge.

Let us show that, up to the bijection α, the in-splitting map relative to f is inverse of the map h_∞. For $e_1, e_2 \in \mathcal{E}$, let $r = f(e_1)$ and $e' = \alpha^{-1}(r, e_2)$. Then $h(e') = e_2$ and thus h_∞ is the inverse of the map $g_\infty^{1,0}$ corresponding to the 2-block substitution $g(e_1e_2) = (r, e_2)$.

□

Symmetrically an *out-splitting map* is defined by the substitution $g(ab) = af(b)$. Its inverse is an out-merging map.

We use the term splitting to mean either a in-splitting or out-splitting. The same convention holds for a merging.

The following result, from [21], is a generalization of the Decomposition Theorem (Theorem 2.12) to arbitrary shift spaces.

Theorem 5.2. *Any conjugacy between shift spaces is a composition of splitting and merging maps.*

The proof is similar to the proof of Theorem 2.12. It relies on the following lemma, similar to Lemma 2.13.

Lemma 5.3. *Let $\varphi : X \to Y$ be a 1-block conjugacy whose inverse has memory $m \geqslant 1$ and anticipation $n \geqslant 0$. There are in-splitting maps from X, Y to $\tilde{X}, \tilde{Y}$ respectively such that the 1-block conjugacy $\tilde{\varphi}$ making the diagram below commutative has an inverse with memory $m-1$ and anticipation n.*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\varphi"] & \tilde{X} \arrow[d, "\tilde{\varphi}"] \\ Y \arrow[r] & \tilde{Y} \end{tikzcd}$$

Proof. Let A, B the alphabets of X and Y respectively. Let $h : A \to B$ be the 1-block substitution such that $\varphi = h_\infty$. Let $\tilde{X}$ be the in-splitting of X relative to the map h. Set $A' = \{h(a_1)a_2 \mid a_1a_2 \in \mathcal{B}_2(X)\}$. Let $\tilde{Y} = Y^{[2]}$ be the second higher block shift of Y and let $B' = \mathcal{B}_2(Y)$. Let $\tilde{h} : A' \to B'$ be the 1-block substitution defined by $\tilde{h}(h(a_1)a_2) = h(a_1)h(a_2)$. Then the 1-block map $\tilde{\varphi} = \tilde{h}_\infty$ has the required properties. $\square$

Lemma 5.3 has a dual where φ is a 1-block map whose inverse has memory $m \geqslant 0$ and anticipation $n \geqslant 1$ and where in-splits are replaced by out-splits.

Proof of Theorem 5.2. Let $\varphi : X \to Y$ be a conjugacy from X onto Y. Replacing X by a higher block shift, we may assume that φ is a 1-block map. Using iteratively Lemma 5.3, we can replace φ by a 1-block map whose inverse has memory 0. Using then iteratively the dual of Lemma 5.3, we finally obtain a 1-block map whose inverse is also 1-block and is thus just a renaming of the symbols. $\square$

Symbolic strong shift equivalence. Let M and M' be two alphabetic $Q \times Q$-matrices over the alphabets A and B, respectively. We say that M and M' are *similar* if they are equal up to a bijection of A onto B. We write $M \leftrightarrow M'$ when M and M' are similar. We say that two alphabetic square matrices M and M' over the alphabets A and B respectively are *symbolic elementary equivalent* if there exist two alphabetic matrices R, S over the alphabets C and D respectively such that

$$M \leftrightarrow RS, \quad M' \leftrightarrow SR\,.$$

In this definition, the sets CD and DC of two letter words are identified with alphabets in bijection with A and B, respectively.

We say that two matrices M, M' are *symbolic strong shift equivalent* if there is a sequence $(M_0, M_1, \ldots, M_n)$ of alphabetic matrices such that M_i and M_{i+1} are symbolic elementary equivalent for $0 \leqslant i < n$ with $M_0 = M$ and $M_n = M'$.

We introduce the following notion. An automaton $\mathcal{A}$ on the alphabet A is said to be *bipartite* if there are partitions $Q = Q_1 \cup Q_2$ of the set of states and $A = A_1 \cup A_2$ of the alphabet such that all edges labeled in A_1 go from Q_1 to Q_2 and all edges labeled in A_2 go from Q_2 to Q_1.

Let $\mathcal{A}$ be a bipartite automaton. Its adjacency matrix has the form

$$M(\mathcal{A}) = \begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix}$$

where M_1 is a $Q_1 \times Q_2$-matrix with elements in $\mathbb{N}\langle A_1 \rangle$ and M_2 is a $Q_2 \times Q_1$-matrix with elements in $\mathbb{N}\langle A_2 \rangle$ The automata $\mathcal{A}_1$ and $\mathcal{A}_2$ which have $M_1 M_2$ and $M_2 M_1$ respectively as adjacency matrix are called the *components* of $\mathcal{A}$ and the pair $\mathcal{A}_1, \mathcal{A}_2$ is a *decomposition* of $\mathcal{A}$. We denote $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ a bipartite automaton $\mathcal{A}$ with components $\mathcal{A}_1, \mathcal{A}_2$. Note that $\mathcal{A}_1, \mathcal{A}_2$ are automata on the alphabets $A_1 A_2$ and $A_2 A_1$ respectively.

Proposition 5.4. *Let $\mathcal{A} = (Q, E)$ be a bipartite deterministic essential automaton. Its components $\mathcal{A}_1, \mathcal{A}_2$ are deterministic essential automata which are symbolic conjugate. If moreover $\mathcal{A}$ is strongly connected (resp. reduced, resp. synchronized), then $\mathcal{A}_1, \mathcal{A}_2$ are strongly connected (resp.reduced, resp. synchronized).*

Proof. Let $Q = Q_1 \cup Q_2$ and $A = A_1 \cup A_2$ be the partitions of the set Q and the alphabet A corresponding to the decomposition $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$. It is clear that $\mathcal{A}_1, \mathcal{A}_2$ are deterministic and that they are strongly connected if $\mathcal{A}$ is strongly connected.

Let $\varphi : X_{\mathcal{A}_1} \to X_{\mathcal{A}_2}$ be the conjugacy defined as follows. For any $y = (y_n)_{n \in \mathbb{Z}}$ in $X_{\mathcal{A}_1}$ there is an $x = (x_n)_{n \in \mathbb{Z}}$ in $X_{\mathcal{A}}$ such that $y_n = x_{2n} x_{2n+1}$. Then $z = (z_n)_{n \in \mathbb{Z}}$ with $z_n = x_{2n+1} x_{2n}$ is an element of $X_{\mathcal{A}_2}$. We define $\varphi(y) = z$. The analogous map $\psi : L_{\mathcal{A}_1} \to L_{\mathcal{A}_2}$ is such that (φ, ψ) is a symbolic conjugacy from $\mathcal{A}_1$ onto $\mathcal{A}_2$.

Assume that $\mathcal{A}$ is reduced. For $p, q \in Q_1$, there is a word w such that $w \in L_p(\mathcal{A})$ and $w \notin L_q(\mathcal{A})$ (or conversely). Set $w = a_1 a_2 \cdots a_n$ with $a_i \in A$. If n is even, then $(a_1 a_2) \cdots (a_{n-1} a_n)$ is in $L_p(\mathcal{A}_1)$ but not in $L_q(\mathcal{A}_1)$. Otherwise, since $\mathcal{A}$ is essential, there is a letter a_{n+1} such that $w a_{n+1}$ is in $L_p(\mathcal{A})$. Then $(a_1 a_2) \cdots (a_n a_{n+1})$ is in $L_p(\mathcal{A}_1)$ but not in $L_q(\mathcal{A}_1)$. Thus $\mathcal{A}_1$ is reduced. One proves in the same way that $\mathcal{A}_2$ is reduced.

Suppose finally that $\mathcal{A}$ is synchronized. Let x be a synchronizing word and set $x = a_1 a_2 \cdots a_n$ with $a_i \in A$. Suppose that all paths labeled x end in $q \in Q_1$. Let a_{n+1} be a letter such that $q \cdot a_{n+1} \neq \emptyset$ and let a_0 be a letter such that $a_0 x$ is the label of at least one path. If n is even, then $(a_1 a_2) \cdots (a_{n-1} a_n)$ is synchronizing for $\mathcal{A}_1$ and $(a_0 a_1) \cdots (a_n a_{n+1})$ is synchronizing for $\mathcal{A}_2$. Otherwise, $(a_0 a_1) \cdots (a_{n-1} a_n)$ is synchronizing for $\mathcal{A}_1$ and $(a_1 a_2) \cdots (a_n a_{n+1})$ is synchronizing for $\mathcal{A}_2$. □

Proposition 5.5. *Let $\mathcal{A}, \mathcal{B}$ be two automata such that $M(\mathcal{A})$ and $M(\mathcal{B})$ are symbolic elementary equivalent. Then there is a bipartite automaton $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ such that $M(\mathcal{C}_1), M(\mathcal{C}_2)$ are similar to $M(\mathcal{A}), M(\mathcal{B})$ respectively.*

Proof. Let R, S be alphabetic matrices over alphabets C and D respectively such that $M(\mathcal{A}) \leftrightarrow RS$ and $M(\mathcal{B}) \leftrightarrow SR$. Let $\mathcal{C}$ be the bipartite automaton on the alphabet $C \cup D$ which is defined by the adjacency matrix

$$M(\mathcal{C}) = \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}$$

Then $M(\mathcal{A})$ is similar to $M(\mathcal{C}_1)$ and $M(\mathcal{B})$ is similar to $M(\mathcal{C}_2)$. □

Proposition 5.6. *If the adjacency matrices of two automata are symbolic strong shift equivalent, the automata are symbolic conjugate.*

Proof. Since a composition of conjugacies is a conjugacy, it is enough to consider the case where the adjacency matrices are symbolic elementary equivalent. Let $\mathcal{A}, \mathcal{B}$ be such that $M(\mathcal{A}), M(\mathcal{B})$ are symbolic elementary equivalent. By Proposition 5.5, there is a bipartite automaton $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ such that $M(\mathcal{C}_1), M(\mathcal{C}_2)$ are similar to $M(\mathcal{A})$ and $M(\mathcal{B})$ respectively. By Proposition 5.4, the automata $\mathcal{C}_1, \mathcal{C}_2$ are symbolic conjugate. Since automata with similar adjacency matrices are obviously symbolic conjugate, the result follows. □

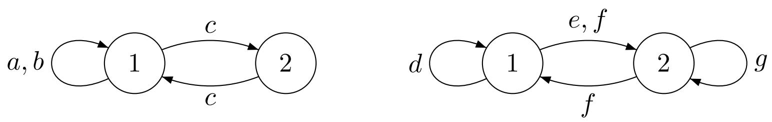

Figure 13. Two symbolic conjugate automata.

Example 5.2. Let $\mathcal{A}, \mathcal{B}$ be the automata represented on Figure 13. The matrices $M(\mathcal{A})$ and $M(\mathcal{B})$ are symbolic elementary equivalent. Indeed, we have $M(\mathcal{A}) \leftrightarrow RS$ and $M(\mathcal{B}) \leftrightarrow SR$ for

$$R = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}, \quad S = \begin{bmatrix} z & t \\ t & 0 \end{bmatrix}.$$

Indeed, one has

$$RS = \begin{bmatrix} xz + yt & xt \\ xt & 0 \end{bmatrix}, \quad SR = \begin{bmatrix} zx & zy + tx \\ tx & ty \end{bmatrix}.$$

Thus the following tables give two bijections between the alphabets.

a	b	c
xz	yt	xt

,

d	e	f	g
zx	zy	tx	ty

.

The following result is due to Nasu [21]. The equivalence between conditions (i) and (ii) is a version, for sofic shifts, of the Classification Theorem (Theorem 7.2.12 in [19]). The equivalence between conditions (i) and (iii) is due to Krieger [18].

Theorem 5.7. *Let X, X' be two sofic shifts (resp. irreducible sofic shifts) and let $\mathcal{A}, \mathcal{A}'$ be their Krieger (resp. Fischer) automata. The following conditions are equivalent.*

(i) X, X' are conjugate.

(ii) The adjacency matrices of $\mathcal{A}, \mathcal{A}'$ are symbolic strong shift equivalent.

(iii) $\mathcal{A}, \mathcal{A}'$ are symbolic conjugate.

Proof. We prove the result for irreducible shifts. The proof of the general case is in [21].

Assume that X, X' are conjugate. By the Decomposition Theorem (Theorem 5.2), it is enough to consider the case where X' is an in-splitting of X. Let $f : A \to B$ be a map and let $A' = \{f(a_1)a_2 \mid a_1a_2 \in \mathcal{B}_2(X)\}$ in such a way that X' is the in-splitting of

X relative to f. Let $C = A \cup B$ and let Z be the shift space composed of all biinfinite sequences $\cdots a_i f(a_i) a_{i+1} f(a_{i+1}) \cdots$ such that $\cdots a_i a_{i+1} \cdots$ is in X. Then Z is an irreducible sofic shift. Let $\mathcal{A}$ be the Fischer automaton of Z. Then $\mathcal{A}$ is bipartite and its components recognize, up to a bijection of the alphabets, X and X' respectively. By Proposition 5.4 the components are the Fischer automata of X and X' respectively. Since the components of a bipartite automaton have symbolic elementary equivalent adjacency matrices, this proves that (i) implies (ii).

That (ii) implies (iii) is Proposition 5.6. Finally, (iii) implies (i) by definition of symbolic conjugacy. □

5.2 Symbolic conjugate automata

The following result is due to Hamachi and Nasu [16]. It shows that, in Theorem 5.7, the equivalence between conditions (ii) and (iii) holds for automata which are not reduced.

Theorem 5.8. *Two essential automata are symbolic conjugate if and only if their adjacency matrices are symbolic strong shift equivalent.*

The first element of the proof is a version of the Decomposition Theorem for automata.

Let $\mathcal{A}, \mathcal{A}'$ be two automata. An *in-split* from $\mathcal{A}$ onto $\mathcal{A}'$ is a symbolic conjugacy (φ, ψ) such that $\varphi : X_{\mathcal{A}} \to X_{\mathcal{A}'}$ and $\psi : L_{\mathcal{A}} \to L_{\mathcal{A}'}$ are in-splitting maps. A similar definition holds for out-splits.

Theorem 5.9. *Any symbolic conjugacy between automata is a composition of splits and merges.*

The proof relies on the following variant of Lemma 5.3.

Lemma 5.10. *Let α, β be 1-block maps and φ, ψ be 1-block conjugacies such such that the diagram below on the left is commutative.*

If the inverses of φ, ψ have memory $m \geqslant 1$ and anticipation $n \geqslant 0$, there exist in-splits $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{T}$ of X, Y, Z, T respectively and 1-block maps $\tilde{\alpha} : \tilde{X} \to \tilde{Z}$, $\tilde{\beta} : \tilde{Y} \to \tilde{T}$ such that the 1-block conjugacies $\tilde{\varphi}, \tilde{\psi}$ making the diagram below on the right commutative have inverses with memory $m - 1$ and anticipation n.

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\
Z \arrow[r, "\psi"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rrr, "\varphi"] \arrow[dr] \arrow[ddd, "\alpha"'] & & & Y \arrow[ddd, "\beta"] \\
& \tilde{X} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{\alpha}"] & \tilde{Y} \arrow[ur] \arrow[d, "\tilde{\beta}"] & \\
& \tilde{Z} \arrow[r, "\tilde{\psi}"] & \tilde{T} \arrow[dr] & \\
Z \arrow[ur] \arrow[rrr, "\psi"] & & & T
\end{tikzcd}$$

Proof. Let A, B, C, D be the alphabets of X, Y, Z and T respectively. Let $h : A \to B$ and $k : C \to D$ be the 1-block substitutions such that $\varphi = h_\infty$ and $\psi = k_\infty$. Set $\tilde{A} = \{h(a_1)a_2 \mid a_1a_2 \in \mathcal{B}_2(X)\}$ and $\tilde{C} = \{k(c_1)c_2 \mid c_1c_2 \in \mathcal{B}_2(Z)\}$. Let $\tilde{X}$ (resp. $\tilde{Z}$) be the image of X (resp. of Z) under the in-splitting map relative to h (resp. k). Set $\tilde{Y} = Y^{[2]}$, $\tilde{B} = \mathcal{B}_2(Y)$, $\tilde{T} = T^{[2]}$ and $\tilde{D} = \mathcal{B}_2(T)$. Define $\tilde{\alpha}$ and $\tilde{\beta}$ by

$$\tilde{\alpha}(h(a_1)a_2) = k\alpha(a_1)\alpha(a_2), \quad \tilde{\beta}(b_1b_2) = \beta(b_1)\beta(b_2)$$

and $\tilde{h} : \tilde{A} \to \tilde{B}$, $\tilde{k} : \tilde{C} \to \tilde{D}$ by

$$\tilde{h}(h(a_1)a_2) = h(a_1)h(a_2), \quad \tilde{k}(k(c_1)c_2) = k(c_1)k(c_2)$$

Then the 1-block conjugacies $\tilde{\varphi} = \tilde{h}_\infty$ and $\tilde{\psi} = \tilde{k}_\infty$ satisfy the conditions of the statement. □

Proof of Theorem 5.9. Let $\mathcal{A} = (G, \lambda)$ and $\mathcal{A}' = (G', \lambda')$ be two automata with $G = (Q, \mathcal{E})$ and $G' = (Q', \mathcal{E}')$. Let (φ, ψ) be a symbolic conjugacy from $\mathcal{A}$ onto $\mathcal{A}'$. Replacing $\mathcal{A}$ and $\mathcal{B}$ by some extension $\mathcal{A}^{[m,n]}$ and $\mathcal{B}^{[m,n]}$ we may reduce to the case where φ, ψ are 1-block conjugacies. By using repeatedly Lemma 5.10, we may reduce to the case where the inverses of φ, ψ have memory 0. Using repeatedly the dual version of Lemma 5.10, we are reduced to the case where φ, ψ are renaming of the alphabets. □

The second step for the proof of Theorem 5.8 is the following statement.

Proposition 5.11. *Let $\mathcal{A}, \mathcal{A}'$ be two essential automata. If $\mathcal{A}'$ is an in-split of $\mathcal{A}$, the matrices $M(\mathcal{A})$ and $M(\mathcal{A}')$ are symbolic elementary equivalent.*

Proof. Set $\mathcal{A} = (G, \lambda)$ and $\mathcal{A}' = (G', \lambda')$. Let $A' = \{f(a)b \mid ab \in \mathcal{B}_2(L_\mathcal{A})\}$ be the alphabet of $\mathcal{A}'$ for a map $f : A \to B$. By Proposition 5.1, the symbolic in-splitting map from X_G onto $X_{G'}$ is also an in-splitting map. Thus there is an in-merge (h, k) from G' onto G such that the in-split from $\mathcal{A}$ onto $\mathcal{A}'$ has the form (h_∞^{-1}, ψ). We define an alphabetic $Q' \times Q$-matrix R and a $Q \times Q'$-matrix S as follows. Let $r, t \in Q'$ and let $p = k(r)$, $q = k(t)$. Let e be an edge of $\mathcal{A}'$ ending in r, and set $a = \lambda(h(e))$. Then the label of any edge going out of r is of the form $f(a)b$ for some $b \in A$. Thus $f(a)$ does not depend on e but only on r. We define a map $\pi : Q' \to B$ by $\pi(r) = f(a)$. Then, we set

$$R_{rp} = \begin{cases} \pi(r) & \text{if } k(r) = p \\ 0 & \text{otherwise} \end{cases}, \quad S_{pt} = M(\mathcal{A})_{pq}$$

Let us verify that $M(\mathcal{A}') = RS$ and $M(\mathcal{A}) \leftrightarrow SR$. We first have for $r, t \in Q'$

$$(RS)_{rt} = \sum_{p \in Q} R_{rp}S_{pt} = \pi(r)M_{k(r)k(q)} = M(\mathcal{A}')_{rt}$$

and thus $RS = M(\mathcal{A}')$. Next, for $p, q \in Q$

$$(SR)_{pq} = \sum_{p \in Q} R_{rp}S_{pt} = \sum_{t \in k^{-1}(q)} M(\mathcal{A})_{pq}\pi(t) = \sum_{a \in A} (M(\mathcal{A})_{pq}, a)af(a)$$

and thus $SR \leftrightarrow M(\mathcal{A})$ using the bijection $a \to af(a)$ between A and AB. □

Proof of Theorem 5.8. The condition is sufficient by Proposition 5.6. Conversely, let $\mathcal{A}, \mathcal{A}'$ be two symbolic conjugate essential automata. By Theorem 5.9, we may assume that $\mathcal{A}'$ is a split of $\mathcal{A}$. We assume that $\mathcal{A}'$ is an in-split of $\mathcal{A}$. By Proposition 5.11, the adjacency matrices of $\mathcal{A}$ and $\mathcal{A}'$ are symbolic elementary equivalent. □

6 Special families of automata

In this section, we consider two particular families of automata: local automata and automata with finite delay. Local automata are closely related to shifts of finite type. The main result is an embedding theorem (Theorem 6.4) related to Nasu's Masking Lemma (Proposition 6.5). Automata with finite left and right delay are related to a class of shifts called shifts of almost finite type (Proposition 6.10).

6.1 Local automata

Let $m, n \geqslant 0$. An automaton $\mathcal{A} = (Q, E)$ is said to be (m, n)*-local* if whenever $p \xrightarrow{u} q \xrightarrow{v} r$ and $p' \xrightarrow{u} q' \xrightarrow{v} r'$ are two paths with $|u| = m$ and $|v| = n$, then $q = q'$. It is *local* if it is (m, n)-local for some m, n.

Example 6.1. The automaton represented in Figure 14 is $(3, 0)$-local. Indeed, a simple inspection shows that each of the six words of length 3 which are labels of paths uniquely determines its terminal vertex. It is also $(0, 3)$-local. It is not $(2, 0)$-local (check the word ab), but it is $(2, 1)$-local and also $(1, 2)$-local.

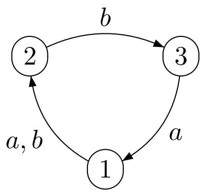

Figure 14. A local automaton.

We say that an automaton $\mathcal{A} = (Q, E)$ is *contained* in an automaton $\mathcal{A}' = (Q', E')$ if $Q \subset Q'$ and $E \subset E'$. We note that if $\mathcal{A}$ is contained in $\mathcal{A}'$ and if $\mathcal{A}'$ is local, then $\mathcal{A}$ is local.

Proposition 6.1. *An essential automaton $\mathcal{A}$ is local if and only if the map $\lambda_{\mathcal{A}} : X_{\mathcal{A}} \to L_{\mathcal{A}}$ is a conjugacy from $X_{\mathcal{A}}$ onto $L_{\mathcal{A}}$.*

Proof. Suppose first that $\mathcal{A}$ is (m, n)-local. Consider an $m+1+n$-block $w = uav$ of $L_{\mathcal{A}}$, with $|u| = m$, $|v| = n$. All finite paths of $\mathcal{A}$ labeled w have the form $r \xrightarrow{u} p \xrightarrow{a} q \xrightarrow{v} s$ and share the same edge $p \xrightarrow{a} q$. This shows that $\lambda_{\mathcal{A}}$ is injective and that $\lambda_{\mathcal{A}}^{-1}$ is a map with memory m and anticipation n.

Conversely, assume that $\lambda_{\mathcal{A}}^{-1}$ exists, and that it has memory m and anticipation n. We show that $\mathcal{A}$ is $(m+1,n)$-local. Let

$$r \xrightarrow{u} p \xrightarrow{a} q \xrightarrow{v} s \quad \text{and} \quad r' \xrightarrow{u} p' \xrightarrow{a} q' \xrightarrow{v} s'$$

and be two paths of length $m+1+n$, with $|u|=m$, $|v|=n$ and a a letter. Since $\mathcal{A}$ is essential, there exist two biinfinite paths which contain these finite paths, respectively. Since $\lambda_{\mathcal{A}}^{-1}$ has memory m and anticipation n, the blocks uav of the biinfinite words carried by these paths are mapped by $\lambda_{\mathcal{A}}^{-1}$ onto the edges $p \xrightarrow{a} q$ and $p' \xrightarrow{a} q'$ respectively. This shows that $p=p'$ and $q=q'$. □

The next statement is Proposition 10.3.10 in [4].

Proposition 6.2. *The following conditions are equivalent for a strongly connected finite automaton $\mathcal{A}$.*

(i) *$\mathcal{A}$ is local;*
(ii) *distinct cycles have distinct labels.*

Two cycles in this statement are considered to be distinct if, viewed as paths, they are distinct.

The following result shows the strong connection between shifts of finite type and local automata. It gives an effective method to verify whether or not a shift space is of finite type.

Proposition 6.3. *A shift space (resp. an irreducible shift space) is of finite type if and only if its Krieger automaton (resp. its Fischer automaton) is local.*

Proof. Let $X = X^{(W)}$ for a finite set $W \subset A^*$. We may assume that all words of W have the same length n. Let $\mathcal{A} = (Q,i,Q)$ be the $(n,0)$-local deterministic automaton defined as follows. The set of states is $Q = A^n \setminus W$ and there is an edge (u,a,v) for every $u,v \in Q$ and $a \in A$ such that $ua \in Av$. Then $\mathcal{A}$ recognizes the set $\mathcal{B}(X)$. Since the reduction of a local automaton is local, the minimal automaton of $\mathcal{B}(X)$ is local. Since the Krieger automaton of X is contained in the minimal automaton of $\mathcal{B}(X)$, it is local. If X is irreducible, then its Fischer automaton is also local since it is contained in the Krieger automaton.

Conversely, Proposition 6.1 implies that a shift space recognized by a local automaton is conjugate to a shift of finite type and thus is of finite type. □

Example 6.2. Let X be the shift of finite type on the alphabet $A=\{a,b\}$ defined by the forbidden factor ba. The Krieger automaton of X is represented on Figure 15. It is $(1,0)$-local.

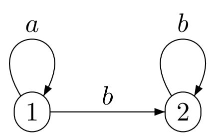

Figure 15. The Krieger automaton of a reducible shift of finite type.

For $m, n \geqslant 0$, the *standard* (m, n)-local automaton is the automaton with states the set of words of length $m + n$ and edges the triples $(uv, a, u'v')$ for $u, u' \in A^m$, $a \in A$ and $v, v' \in A^n$ such that for some letters $b, c \in A$, one has $uvc = bu'v'$ and a is the first letter of vc.

The standard $(m, 0)$-local automaton is also called the De Bruijn automaton of order m.

Example 6.3. The standard $(1, 1)$-local automaton on the alphabet $\{a, b\}$ is represented on Figure 16.

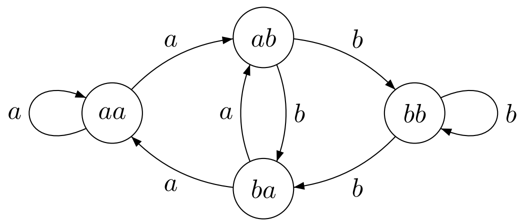

Figure 16. The standard $(1, 1)$-local automaton.

Complete automata. An automaton $\mathcal{A}$ on the alphabet A is called *complete* if any word on A is the label of some path in $\mathcal{A}$. As an example, the standard (m, n)-local automaton is complete.

The following result is from [3].

Theorem 6.4. *Any local automaton is contained in a complete local automaton.*

The proof relies on the following version of the masking lemma.

Proposition 6.5 (Masking lemma). *Let $\mathcal{A}$ and $\mathcal{B}$ be two automata and assume that $M(\mathcal{A})$ and $M(\mathcal{B})$ are elementary equivalent. If $\mathcal{B}$ is contained in an automaton $\mathcal{B}'$, then $\mathcal{A}$ is contained in some automaton $\mathcal{A}'$ which is conjugate to $\mathcal{B}'$.*

Proof. Let $\mathcal{A} = (Q, E)$, $\mathcal{B} = (R, F)$ and $\mathcal{B}' = (R', F')$. Let D be an $R \times Q$ nonnegative integral matrix and N be an alphabetic $Q \times R$ matrix such that $M(\mathcal{A}) = ND$ and $M(\mathcal{B}) = DN$. Set $Q' = Q \cup (F' \setminus F)$. Let D' be the $R' \times Q'$ nonnegative integral matrix defined for $r \in R'$ and $u \in Q'$ by

$$D'_{ru} = \begin{cases} D_{ru} & \text{if } r \in R, u \in Q \\ 1 & \text{if } u \in F' \setminus F \text{ and } u \text{ starts in } r \\ 0 & \text{otherwise} \end{cases}$$

Let N' be the alphabetic $Q' \times R'$ matrix defined for $a \in A$ for $u \in Q'$ and $s \in R'$ by

$$(N'_{us}, a) = \begin{cases} (N_{us}, a) & \text{if } u \in Q, s \in R \\ 1 & \text{if } u \in F' \setminus F \text{ and } u \text{ is labeled with } a \text{ and ends in } s, \\ 0 & \text{otherwise.} \end{cases}$$

Then $N'D'$ is the adjacency matrix of an automaton $\mathcal{A}'$. By definition, $\mathcal{A}'$ contains $\mathcal{A}$ and it is conjugate to $\mathcal{B}'$ by Proposition 3.10. $\square$

We illustrate the proof of Proposition 6.5 by the following example.

Example 6.4. Consider the automata $\mathcal{A}$ and $\mathcal{B}$ given of Figure 17. The automaton $\mathcal{A}$ is the local automaton of Example 6.1. The automaton $\mathcal{B}$ is an in-split of $\mathcal{A}$. Indeed, we have $M(\mathcal{A}) = ND$, $M(\mathcal{B}) = DN$ with

$$N = \begin{bmatrix} 0 & a & b & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

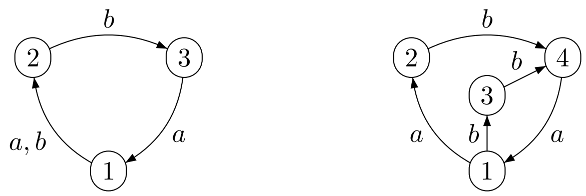

Figure 17. The automaton $\mathcal{B}$ on the right is an in-split of the local automaton $\mathcal{A}$ on the left.

We have represented on the right of Figure 18 the completion of $\mathcal{B}$ as a complete local automaton with the same number of states. On the left, the construction of the proof of Proposition 6.5 has been carried on to produce a local automaton containing $\mathcal{A}$. In terms

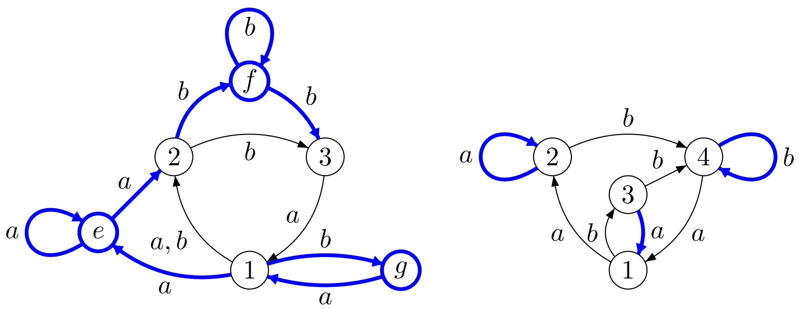

Figure 18. The automata $\mathcal{A}'$ and $\mathcal{B}'$. Additional edges are drawn thick.

of adjacency matrices, we have $M(\mathcal{A}') = N'D'$, $M(\mathcal{B}') = D'N'$ with

$$N' = \begin{bmatrix} 0 & a & b & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \end{bmatrix}, \qquad D' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Proof of Theorem 6.4. Since $\mathcal{A}$ is local, the map $\lambda_{\mathcal{A}}$ is a conjugacy from $X_{\mathcal{A}}$ to $L_{\mathcal{A}}$. Let (m, n) be the memory and anticipation of $\lambda_{\mathcal{A}}^{-1}$. There is a sequence $(\mathcal{A}_0, \ldots, \mathcal{A}_{m+n})$ of automata such that $\mathcal{A}_0 = \mathcal{A}$, each $\mathcal{A}_i$ is a split or a merge of $\mathcal{A}_{i-1}$ and $\mathcal{A}_{n+m}$ is contained in the standard $(n+m)$-local automaton. Applying iteratively Proposition 6.5, we obtain that $\mathcal{A}$ is contained in an automaton which is conjugate to the standard (m, n)-local automaton and which is thus complete. $\square$

6.2 Automata with finite delay

An automaton is said to have *right delay* $d \geqslant 0$ if for any pair of paths

$$p \xrightarrow{a} q \xrightarrow{z} r, \quad p \xrightarrow{a} q' \xrightarrow{z} r'$$

with $a \in A$, if $|z| = d$, then $q = q'$. Thus a deterministic automaton has right delay 0. An automaton has *finite right delay* if it has right delay d for some (finite) integer d. Otherwise, it is said to have *infinite right delay*.

Example 6.5. The automaton represented on Figure 19 has right delay 1.

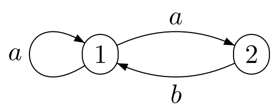

Figure 19. A automaton with right delay 1

Proposition 6.6. *An automaton has infinite right delay if and only if there exist paths* $p \xrightarrow{v} q \xrightarrow{u} q$ *and* $p \xrightarrow{v} q' \xrightarrow{u} q'$ *with* $q \neq q'$ *and* $|u| > 0$.

The following statement is Proposition 5.1.11 in [19].

Proposition 6.7. *An automaton has finite right delay if and only if it is conjugate to a deterministic automaton.*

In the same way the automaton is said to have *left delay* $d \geqslant 0$ if for any pair of paths $p \xrightarrow{z} q \xrightarrow{a} r$ and $p' \xrightarrow{z} q' \xrightarrow{a} r$ with $a \in A$, if $|z| = d$, then $q = q'$.

Corollary 6.8. *If two automata are conjugate, and if one has finite right (left) delay, then the other also has.*

Proposition 6.9. *An essential* (m, n)*-local automaton has right delay* n *and left delay* m.

Proof. Let $p \xrightarrow{a} q \xrightarrow{z} r$ and $p \xrightarrow{a} q' \xrightarrow{z} r'$ be two paths with $a \in A$ and $|z| = n$. Since $\mathcal{A}$ is essential there is a path $u \xrightarrow{y} p$ of length m in $\mathcal{A}$. Since $\mathcal{A}$ is (m, n)-local, we have $q = q'$. Thus $\mathcal{A}$ has right delay n. The proof for the left delay m is symmetrical. $\square$

A shift space is said to have *almost finite type* if it can be recognized by a strongly connected automaton with both finite left and finite right delay.

An irreducible shift of finite type is also of almost finite type since a local automaton has finite right and left delay by Proposition 6.9.

Example 6.6. The even shift has almost finite type. Indeed, the automaton of Figure 5 on the right has right and left delay 0.

The following result is from [20].

Proposition 6.10. *An irreducible shift space is of almost finite type if and only if its Fischer automaton has finite left delay.*

Proof. The condition is obviously sufficient. Conversely, let X be a shift of almost finite type. Assume the Fischer automaton $\mathcal{A} = (Q, E)$ of X does not have finite left delay. Let, in view of Proposition 6.6 $u, v \in A^*$ and $p, q, q' \in Q$ with $q \neq q'$ be such that $q \cdot u = q$, $q' \cdot u = q'$ and $p = q \cdot v = q' \cdot v$. Since $\mathcal{A}$ is strongly connected, there is a word w such that $p \cdot w = q$.

Let $\mathcal{B} = (R, F)$ be an automaton with finite right and left delay which recognizes X. By Proposition 6.7, we may assume that $\mathcal{B}$ is deterministic. Let $\varphi : R \to Q$ be a reduction from $\mathcal{B}$ onto $\mathcal{A}$. Since R is finite, there is an $x \in u^+$ such that $r \cdot x = r \cdot x^2$ for all $r \in R$ (this means that the map $r \mapsto r \cdot x$ is idempotent; such a word exists since each element in the finite transition semigroup of the automaton $\mathcal{B}$ has a power which is an idempotent). Set

$$S = R \cdot x, \quad T = \varphi^{-1}(q) \cap S, \quad T' = \varphi^{-1}(q') \cap S$$

Since $q \neq q'$, we have $T \cap T' = \emptyset$. For any $t \in T$, we have $\varphi(t \cdot vw) = q$ and thus $t \cdot vwx \in T$. For $t, t' \in T$ with $t \neq t'$, we cannot have $t \cdot vwx = t' \cdot vwx$ since otherwise $\mathcal{B}$ would have infinite left delay. Thus the map $t \mapsto t \cdot vwx$ is a bijection of T.

Let $t' \in T'$. Since $\varphi(t' \cdot vw) = q$, we have $t' \cdot vwx \in T$. Since the action of vwx induces a permutation on T, there exists $t \in T$ such that $t \cdot vwx = t' \cdot vwx$. This contradicts the fact that $\mathcal{B}$ has finite left delay. $\square$

Example 6.7. The deterministic automaton represented on Figure 20 has infinite left delay. Indeed, there are paths $\cdots 1 \xrightarrow{b} 1 \xrightarrow{a} 1$ and $\cdots 2 \xrightarrow{b} 2 \xrightarrow{a} 1$. Since this automaton cannot be reduced, $X = L_{\mathcal{A}}$ is not of almost finite type.

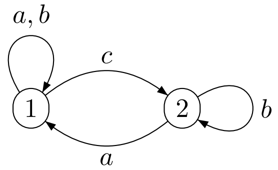

Figure 20. An automaton with infinite left delay

7 Syntactic invariants

We introduce in this section the syntactic graph of an automaton. It uses the Green relations in the transition semigroup of the automaton. We show that the syntactic graph is an invariant for symbolic conjugacy (Theorem 7.4). The proof uses bipartite automata.

The final subsection considers the characterization of sofic shifts with respect to the families of ordered semigroups known as pseudovarieties.

7.1 The syntactic graph

Let $\mathcal{A} = (Q, E)$ be a deterministic automaton on the alphabet A. Each word $w \in A^*$ defines a partial map denoted by $\varphi_{\mathcal{A}}(w)$ from Q to Q which maps $p \in Q$ to $q \in Q$ if $p \cdot w = q$. The transition semigroup of $\mathcal{A}$, already defined in Section 4.2, is the image of A^+ by the morphism $\varphi_{\mathcal{A}}$ (in this subsection, we will not use the order on the transition semigroup).

We give a short summary of *Green relations* in a semigroup (see [17] for example). Let S be a semigroup and let $S^1 = S \cup 1$ be the monoid obtained by adding an identity to S. Two elements s, t of S are $\mathcal{R}$-equivalent if $sS^1 = tS^1$. They are $\mathcal{L}$-equivalent if $S^1 s = S^1 t$. It is a classical result (see [17]) that $\mathcal{LR} = \mathcal{RL}$. Thus $\mathcal{LR} = \mathcal{RL}$ is an equivalence on the semigroup S called the $\mathcal{D}$-equivalence. A class of the $\mathcal{R}, \mathcal{L}$ or $\mathcal{D}$-equivalence is called an $\mathcal{R}, \mathcal{L}$ or $\mathcal{D}$-class. An *idempotent* of S is an element e such that $e^2 = e$. A $\mathcal{D}$-class is *regular* if it contains an idempotent. The equivalence $\mathcal{H}$ is defined as $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. It is classical result that the $\mathcal{H}$-class of an idempotent is a group. The $\mathcal{H}$-class of idempotents in the same $\mathcal{D}$-class are isomorphic groups. The *structure group* of a regular $\mathcal{D}$-class is any of the $\mathcal{H}$-classes of an idempotent of the $\mathcal{D}$-class.

When S is a semigroup of partial maps from a set Q into itself, each element of S has a rank which is the cardinality of its image. The elements of a $\mathcal{D}$-class all have the same rank, which is called the *rank* of the $\mathcal{D}$-class. There is at most one element of rank 0 which is the *zero* of the semigroup S and is denoted 0.

A *fixpoint* of a partial map s from Q into itself is an element q such that the image of q by s is q. The rank of an idempotent is equal to the number of its fixpoints. Indeed, in this case, every element in the image is a fixpoint.

The preorder $\leqslant_{\mathcal{J}}$ on S is defined by $s \leqslant_{\mathcal{J}} t$ if $S^1 s S^1 \subset S^1 t S^1$. Two elements $s, t \in S$ are $\mathcal{J}$-equivalent if $S^1 s S^1 = S^1 t S^1$. One has $\mathcal{D} \subset \mathcal{J}$ and it is a classical result that in a finite semigroup $\mathcal{D} = \mathcal{J}$. The preorder $\leqslant_{\mathcal{J}}$ induces a partial order on the $\mathcal{D}$-classes, still denoted $\leqslant_{\mathcal{J}}$.

We associate with $\mathcal{A}$ a labeled graph $G(\mathcal{A})$ called its *syntactic graph*. The vertices of $G(\mathcal{A})$ are the regular $\mathcal{D}$-classes of the transition semigroup of $\mathcal{A}$. Each vertex is labeled by the rank of the $\mathcal{D}$-class and its structure group. There is an edge from the vertex associated with a $\mathcal{D}$-class D to the vertex associated to a $\mathcal{D}$-class D' if and only if $D \geqslant_{\mathcal{J}} D'$.

Example 7.1. The automaton $\mathcal{A}$ of Figure 21 on the left is the Fischer automaton of the even shift (Example 4.3). The semigroup of transitions of $\mathcal{A}$ has 3 regular $\mathcal{D}$-classes of ranks 2 (containing $\varphi_{\mathcal{A}}(b)$), 1 (containing $\varphi_{\mathcal{A}}(a)$), and 0 (containing $\varphi_{\mathcal{A}}(aba)$). Its syntactic graph is represented on the right.

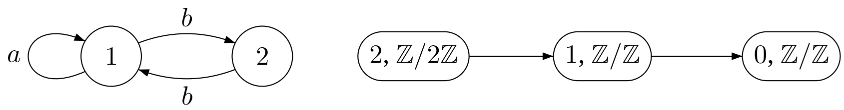

Figure 21. The syntactic graph of the even shift

The following result shows that one may reduce to the case of essential automata.

Proposition 7.1. *The syntactic graphs of an automaton and of its essential part are isomorphic.*

Proof. Let $\mathcal{A} = (Q, E)$ be a deterministic automaton on the alphabet A and let $\mathcal{A}' = (Q', E')$ be its essential part. Let $w \in A^+$ be such that $e = \varphi_{\mathcal{A}}(w)$ is an idempotent. Then any fixpoint of e is in Q' and thus $e' = \varphi_{\mathcal{A}'}(w)$ an idempotent of the same rank as e. This shows that $G(\mathcal{A})$ and $G(\mathcal{A}')$ are isomorphic. $\square$

The following result shows that the syntactic graph characterizes irreducible shifts of finite type.

Proposition 7.2. *A sofic shift (resp. an irreducible sofic shift) is of finite type if and only if the syntactic graph of its Krieger automaton (resp. its Fischer automaton) has nodes of rank at most* 1.

In the proof, we use the following classical property of finite semigroups.

Proposition 7.3. *Let S be a finite semigroup and let J be an ideal of S. The following conditions are equivalent.*

(i) All idempotents of S are in J.
(ii) There exists an integer $n \geqslant 1$ such that $S^n \subset J$.

Proof. Assume that (i) holds. Let $n = \mathrm{Card}(S) + 1$ and let $s = s_1 s_2 \cdots s_n$ with $s_i \in S$. Then there exist i, j with $1 \leqslant i < j \leqslant n$ such that $s_1 s_2 \cdots s_i = s_1 s_2 \cdots s_i \cdots s_j$. Let $t, u \in S^1$ be defined by $t = s_1 \cdots s_i$ and $u = s_{i+1} \cdots s_j$. Since $tu = t$, we have $tu^k = t$ for all $k \geqslant 1$. Since S is finite, there is a $k \geqslant 1$ such that u^k is idempotent and thus $u^k \in J$. This implies that $t \in J$ and thus $s \in J$. Thus (ii) holds.
It is clear that (ii) implies (i). $\square$

Proof of Proposition 7.2. Let X be a shift space (resp. an irreducible shift space), let $\mathcal{A}$ be its Krieger automaton (resp. its Fischer automaton) and let S be the transition semigroup of $\mathcal{A}$.

If X is of finite type, by Proposition 6.3, the automaton $\mathcal{A}$ is local. Any idempotent in S has rank 1 and thus the condition is satisfied.

Conversely, assume that the graph $G(\mathcal{A})$ has nodes of rank at most 1. Let J be the ideal of S formed of the elements of rank at most 1. Since all idempotents of S belong to J, by Proposition 7.3, the semigroup S satisfies $S^n = J$ for some $n \geqslant 1$. This shows that for any sufficiently long word x, the map $\varphi_{\mathcal{A}}(x)$ has rank at most 1. Thus for $p, q, r, s \in Q$, if $p \cdot x = r$ and $q \cdot x = s$ then $r = s$. This implies that $\mathcal{A}$ is $(n, 0)$-local. $\square$

The following result is from [2].

Theorem 7.4. *Two symbolic conjugate automata have isomorphic syntactic graphs.*

We use the following intermediary result.

Proposition 7.5. *Let $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ be a bipartite automaton. The syntactic graphs of $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic.*

Proof. Let $Q = Q_1 \cup Q_2$ and $A = A_1 \cup A_2$ be the partitions of the set of of states and of the alphabet of $\mathcal{A}$ corresponding to the decomposition $(\mathcal{A}_1, \mathcal{A}_2)$. Set $B_1 = A_1A_2$ and $B_2 = A_2A_1$. The semigroups $S_1 = \varphi_{\mathcal{A}_1}(B_1^+)$ and $S_2 = \varphi_{\mathcal{A}_2}(B_2^+)$ are included in the semigroup $S = \varphi_{\mathcal{A}}(A^+)$. Thus the Green relations of S are refinements of the corresponding Green relations in S_1 or in S_2. Any idempotent e of S belongs either to S_1 or to S_2. Indeed, if $e = 0$ then e is in $S_1 \cap S_2$. Otherwise, it has at least one fixpoint $p \in Q_1 \cup Q_2$. If $p \in Q_1$, then e is in $\varphi_A(B_1^+)$ and thus $e \in S_1$. Similarly if $p \in Q_2$ then $e \in S_2$.

Let e be an idempotent in S_1 and let $e = \varphi_{\mathcal{A}}(u)$. Since $u \in B_1^+$, we have $u = au'$ with $a \in A_1$ and $u' \in B_2^*A_2$. Let $v = u'a$. Then $f = \varphi_{\mathcal{A}}(v)^2$ is idempotent. Indeed, we have

$$\varphi_{\mathcal{A}}(v^3) = \varphi_{\mathcal{A}}(u'au'au'a) = \varphi_{\mathcal{A}}(u'uua) = \varphi_{\mathcal{A}}(u'ua) = \varphi_{\mathcal{A}}(v^2)$$

Moreover e, f belong the same $\mathcal{D}$-class. Similarly, if $e \in S_2$, there is an idempotent in S_1 which is $\mathcal{D}$ equivalent to e. This shows that a regular $\mathcal{D}$-class of $\varphi_{\mathcal{A}}(A^+)$ contains idempotents in S_1 and in S_2.

Finally, two elements of S_1 which are $\mathcal{D}$-equivalent in S are also $\mathcal{D}$-equivalent in S_1. Indeed, let $s, t \in S_1$ be such that $s\mathcal{RL}t$. Let $u, u', v, v' \in S$ be such that

$$suu' = s, \quad v'vt = t, \quad su = tv$$

in such a way that $s\mathcal{R}su$ and $vt\mathcal{L}t$. Then $su = vt$ implies that u, v are both in S_1. Similarly $suu' = s$ and $v'vt = t$ imply that $u'v' \in S_1$. Thus $s\mathcal{D}t$ in S_1. This shows that a regular $\mathcal{D}$ class D of S contains exactly one $\mathcal{D}$-class D_1 of S_1 (resp. D_2 of S_2). Moreover, an $\mathcal{H}$-class of D_1 is also an $\mathcal{H}$-class of D.

Thus the three syntactic graphs are isomorphic. □

Proof of Theorem 7.4. Let $\mathcal{A} = (Q, E)$ and $\mathcal{B} = (R, F)$ be two symbolic conjugate automata on the alphabets A and B, respectively. By the Decomposition Theorem (Theorem 5.9), we may assume that the symbolic conjugacy is a split or a merge. Assume that $\mathcal{A}'$ is an in-split of $\mathcal{A}$. By Proposition 7.1, we may assume that $\mathcal{A}$ and $\mathcal{A}'$ are essential. By Proposition 5.11, the adjacency matrices of $\mathcal{A}$ and $\mathcal{A}'$ are symbolic elementary equivalent.

By Proposition 5.5, there is a bipartite automaton $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ such that $M(\mathcal{C}_1), M(\mathcal{C}_2)$ are similar to $M(\mathcal{A}), M(\mathcal{B})$ respectively. By Proposition 7.5, the syntactic graphs of $\mathcal{C}_1, \mathcal{C}_2$ are isomorphic. Since automata with similar adjacency matrices have obviously isomorphic syntactic graphs, the result follows. □

A refinement of the syntactic graph which is also invariant by flow equivalence has been introduced in [9]. The vertices of the graph are the *idempotent-bound* $\mathcal{D}$ classes, where an element s of a semigroup S is called idempotent-bound if there exist idempotents $e, f \in S$ such that $s = esf$. The elements of a regular $\mathcal{D}$-class are idempotent-bound.

Flow equivalent automata. Let $\mathcal{A}$ be an automaton on the alphabet A and let G be its underlying graph. An *expansion* of $\mathcal{A}$ is a pair (φ, ψ) of a graph expansion of G and a symbol expansion of $L_{\mathcal{A}}$ such that the diagram below is commutative. The inverse of an

$$\begin{tikzcd}
X_{\mathcal{A}} \arrow[r, "\varphi"] \arrow[d, "\lambda_{\mathcal{A}}"] & X_{\mathcal{B}} \arrow[d, "\lambda_{\mathcal{B}}"] \\
L_{\mathcal{A}} \arrow[r, "\psi"] & L_{\mathcal{B}}
\end{tikzcd}$$

automaton expansion is called a contraction.

Example 7.2. Let $\mathcal{A}$ and $\mathcal{B}$ be the automata represented on Figure 22. The second automaton is an expansion of the first one.

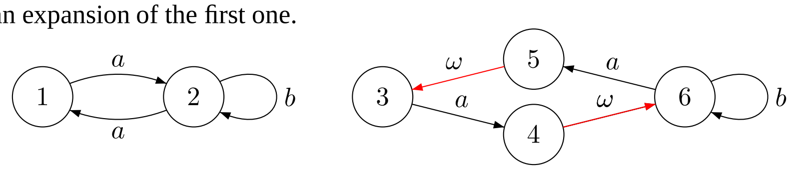

Figure 22. An automaton expansion

The *flow equivalence* of automata is the equivalence generated by symbolic conjugacies, expansions and contractions.

Theorem 7.4 has been generalized by Costa and Steinberg [12] to flow equivalence.

Theorem 7.6. *Two flow equivalent automata have isomorphic syntactic graphs.*

Example 7.3. The syntactic graphs of the automata $\mathcal{A}$, $\mathcal{B}$ of Example 5.2 are isomorphic to the syntactic graph of the Fischer automaton $\mathcal{C}$ of the even shift. Note that the automata $\mathcal{A}, \mathcal{B}$ are not flow equivalent to $\mathcal{C}$. Indeed, the edge shifts $X_{\mathcal{A}}$, $X_{\mathcal{B}}$ on the underlying graphs of the automata $\mathcal{A}$, $\mathcal{B}$ are flow equivalent to the full shift on 3 symbols while the edge shift $X_{\mathcal{C}}$ is flow equivalent to the full shift on 2 symbols. Thus the converse of Theorem 7.6 is false.

7.2 Pseudovarieties

In this subsection, we will see how one can formulate characterizations of some classes of sofic shifts by means of properties of their syntactic semigroup. In order to formulate these syntactic characterizations of sofic shifts, we introduce the notion of pseudovariety of ordered semigroups. For a systematic exposition, see the original articles [25], [27], or the surveys in [26] or [24].

A morphism of ordered semigroups φ from S into T is an order compatible semigroup morphism, that is such that $s \leqslant s'$ implies $\varphi(s) \leqslant \varphi(s')$. An ordered subsemigroup of S is a subsemigroup equipped with the restriction of the preorder.

A *pseudovariety* of finite ordered semigroups is a class of ordered semigroups closed under taking ordered subsemigroups, finite direct products and image under morphisms of ordered semigroups.

Let V be a pseudovariety of ordered semigroups. We say that a semigroup S is *locally* in V if all the submonoids of S are in V. The class of these semigroups is a pseudovariety of ordered semigroups.

The following result is due to Costa [10].

Theorem 7.7. *Let V be a pseudovariety of finite ordered semigroups containing the class of commutative ordered monoids such that every element is idempotent and greater than the identity. The class of shifts whose syntactic semigroup is locally in V is invariant under conjugacy.*

The following statements give examples of pseudovarieties satisfying the above condition.

Proposition 7.8. *An irreducible shift space is of finite type if and only if its syntactic semigroup is locally commutative.*

An *inverse semigroup* is a semigroup which can be represented as a semigroup of partial one-to-one maps from a finite set Q into itself. The family of inverse semigroups does not form a variety (it is not closed under homomorphic image. However, according to Ash's theorem [1], the variety generated by inverse semigroups is characterized by the property that the idempotents commute. Using this result, the following result is proved in [10].

Theorem 7.9. *An irreducible shift space is of almost finite type if and only if its syntactic semigroup is locally in the pseudovariety generated by inverse semigroups.*

The fact that shifts of almost finite type satisfy this condition was proved in [2]. The converse was conjectured in the same paper.

In [12] it is shown that this result implies that the class of shifts of almost finite type is invariant under flow equivalence. This is originally from [15].

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Abstract. This chapter presents some of the links between automata theory and symbolic dynamics. The emphasis is on two particular points. The first one is the interplay between some particular classes of automata, such as local automata and results on embeddings of shifts of finite type. The second one is the connection between syntactic semigroups and the classification of sofic shifts up to conjugacy.

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