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# Symbolic dynamics

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## 1 Introduction

29

30 Symbolic dynamics is part of dynamical systems theory. It studies discrete dynamical  
31 systems called shift spaces and their relations under appropriately defined morphisms,  
32 in particular isomorphisms called conjugacies. A special emphasis has been put on the  
33 classification of shift spaces up to conjugacy or flow equivalence.

34 There is a considerable overlap between symbolic dynamics and automata theory.  
35 Actually, one of the basic objects of symbolic dynamics, the sofic systems, are essentially  
36 the same as finite automata. In addition, the morphisms of shift spaces are a particular  
37 case of rational transductions, that is functions defined by finite automata with output. The  
38 difference is that symbolic dynamics considers mostly infinite words and that all states of  
39 the automata are initial and final. Also, the morphisms are particular transductions which  
40 are given by local maps.

41 This chapter presents some of the links between automata theory and symbolic dy-  
42 namics. The emphasis is on two particular points. The first one is the interplay between  
43 some particular classes of automata, such as local automata and results on embeddings of  
44 shifts of finite type. The second one is the connection between syntactic semigroups and  
45 the classification of sofic shifts up to conjugacy.

46 The chapter is organized as follows. In Section 2, we introduce the basic notions of  
47 symbolic dynamics: shift spaces, conjugacy and flow equivalence. We state without proof  
48 two important results: the Decomposition Theorem and the Classification Theorem.

49 In Section 3, we introduce automata in relation to sofic shifts. In Section 4, we define  
50 two kinds of minimal automata for shift spaces: the Krieger automaton and the Fischer  
51 automaton. We also relate these automata with the syntactic semigroup of a shift space.

52 In Section 5, we state and prove an analogue due to Nasu of the Decomposition The-  
53 orem and of the Classification Theorem.

54 In Section 6 we consider two special families of automata: local automata and au-  
55 tomata with finite delay. We show that they are related to shifts of finite type and of  
56 almost finite type, respectively. We prove an embedding theorem (Theorem 6.4) which is  
57 a counterpart for automata of a result known as Nasu's masking lemma.

58 In Section 7 we study syntactic invariants of sofic shifts. We introduce the syntac-  
59 tic graph of an automaton. We show that that the syntactic graph of an automaton is  
60 invariant under conjugacy (Theorem 7.4) and also under flow equivalence. We finally  
61 state some results concerning the shift spaces corresponding to some pseudovarieties of  
62 ordered semigroups.

63 We follow the notation of the book of Doug Lind and Brian Marcus [19]. In general,  
64 we have not reproduced the proofs of the results which can be found there. We thank  
65 Mike Boyle and Alfredo Costa for their help.

## 2 Shift spaces

66

67 This section contains basic definitions concerning symbolic dynamics.

68 The first subsection gives the definition of shift spaces, and the important case of edge  
69 shifts.

70 The next subsection and thus also under (Section 2.2) introduces conjugacy, and the  
 71 basic notion of state splitting and merging. It contains the statement of two important  
 72 theorems, the Decomposition Theorem (Theorem 2.12) and the Classification Theorem  
 73 (Theorem 2.14).

74 The last subsection (Section 2.3) introduces flow equivalence, and states Frank's char-  
 75 acterization of flow equivalent edge shifts (Theorem 2.16).

## 76 2.1 Shift spaces

77 Let  $A$  be a finite alphabet. We denote by  $A^*$  the set of words on  $A$  and by  $A^+$  the set of  
 78 nonempty words. A word  $v$  is a *factor* of a word  $t$  if  $t = uvw$  for some words  $u, w$ .

79 We denote by  $A^{\mathbb{Z}}$  the set of biinfinite sequences of symbols from  $A$ . This set is a  
 80 topological space in the product topology of the discrete topology on  $A$ . The *shift trans-*  
 81 *formation* on  $A^{\mathbb{Z}}$  is the map  $\sigma_A$  from  $A^{\mathbb{Z}}$  onto itself defined by  $y = \sigma_A(x)$  if  $y_n = x_{n+1}$   
 82 for  $n \in \mathbb{Z}$ . A set  $X \subset A^{\mathbb{Z}}$  is *shift invariant* if  $\sigma(X) = X$ . A *shift space* on the alphabet  
 83  $A$  is a shift-invariant subset of  $A^{\mathbb{Z}}$  which is closed in the topology. The set  $A^{\mathbb{Z}}$  itself is a  
 84 shift space called the *full shift*.

85 For a set  $W \subset A^*$  of words (whose elements are called the *forbidden factors*), we  
 86 denote by  $X^{(W)}$  the set of  $x \in A^{\mathbb{Z}}$  such that no  $w \in W$  is a factor of  $x$ .

87 **Proposition 2.1.** *The shift spaces on the alphabet  $A$  are the sets  $X^{(W)}$ , for  $W \subset A^*$ .*

88 A shift space  $X$  is of *finite type* if there is a finite set  $W \subset A^*$  such that  $X = X^{(W)}$ .

89 **Example 2.1.** Let  $A = \{a, b\}$ , and let  $W = \{bb\}$ . The shift  $X^{(W)}$  is composed of the  
 90 sequences without two consecutive  $b$ 's. It is a shift of finite type, called the *golden mean*  
 91 *shift*.

92 Recall that a set  $W \subset A^*$  is said to be *recognizable* if it can be recognized by a  
 93 finite automaton or, equivalently, defined by a regular expression. A shift space  $X$  is said  
 94 to be *sofic* if there is a recognizable set  $W$  such that  $X = X^{(W)}$ . Since a finite set is  
 95 recognizable, any shift of finite type is sofic.

96 **Example 2.2.** Let  $A = \{a, b\}$ , and let  $W = a(bb)^*ba$ . The shift  $X^{(W)}$  is composed of  
 97 the sequences where two consecutive occurrences of the symbol  $a$  are separated by an  
 98 even number of  $b$ 's. It is a sofic shift called the *even shift*. It is not a shift of finite type.  
 99 Indeed, assume that  $X = X^{(V)}$  for a finite set  $V \subset A^*$ . Let  $n$  be the maximal length of  
 100 the words of  $V$ . A biinfinite repetition of the word  $ab^n$  has the same blocks of length at  
 101 most  $n$  as a biinfinite repetition of the word  $ab^{n+1}$ . However, one is in  $X$  if and only if  
 102 the other is not in  $X$ , a contradiction.

103 **Example 2.3.** Let  $A = \{a, b\}$  and let  $W = \{ba^n b^m a \mid n, m \geq 1, n \neq m\}$ . The shift  
 104  $X^{(W)}$  is composed of infinite sequences of the form  $\dots a^{n_i} b^{n_i} a^{n_{i+1}} b^{n_{i+1}} \dots$ . The set  $W$   
 105 is not recognizable and it can be shown that  $X$  is not sofic.

106 **Edge shifts.** In this chapter, a *graph*  $G = (Q, \mathcal{E})$  is a pair composed of a finite set  $Q$   
 107 of *vertices* (or *states*), and a finite set  $\mathcal{E}$  of *edges*. The graph is equipped with two maps  
 108  $i, t : \mathcal{E} \rightarrow Q$  which associate, to an edge  $e$ , its *initial* and *terminal* vertex<sup>1</sup>. We say that  $e$   
 109 starts in  $i(e)$  and ends in  $t(e)$ . Sometimes,  $i(e)$  is called the *source* and  $t(e)$  is called the  
 110 *target* of  $e$ .

111 We also say that  $e$  is an incoming edge for  $t(e)$ , and an outgoing edge for  $i(e)$ . Two  
 112 edges  $e, e' \in \mathcal{E}$  are *consecutive* if  $t(e) = i(e')$ .

For  $p, q \in Q$ , we denote by  $\mathcal{E}_p^q$  the set of edges of a graph  $G = (Q, \mathcal{E})$  starting in state  $p$  and ending in state  $q$ . The *adjacency matrix* of a graph  $G = (Q, \mathcal{E})$  is the  $Q \times Q$ -matrix  $M(G)$  with elements in  $\mathbb{N}$  defined by

$$M(G)_{pq} = \text{Card}(\mathcal{E}_p^q).$$

113 A (finite or biinfinite) *path* is a (finite or biinfinite) sequence of consecutive edges. The  
 114 *edge shift* on the graph  $G$  is the set of biinfinite paths in  $G$ . It is denoted by  $X_G$  and is a  
 115 shift of finite type on the alphabet of edges. Indeed, it can be defined by taking the set of  
 116 non-consecutive edges for the set of forbidden factors. The converse does not hold, since  
 117 the golden mean shift is not an edge shift. However, we shall see below (Proposition 2.5)  
 118 that every shift of finite type is conjugate to an edge shift.

119 A graph is *essential* if every state has at least one incoming and one outgoing edge.  
 120 This implies that every edge is on a biinfinite path. The *essential part* of a graph  $G$  is the  
 121 subgraph obtained by restricting to the set of vertices and edges which are on a biinfinite  
 122 path.

## 123 2.2 Conjugacy

124 **Morphisms.** Let  $X$  be a shift space on an alphabet  $A$ , and let  $Y$  be a shift space on an  
 125 alphabet  $B$ .

126 A *morphism*  $\varphi$  from  $X$  into  $Y$  is a continuous map from  $X$  into  $Y$  which commutes  
 127 with the shift. This means that  $\varphi \circ \sigma_A = \sigma_B \circ \varphi$ .

Let  $k$  be a positive integer. A *k-block* of  $X$  is a factor of length  $k$  of an element of  $X$ . We denote by  $\mathcal{B}(X)$  the set of all blocks of  $X$  and by  $\mathcal{B}_k(X)$  the set of  $k$ -blocks of  $X$ . A function  $f : \mathcal{B}_k(X) \rightarrow B$  is called a *k-block substitution*. Let now  $m, n$  be fixed nonnegative integers with  $k = m + 1 + n$ . Then the function  $f$  defines a map  $\varphi$  called *sliding block map* with *memory*  $m$  and *anticipation*  $n$  as follows. The image of  $x \in X$  is the element  $y = \varphi(x) \in B^{\mathbb{Z}}$  given by

$$y_i = f(x_{i-m} \cdots x_i \cdots x_{i+n}).$$

128 We denote  $\varphi = f_{\infty}^{[m,n]}$ . It is a sliding block map from  $X$  into  $Y$  if  $y$  is in  $Y$  for all  $x$  in  
 129  $X$ . We also say that  $\varphi$  is a  $k$ -block map from  $X$  into  $Y$ . The simplest case occurs when  
 130  $m = n = 0$ . In this case,  $\varphi$  is a 1-block map.

131 The following result is Theorem 6.2.9 in [19].

132 **Theorem 2.2** (Curtis–Lyndon–Hedlund). *A map from a shift space  $X$  into a shift space*  
 133  *$Y$  is a morphism if and only if it is a sliding block map.*

<sup>1</sup>We avoid the use of the terms ‘initial state’ or ‘terminal state’ of an edge to avoid confusion with the initial or terminal states of an automaton

134 **Conjugacies of shifts.** A morphism from a shift  $X$  onto a shift  $Y$  is called a *conjugacy*  
 135 if it is one-to-one from  $X$  onto  $Y$ . Note that in this case, using standard topological  
 136 arguments, one shows that the inverse mapping is also a morphism, and thus a conjugacy.

137 We define the  $n$ -th *higher block shift*  $X^{[n]}$  of a shift  $X$  over the alphabet  $A$  as follows.  
 138 The alphabet of  $X^{[n]}$  is the set  $B = \mathcal{B}_n(X)$  of blocks of length  $n$  of  $X$ .

139 **Proposition 2.3.** *The shifts  $X$  and  $X^{[n]}$  for  $n \geq 1$  are conjugate.*

140 *Proof.* Let  $f : \mathcal{B}_n(X) \rightarrow B$  be the  $n$ -block substitution which maps the factor  $x_1 \cdots x_n$   
 141 to itself, viewed as a symbol of the alphabet  $B$ . By construction, the shift  $X^{[n]}$  is the  
 142 image of  $X$  by the map  $f_\infty^{[n-1,0]}$ . This map is a conjugacy since it is bijective, and its  
 143 inverse is the 1-block map  $g_\infty$  corresponding to the 1-block map which associates to the  
 144 symbol  $x_1 \cdots x_n$  of  $B$  the symbol  $x_n$  of  $A$ .  $\square$

145 Let  $G = (Q, \mathcal{E})$  be a graph. For an integer  $n \geq 1$ , denote by  $G^{[n]}$  the following graph  
 146 called the  $n$ -th *higher edge graph* of  $G$ . For  $n = 1$ , one has  $G^{[1]} = G$ . For  $n > 1$ , the set  
 147 of states of  $G^{[n]}$  is the set of paths of length  $n - 1$  in  $G$ . The edges of  $G^{[n]}$  are the paths  
 148 of length  $n$  of  $G$ . The start state of an edge  $(e_1, e_2, \dots, e_n)$  is  $(e_1, e_2, \dots, e_{n-1})$  and its  
 149 end state is  $(e_2, e_3, \dots, e_n)$ .

150 The following result shows that the higher block shifts of an edge shift are again edge  
 151 shifts.

152 **Proposition 2.4.** *Let  $G$  be a graph. For  $n \geq 1$ , one has  $X_G^{[n]} = X_{G^{[n]}}$ .*

153 A shift of finite type need not be an edge shift. For example the golden mean shift of  
 154 Example 2.1 is not an edge shift. However, any shift of finite type comes from an edge  
 155 shift in the following sense.

156 **Proposition 2.5.** *Every shift of finite type is conjugate to an edge shift.*

157 *Proof.* We show that for every shift of finite type  $X$  there is an integer  $n$  such that  $X^{[n]}$  is  
 158 an edge shift. Let  $W \subset A^*$  be a finite set of words such that  $X = X^{(W)}$ , and let  $n$  be the  
 159 maximal length of the words of  $W$ . If  $n = 0$ ,  $X$  is the full shift. Thus we assume  $n \geq 1$ .  
 160 Define a graph  $G$  whose vertices are the blocks of length  $n - 1$  of  $X$ , and whose edges  
 161 are the block of length  $n$  of  $X$ . For  $w \in \mathcal{B}_n(X)$ , the initial (resp. terminal) vertex of  $w$  is  
 162 the prefix (resp. suffix) of length  $n - 1$  of  $w$ .

163 We show that  $X_G = X^{[n]}$ . An element of  $X^{[n]}$  is always an infinite path in  $G$ . To  
 164 show the other inclusion, consider an infinite path  $y$  in  $G$ . It is the sequence of  $n$ -blocks  
 165 of an element  $x$  of  $A^{\mathbb{Z}}$  which does not contain any block on  $W$ . Since  $X = X^{(W)}$ , we  
 166 get that  $x$  is in  $X$ . Consequently,  $y$  is in  $X^{[n]}$ . This proves the equality.  $\square$

167 **Proposition 2.6.** *A shift space that is conjugate to a shift of finite type is itself of finite  
 168 type.*

169 *Proof.* Let  $\varphi : X \rightarrow Y$  be a conjugacy from a shift of finite type  $X$  onto a shift space  
 170  $Y$ . By Proposition 2.5, we may assume that  $X = X_G$  for some graph  $G$ . Changing  
 171  $G$  into some higher edge graph, we may assume that  $\varphi$  is 1-block. We may consider

172  $G$  as a graph labeled by  $\varphi$ . Suppose that  $\varphi^{-1}$  has memory  $m$  and anticipation  $n$ . Set  
 173  $\varphi^{-1} = f_{\infty}^{[m,n]}$ . Let  $W$  be the set of words of length  $m + n + 2$  which are not the label  
 174 of a path in  $G$ . We show that  $Y = X^{(W)}$ , which implies that  $Y$  is of finite type. Indeed,  
 175 the inclusion  $Y \subset X^{(W)}$  is clear. Conversely, consider  $y$  in  $X^{(W)}$ . For each  $i \in \mathbb{Z}$ , set  
 176  $x_i = f(y_{i-m} \cdots y_i \cdots y_{i+n})$ . Since  $y_{i-m} \cdots y_i \cdots y_{i+n} y_{i+n+1}$  is the label of a path in  
 177  $G$ , the edges  $x_i$  and  $x_{i+1}$  are consecutive. Thus  $x = (x_i)_{i \in \mathbb{Z}}$  is in  $X$  and  $y = \varphi(x)$  is in  
 178  $Y$ .  $\square$

179 **Conjugacy invariants.** No effective characterization of conjugate shift spaces is known,  
 180 even for shifts of finite type. There are however several quantities that are known to be  
 181 invariant under conjugacy.

The *entropy* of a shift space  $X$  is defined by

$$h(X) = \lim_{n \rightarrow \infty} \frac{1}{n} \log s_n,$$

182 where  $s_n = \text{Card}(\mathcal{B}_n(X))$ . The limit exists because the sequence  $s_n$  is sub-additive  
 183 (see [19] Lemma 4.1.7). Note that since  $\text{Card}(\mathcal{B}_n(X)) \leq \text{Card}(A)^n$ , we have  $h(X) \leq$   
 184  $\log \text{Card}(A)$ . If  $X$  is nonempty, then  $0 \leq h(X)$ .

185 The following statement shows that the entropy is invariant under conjugacy (see [19]  
 186 Corollary 4.1.10).

187 **Theorem 2.7.** *If  $X, Y$  are conjugate shift spaces, then  $h(X) = h(Y)$ .*

188 **Example 2.4.** Let  $X$  be the golden mean shift of Example 2.1. Then a block of length  
 189  $n + 1$  is either a block of length  $n - 1$  followed by  $ab$  or a block of length  $n$  followed by  
 190  $a$ . Thus  $s_{n+1} = s_n + s_{n-1}$ . As a classical result,  $h(X) = \log \lambda$  where  $\lambda = (1 + \sqrt{5})/2$   
 191 is the golden mean.

192 An element  $x$  of a shift space  $X$  over the alphabet  $A$  has *period*  $n$  if  $\sigma_A^n(x) = x$ . If  
 193  $\varphi : X \rightarrow Y$  is a conjugacy, then an element  $x$  of  $X$  has period  $n$  if and only if  $\varphi(x)$  has  
 194 period  $n$ .

The *zeta function* of a shift space  $X$  is the power series

$$\zeta_X(z) = \exp \sum_{n \geq 0} \frac{p_n}{n} z^n,$$

195 where  $p_n$  is the number of elements  $x$  of  $X$  of period  $n$ .

196 It follows from the definition that the sequence  $(p_n)_{n \in \mathbb{N}}$  is invariant under conjugacy,  
 197 and thus the zeta function of a shift space is invariant under conjugacy.

198 Several other conjugacy invariants are known. One of them is the Bowen-Franks group  
 199 of a matrix which defines an invariant of the associated shift space. This will be defined  
 200 below.

201 **Example 2.5.** Let  $X = A^{\mathbb{Z}}$ . Then  $\zeta_X(z) = \frac{1}{1-kz}$ , where  $k = \text{Card}(A)$ . Indeed, one has  
 202  $p_n = k^n$ , since an element  $x$  of  $A^{\mathbb{Z}}$  has period  $n$  if and only if it is a biinfinite repetition  
 203 of a word of length  $n$  over  $A$ .

204 **State splitting.** Let  $G = (Q, \mathcal{E})$  and  $H = (R, \mathcal{F})$  be graphs. A pair  $(h, k)$  of surjective  
 205 maps  $k : R \rightarrow Q$  and  $h : \mathcal{F} \rightarrow \mathcal{E}$  is called a *graph morphism* from  $H$  onto  $G$  if the two  
 206 diagrams in Figure 1 are commutative.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{h} & \mathcal{E} \\ \downarrow i & & \downarrow i \\ R & \xrightarrow{k} & Q \end{array} \qquad \begin{array}{ccc} \mathcal{F} & \xrightarrow{h} & \mathcal{E} \\ \downarrow t & & \downarrow t \\ R & \xrightarrow{k} & Q \end{array}$$

**Figure 1.** Graph morphism.

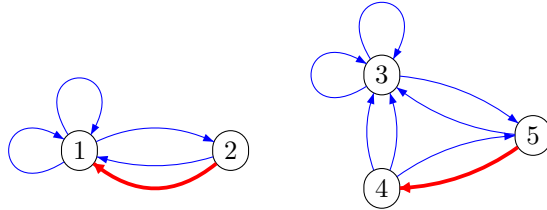
207 A graph morphism  $(h, k)$  from  $H$  onto  $G$  is an *in-merge* from  $H$  onto  $G$  if for each  
 208  $p, q \in Q$  there is a partition  $(\mathcal{E}_p^q(t))_{t \in k^{-1}(q)}$  of the set  $\mathcal{E}_p^q$  such that for each  $r \in k^{-1}(p)$   
 209 and  $t \in k^{-1}(q)$ , the map  $h$  is a bijection from  $\mathcal{F}_r^t$  onto  $\mathcal{E}_p^q(t)$ . If this holds, then  $G$  is  
 210 called an *in-merge* of  $H$ , and  $H$  is an *in-split* of  $G$ .<sup>2</sup>

211 Thus an in-split  $H$  is obtained from a graph  $G$  as follows: each state  $q \in Q$  is split  
 212 into copies which are the states of  $H$  in the set  $k^{-1}(q)$ . Each of these states  $t$  receives a  
 213 copy of  $\mathcal{E}_p^q(t)$  starting in  $r$  and ending in  $t$  for each  $r$  in  $k^{-1}(p)$ .

214 Each  $r$  in  $k^{-1}(p)$  has the same number of edges going out of  $r$  and coming in  $s$ , for  
 215 any  $s \in R$ .

216 Moreover, for any  $p, q \in Q$  and  $e \in \mathcal{E}_p^q$ , all edges in  $h^{-1}(e)$  have the same terminal  
 217 vertex, namely the state  $t$  such that  $e \in \mathcal{E}_p^q(t)$ .

218 **Example 2.6.** Let  $G$  and  $H$  be the graphs represented on Figure 2. Here  $Q = \{1, 2\}$  and  
 $R = \{3, 4, 5\}$ . The graph  $H$  is an in-split of the graph  $G$ . The graph morphism  $(h, k)$



**Figure 2.** An in-split from  $G$  (on the left) onto  $H$  (on the right).

219 is defined by  $k(3) = k(4) = 1$  and  $k(5) = 2$ . Thus the state 1 of  $G$  is split into two  
 220 states 3 and 4 of  $H$ , and the map  $h$  is associated to the partition obtained as follows: the  
 221 edges from 2 to 1 are partitioned into two classes, indexed by 3 and 4 respectively, and  
 222 containing each one edge from 2 to 1.  
 223

224 The following result is well-known (see [19]). It shows that if  $H$  is an in-split of a  
 225 graph  $G$ , then  $X_G$  and  $X_H$  are conjugate.

226 **Proposition 2.8** ([19, Theorem 2.4.1]). *If  $(h, k)$  is an in-merge of a graph  $H$  onto a graph  
 227  $G$ , then  $h_\infty$  is a 1-block conjugacy from  $X_H$  onto  $X_G$  and its inverse is 2-block.*

<sup>2</sup>In this chapter, a *partition* of a set  $X$  is a family  $(X_i)_{i \in I}$  of pairwise disjoint, possibly empty subsets of  $X$ , indexed by a set  $I$ , such that  $X$  is the union of the sets  $X_i$  for  $i \in I$ .

228 The map  $h_\infty$  from  $X_H$  to  $X_G$  is called an *edge in-merging map* and its inverse an  
 229 *edge in-splitting map*.

230 A *column division matrix* over two sets  $R, Q$  is an  $R \times Q$ -matrix  $D$  with elements in  
 231  $\{0, 1\}$  such that each column has at least one 1 and each row has exactly one 1. Thus, the  
 232 columns of such a matrix represent a partition of  $R$  into  $\text{Card}(Q)$  sets.

233 The following result is Theorem 2.4.14 of [19].

**Proposition 2.9.** *Let  $G$  and  $H$  be essential graphs. The graph  $H$  is an in-split of the graph  $G$  if and only if there is an  $R \times Q$ -column division matrix  $D$  and a  $Q \times R$ -matrix  $E$  with nonnegative integer entries such that*

$$M(G) = ED, \quad M(H) = DE. \quad (2.1)$$

**Example 2.7.** For the graphs  $G, H$  of Example 2.6, one has  $M(G) = DE$  and  $M(H) = ED$  with

$$E = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

234 Observe that a particular case of a column division matrix is a permutation matrix.  
 235 The corresponding in-split (or merge) is a renaming of the states of a graph.

236 The notion of an *out-merge* is defined symmetrically. A graph morphism  $(h, k)$  from  
 237  $H$  onto  $G$  is an *out-merge* from  $H$  onto  $G$  if for each  $p, q \in Q$  there is a partition  
 238  $(\mathcal{E}_p^q(r))_{r \in k^{-1}(p)}$  of the set  $\mathcal{E}_p^q$  such that for each  $r \in k^{-1}(p)$  and  $t \in k^{-1}(q)$ , the map  
 239  $h$  is a bijection from  $\mathcal{F}_r^t$  onto  $\mathcal{E}_p^q(r)$ . If this holds, then  $G$  is called an *out-merge* of  $H$ , and  
 240  $H$  is an *out-split* of  $G$ .

241 Proposition 2.8 also has a symmetrical version. Thus if  $(h, k)$  is an out-merge from  
 242  $G$  onto  $H$ , then  $h_\infty$  is a 1-block conjugacy from  $X_H$  onto  $X_G$  whose inverse is 2-block.  
 243 The conjugacy  $h_\infty$  is called an *edge out-merging map* and its inverse an *edge out-splitting*  
 244 *map*.

245 Symmetrically, a *row division matrix* is a matrix with elements in the set  $\{0, 1\}$  such  
 246 that each column has at least one 1 and each row has exactly one 1.

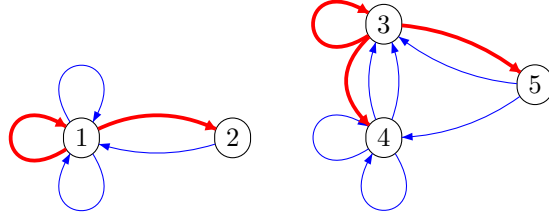
247 The following statement is symmetrical to Proposition 2.9.

**Proposition 2.10.** *Let  $G$  and  $H$  be essential graphs. The graph  $H$  is an out-split of the graph  $G$  if and only if there is a row division matrix  $D$  and a matrix  $E$  with nonnegative integer entries such that*

$$M(G) = DE, \quad M(H) = ED. \quad (2.2)$$

**Example 2.8.** Let  $G$  and  $H$  be the graphs represented on Figure 3. Here  $Q = \{1, 2\}$  and  
 $R = \{3, 4, 5\}$ . The graph  $H$  is an out-split of the graph  $G$ . The graph morphism  $(h, k)$   
 is defined by  $k(3) = k(4) = 1$  and  $k(5) = 2$ . The map  $h$  is associated with the partition  
 indicated by the colors. One has  $M(G) = ED$  and  $M(H) = DE$  with

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}.$$



**Figure 3.** The graphs  $G$  and  $H$ .

248 We use the term *split* to mean either an in-split or an out-split. The same convention  
249 holds for a *merge*.

250 **Proposition 2.11.** For  $n \geq 2$ , the graph  $G^{[n-1]}$  is an in-merge of the graph  $G^{[n]}$ .

251 *Proof.* Consider for  $n \geq 2$  the equivalence on the states of  $G^{[n]}$  which relates two paths  
252 of length  $n - 1$  which differ only by the first edge. It is clear that this equivalence is  
253 such that two equivalent elements have the same output. Thus  $G^{[n-1]}$  is an in-merge of  
254  $G^{[n]}$ .  $\square$

255 **The Decomposition Theorem.** The following result is known as the *Decomposition*  
256 *Theorem* (Theorem 7.1.2 in [19]).

257 **Theorem 2.12.** Every conjugacy from an edge shift onto another is the composition of a  
258 sequence of edge splitting maps followed by a sequence of edge merging maps.

259 The statement of Theorem 2.12 given in [19] is less precise, since it does not specify  
260 the order of splitting and merging maps.

261 The proof relies on the following statement (Lemma 7.1.3 in [19]).

262 **Lemma 2.13.** Let  $G, H$  be graphs and let  $\varphi : X_G \rightarrow X_H$  be a 1-block conjugacy  
263 whose inverse has memory  $m \geq 1$  and anticipation  $n \geq 0$ . There are in-splittings  $\overline{G}, \overline{H}$   
264 of the graphs  $G, H$  and a 1-block conjugacy with memory  $m - 1$  and anticipation  $n$   
 $\overline{\varphi} : X_{\overline{G}} \rightarrow X_{\overline{H}}$  such that the following diagram commutes.

$$\begin{array}{ccc}
 X_G & \longrightarrow & X_{\overline{G}} \\
 \downarrow \varphi & & \downarrow \overline{\varphi} \\
 X_H & \longrightarrow & X_{\overline{H}}
 \end{array}$$

265

266 The horizontal edges in the above diagram represent the edge in-splitting maps from  
267  $X_G$  to  $X_{\overline{G}}$  and from  $X_H$  to  $X_{\overline{H}}$  respectively.

**The Classification Theorem.** Two nonnegative integral square matrices  $M, N$  are *elementary equivalent* if there exists a pair  $R, S$  of nonnegative integral matrices such that

$$M = RS, \quad N = SR.$$

268 Thus if a graph  $H$  is a split of a graph  $G$ , then, by Proposition 2.9, the matrices  $M(G)$   
 269 and  $M(H)$  are elementary equivalent. The matrices  $M$  and  $N$  are *strong shift equivalent*  
 270 if there is a sequence  $(M_0, M_1, \dots, M_n)$  of nonnegative integral matrices such that  $M_i$   
 271 and  $M_{i+1}$  are elementary equivalent for  $0 \leq i < n$  with  $M_0 = M$  and  $M_n = N$ .

272 The following theorem is Williams' Classification Theorem (Theorem 7.2.7 in [19]).

273 **Theorem 2.14.** *Let  $G$  and  $H$  be two graphs. The edge shifts  $X_G$  and  $X_H$  are conjugate*  
 274 *if and only if the matrices  $M(G)$  and  $M(H)$  are strong shift equivalent.*

275 Note that one direction of this theorem is contained in the Decomposition Theorem.  
 276 Indeed, if  $X_G$  and  $X_H$  are conjugate, there is a sequence of edge splitting and edge  
 277 merging maps from  $X_G$  to  $X_H$ . And if  $G$  is a split or a merge of  $H$ , then  $M(G)$  and  
 278  $M(H)$  are elementary equivalent, whence the result in one direction follows. Note also  
 279 that, in spite of the easy definition of strong shift equivalence, it is not even known whether  
 280 there exists a decision procedure for determining when two nonnegative integral matrices  
 281 are strong shift equivalent.

## 282 2.3 Flow equivalence

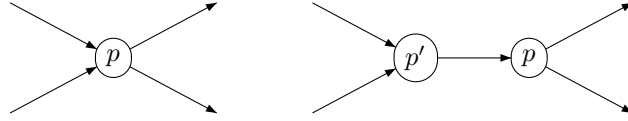
283 In this section, we give basic definitions and properties concerning flow equivalence of  
 284 shift spaces. The notion comes from the notion of equivalence of continuous flows, see  
 285 Section 13.6 of [19]. A characterization of flow equivalence for shift spaces (which we  
 286 will take below as our definition of flow equivalence for shift spaces) is due to Parry and  
 287 Sullivan [23]. It is noticeable that the flow equivalence of irreducible shifts of finite type  
 288 has an effective characterization, by Franks' Theorem (Theorem 2.16).

289 Let  $A$  be an alphabet and  $a$  be a letter in  $A$ . Let  $\omega$  be a letter which does not belong  
 290 to  $A$ . Set  $B = A \cup \omega$ . The *symbol expansion* of a set  $W \subset A^+$  relative to  $a$  is the image  
 291 of  $W$  by the semigroup morphism  $\varphi : A^+ \rightarrow B^+$  such that  $\varphi(a) = a\omega$  and  $\varphi(b) = b$   
 292 for all  $b \in A \setminus a$ . Recall that a *semigroup morphism*  $f : A^+ \rightarrow B^+$  is a map satisfying  
 293  $f(xy) = f(x)f(y)$  for all words  $x, y$ . It should not be confused with the morphisms of  
 294 shift spaces defined earlier. The semigroup morphism  $\varphi$  is also called a symbol expansion.  
 295 Let  $X$  be a shift space on the alphabet  $A$ . The *symbol expansion* of  $X$  relative to  $a$  is the  
 296 least shift space  $X'$  on the alphabet  $B = A \cup \omega$  which contains the symbol expansion of  
 297  $\mathcal{B}(X)$ . Note that if  $\varphi$  is a symbol expansion, it defines a bijection from  $\mathcal{B}(X)$  onto  $\mathcal{B}(X')$ .  
 298 The inverse of a symbol expansion is called a *symbol contraction*.

299 Two shift spaces  $X, Y$  are said to be *flow equivalent* if there is a sequence  $X_0, \dots, X_n$   
 300 of shift spaces such that  $X_0 = X$ ,  $X_n = Y$  and for  $0 \leq i \leq n - 1$ , either  $X_{i+1}$  is the  
 301 image of  $X_i$  by a conjugacy, a symbol expansion or a symbol contraction.

302 **Example 2.9.** Let  $A = \{a, b\}$ . The symbol expansion of the full shift  $A^{\mathbb{Z}}$  relative to  $b$   
 303 is conjugate to the golden mean shift. Thus the full shift on two symbols and the golden  
 304 mean shift are flow equivalent.

For edge shifts, symbol expansion can be replaced by another operation. Let  $G$  be a graph and let  $p$  be a vertex of  $G$ . The *graph expansion* of  $G$  relative to  $p$  is the graph  $G'$  obtained by replacing  $p$  by an edge from a new vertex  $p'$  to  $p$  and replacing all edges coming in  $p$  by edges coming in  $p'$  (see Figure 4). The inverse of a graph expansion is called a *graph contraction*. Note that graph expansion (relative to vertex 1) changes the



**Figure 4.** Graph expansion

adjacency matrix of a graph as indicated below.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & a_{11} & a_{12} & \dots & a_{1n} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & & \\ 0 & a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

305 **Proposition 2.15.** *The flow equivalence relation on edge shifts is generated by conjugacies and graph expansions.*  
306

307 *Proof.* Let  $G = (Q, E)$  be a graph and let  $p$  be a vertex of  $G$ . The graph expansion  
308 of  $G$  relative to  $p$  can be obtained by a symbol expansion of each of the edges coming  
309 into  $p$  followed by a conjugacy which merges all the new symbols into one new symbol.  
310 Conversely, let  $e$  be an edge of  $G$ . The symbol expansion of  $X_G$  relative to  $e$  can be  
311 obtained by an input split which makes  $e$  the only edge going into its end vertex  $q$  followed  
312 by a graph expansion relative to  $q$ .  $\square$

The *Bowen-Franks group* of a square  $n \times n$ -matrix  $M$  with integer elements is the Abelian group

$$BF(M) = \mathbb{Z}^n / \mathbb{Z}^n(I - M)$$

313 where  $\mathbb{Z}^n(I - M)$  is the image of  $\mathbb{Z}^n$  under the matrix  $I - M$  acting on the right. In other  
314 terms,  $\mathbb{Z}^n(I - M)$  is the Abelian group generated by the rows of the matrix  $I - M$ . This  
315 notion is due to Bowen and Franks [5], who have shown that it is an invariant for flow  
316 equivalence.

317 The following result is due to Franks [14]. We say that a graph is *trivial* if it is reduced  
318 to one cycle.

319 **Theorem 2.16.** *Let  $G, G'$  be two strongly connected nontrivial graphs and let  $M, M'$   
320 be their adjacency matrices. The edge shifts  $X_G, X_{G'}$  are flow equivalent if and only if  
321  $\det(I - M) = \det(I - M')$  and the groups  $BF(M), BF(M')$  are isomorphic.*

322 In the case trivial graphs, the theorem is false. Indeed, any two edge shifts on strongly  
 323 connected trivial graphs are flow equivalent and are not flow equivalent to any edge shift  
 324 on a nontrivial irreducible graph. For any trivial graph  $G$  with adjacency matrix  $M$ , one  
 325 has  $\det(I - M) = 0$  and  $BF(M) \sim \mathbb{Z}$ . However there are nontrivial strongly connected  
 326 graphs such that  $\det(I - M) = 0$  and  $BF(M) \sim \mathbb{Z}$ .

327 The case of arbitrary shifts of finite type has been solved by Huang (see [6, 8]). A  
 328 similar characterization for sofic shifts is not known (see [7]).

**Example 2.10.** Let

$$M = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}, \quad M' = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}.$$

329 One has  $\det(I - M) = \det(I - M') = -4$ . Moreover  $BF(M) \sim \mathbb{Z}/4\mathbb{Z}$ . Indeed, the  
 330 rows of the matrix  $I - M$  are  $[-3 \ -1]$  and  $[-1 \ 1]$ . They generate the same group  
 331 as  $[4 \ 0]$  and  $[-1 \ 1]$ . Thus  $BF(M) \sim \mathbb{Z}/4\mathbb{Z}$ . In the same way,  $BF(M') \sim \mathbb{Z}/4\mathbb{Z}$ .  
 332 Thus, according to Theorem 2.16, the edge shifts  $X_G$  and  $X_{G'}$  are flow equivalent.

333 Actually  $X_G$  and  $X_{G'}$  are both flow equivalent to the full shift on 5 symbols.

### 3 Automata

334  
 335 In this section, we start with the definition and notation for automata recognizing shifts,  
 336 and we show that sofic shifts are precisely the shifts recognized by finite automata (Propo-  
 337 sition 3.3).

338 We introduce the notion of labeled conjugacy; it is a conjugacy preserving the label-  
 339 ing. We extend the Decomposition Theorem and the Classification Theorem to labeled  
 340 conjugacies (Theorems 3.8 and 3.9).

#### 3.1 Automata and sofic shifts

341  
 342 The automata considered in this section are finite automata. We do not mention the initial  
 343 and final states in the notation when all states are both initial and final. Thus, an automaton  
 344 is denoted by  $\mathcal{A} = (Q, E)$  where  $Q$  is the finite set of *states* and  $E \subset Q \times A \times Q$  is the set  
 345 of *edges*. The edge  $(p, a, q)$  has initial state  $p$ , label  $a$  and terminal state  $q$ . The underlying  
 346 graph of  $\mathcal{A}$  is the same as  $\mathcal{A}$  except that the labels of the edges are not used.

347 An automaton is *essential* if its underlying graph is essential. The *essential part* of an  
 348 automaton is its restriction to the essential part of its underlying graph.

349 We denote by  $X_{\mathcal{A}}$  the set of biinfinite paths in  $\mathcal{A}$ . It is the edge shift of the underlying  
 350 graph of  $\mathcal{A}$ . Note that since the automaton is supposed finite, the shift space  $X_{\mathcal{A}}$  is on  
 351 a finite alphabet, as required for a shift space. We denote by  $L_{\mathcal{A}}$  the set of labels of  
 352 biinfinite paths in  $\mathcal{A}$ . We denote by  $\lambda_{\mathcal{A}}$  the 1-block map from  $X_{\mathcal{A}}$  into the full shift  $A^{\mathbb{Z}}$   
 353 which assigns to a path its label. Thus  $L_{\mathcal{A}} = \lambda_{\mathcal{A}}(X_{\mathcal{A}})$ . If this holds, we say that  $X_{\mathcal{A}}$   
 354 is the shift space *recognized* by  $\mathcal{A}$ .

355 The following propositions describe how this notion of recognition is related to that for  
 356 finite words. In the context of finite words, we denote by  $\mathcal{A} = (Q, I, E, T)$  an automaton

357 with distinguished subsets  $I$  (resp.  $T$ ) of initial (resp. terminal) states. A word  $w$  is  
 358 *recognized* by  $\mathcal{A}$  if there is a path from a state in  $I$  to a state in  $T$  labeled  $w$ . Recall that a  
 359 set is recognizable if it is the set of words recognized by a finite automaton. An automaton  
 360  $\mathcal{A} = (Q, I, T)$  is *trim* if, for every state  $p$  in  $Q$ , there is a path from a state in  $I$  to  $p$  and a  
 361 path from  $p$  to a state in  $T$ .

362 **Proposition 3.1.** *Let  $W \subset A^*$  be a recognizable set and let  $\mathcal{A} = (Q, I, T)$  be a trim*  
 363 *finite automaton recognizing the set  $A^* \setminus A^*WA^*$ . Then  $L_{\mathcal{A}} = X^{(W)}$ .*

364 *Proof.* The label of a biinfinite path in the automaton  $\mathcal{A}$  does not contain a factor  $w$  in  
 365  $W$ . Otherwise, there is a finite path  $p \xrightarrow{w} q$  which is a segment of this infinite path. The  
 366 path  $p \xrightarrow{w} q$  can be extended to a path  $i \xrightarrow{u} p \xrightarrow{w} q \xrightarrow{v} t$  for some  $i \in I, t \in T$ , and  $uvw$   
 367 is accepted by  $\mathcal{A}$ , which is a contradiction.

368 Next, consider a biinfinite word  $x = (x_i)_{i \in \mathbb{Z}}$  in  $X^{(W)}$ . For every  $n \geq 0$ , there is a  
 369 path  $\pi_n$  in the automaton  $\mathcal{A}$  labeled  $w_n = x_{-n} \cdots x_0 \cdots x_n$  because the word  $w_n$  has no  
 370 factor in  $W$ . By compactness (König's lemma) there is an infinite path in  $\mathcal{A}$  labeled  $x$ .  
 371 Thus  $x$  is in  $L_{\mathcal{A}}$ .  $\square$

372 The following proposition states in some sense the converse.

373 **Proposition 3.2.** *Let  $X$  be a sofic shift over  $A$ , and let  $\mathcal{A} = (Q, I, T)$  be a trim finite*  
 374 *automaton recognizing the set  $\mathcal{B}(X)$  of blocks of  $X$ . Then  $L_{\mathcal{A}} = X$ .*

375 *Proof.* Set  $W = A^* \setminus \mathcal{B}(X)$ . Then one easily checks that  $X = X^{(W)}$ . Next,  $\mathcal{A}$  recognizes  
 376  $A^* \setminus A^*WA^*$ . By Proposition 3.1, one has  $L_{\mathcal{A}} = X$ .  $\square$

377 **Proposition 3.3.** *A shift  $X$  over  $A$  is sofic if and only if there is a finite automaton  $\mathcal{A}$  such*  
 378 *that  $X = L_{\mathcal{A}}$ .*

379 *Proof.* The forward implication results from Proposition 3.1. Conversely, assume that  
 380  $X = L_{\mathcal{A}}$  for some finite automaton  $\mathcal{A}$ . Let  $W$  be the set of finite words which are not  
 381 labels of paths in  $\mathcal{A}$ . Clearly  $X \subset X^{(W)}$ . Conversely, if  $x \in X^{(W)}$ , then all its factors  
 382 are labels of paths in  $\mathcal{A}$ . Again by compactness,  $x$  itself is the label of a biinfinite path  
 383 in  $\mathcal{A}$ .  $\square$

384 **Example 3.1.** The golden mean shift of Example 2.1 is recognized by the automaton of  
 385 Figure 5 on the left while the even shift of Example 2.2 is recognized by the automaton  
 of Figure 5 on the right.



**Figure 5.** Automata recognizing the golden mean and the even shift

The *adjacency matrix* of the automaton  $\mathcal{A} = (Q, E)$  is the  $Q \times Q$ -matrix  $M(\mathcal{A})$  with elements in  $\mathbb{N}\langle A \rangle$  defined by

$$(M(\mathcal{A}))_{pq, a} = \begin{cases} 1 & \text{if } (p, a, q) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

387 We write  $M$  for  $M(\mathcal{A})$  when the automaton is understood. The entries in the matrix  $M^n$ ,  
 388 for  $n \geq 0$ , have an easy combinatorial interpretation: for each word  $w$  of length  $n$ , the  
 389 coefficient  $(M^n)_{p,q, w}$  is the number of distinct paths from  $p$  to  $q$  carrying the label  $w$ .

390 A matrix  $M$  is called *alphabetic* over the alphabet  $A$  if its elements are homogeneous  
 391 polynomials of degree 1 over  $A$  with nonnegative coefficients. Adjacency matrices are  
 392 special cases of alphabetic matrices. Indeed, its elements are homogeneous polynomials  
 393 of degree 1 with coefficients 0 or 1.

### 3.2 Labeled conjugacy

394  
 395 Let  $\mathcal{A}$  and  $\mathcal{B}$  be two automata on the alphabet  $A$ . A *labeled conjugacy* from  $X_{\mathcal{A}}$  onto  
 396  $X_{\mathcal{B}}$  is a conjugacy  $\varphi$  such that  $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}}\varphi$ , that is such that the following diagram is  
 commutative. We say that  $\mathcal{A}$  and  $\mathcal{B}$  are *conjugate* if there exists a labeled conjugacy

$$\begin{array}{ccc} X_{\mathcal{A}} & \xrightarrow{\varphi} & X_{\mathcal{B}} \\ & \searrow \lambda_{\mathcal{A}} & \swarrow \lambda_{\mathcal{B}} \\ & & A^{\mathbb{Z}} \end{array}$$

397  
 398 from  $X_{\mathcal{A}}$  to  $X_{\mathcal{B}}$ . The aim of this paragraph is to give two characterizations of labeled  
 399 conjugacy.

400 **Labeled split and merge.** Let  $\mathcal{A} = (Q, E)$  and  $\mathcal{B} = (R, F)$  be two automata. Let  $G, H$   
 401 be the underlying graphs of  $\mathcal{A}$  and  $\mathcal{B}$  respectively.

402 A *labeled in-merge* from  $\mathcal{B}$  onto  $\mathcal{A}$  is an in-merge  $(h, k)$  from  $H$  onto  $G$  such that for  
 403 each  $f \in F$  the labels of  $f$  and  $h(f)$  are equal. We say that  $\mathcal{B}$  is a *labeled in-split* of  $\mathcal{A}$ ,  
 404 or that  $\mathcal{A}$  is a *labeled in-merge* of  $\mathcal{B}$ .

405 The following statement is the analogue of Proposition 2.8 for automata.

406 **Proposition 3.4.** *If  $(h, k)$  is a labeled in-merge from the automaton  $\mathcal{B}$  onto the automaton*  
 407  *$\mathcal{A}$ , then the map  $h_{\infty}$  is a labeled conjugacy from  $X_{\mathcal{B}}$  onto  $X_{\mathcal{A}}$ .*

408 *Proof.* Let  $(h, k)$  be a labeled in-merge from  $\mathcal{B}$  onto  $\mathcal{A}$ . By Proposition 2.8, the map  $h_{\infty}$   
 409 is a 1-block conjugacy from  $X_{\mathcal{B}}$  onto  $X_{\mathcal{A}}$ . Since the labels of  $f$  and  $h(f)$  are equal for  
 410 each edge  $f$  of  $\mathcal{B}$ , this map is a labeled conjugacy.  $\square$

411 The next statement is the analogue of Proposition 2.9 for automata.

**Proposition 3.5.** *An automaton  $\mathcal{B} = (R, F)$  is a labeled in-split of the automaton  $\mathcal{A} = (Q, E)$  if and only if there is an  $R \times Q$ -column division matrix  $D$  and an alphabetic  $Q \times R$ -matrix  $N$  such that*

$$M(\mathcal{A}) = ND, \quad M(\mathcal{B}) = DN. \quad (3.1)$$

*Proof.* Suppose first that  $D$  and  $N$  are as described in the statement, and define a map  $k : R \rightarrow Q$  by  $k(r) = q$  if  $D_{rq} = 1$ . We define  $h : F \rightarrow E$  as follows. Consider an edge  $(r, a, s) \in F$ . Set  $p = k(r)$  and  $q = k(s)$ . Since  $M(\mathcal{B}) = DN$ , we have  $(N_{ps}, a) = 1$ . Since  $M(\mathcal{A}) = ND$ , this implies that  $(M(\mathcal{A})_{pq}, a) = 1$  or, equivalently, that  $(p, a, q) \in E$ . We set  $h(r, a, s) = (p, a, q)$ . Then  $(h, k)$  is a labeled in-merge. Indeed  $h$  is associated with the partitions defined by

$$E_p^q(t) = \{(p, a, q) \in E \mid (N_{pt}, a) = 1 \text{ and } k(t) = q\}.$$

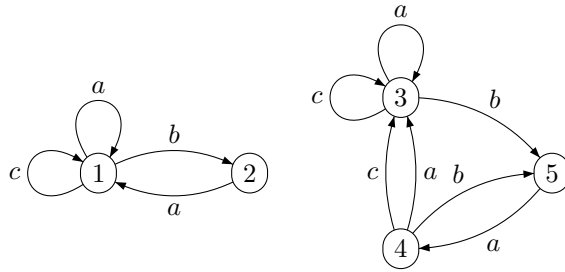
Suppose conversely that  $(h, k)$  is a labeled in-merge from  $\mathcal{B}$  onto  $\mathcal{A}$ . Let  $D$  be the  $R \times Q$ -column division matrix defined by

$$D_{rq} = \begin{cases} 1 & \text{if } k(r) = q \\ 0 & \text{otherwise} \end{cases}$$

For  $p \in Q$  and  $t \in R$ , we define  $N_{rt}$  as follows. Set  $q = k(t)$ . By definition of an in-merge, there is a partition  $(E_p^q(t))_{t \in k^{-1}(q)}$  of  $E_p^q$  such that  $h$  is a bijection from  $F_r^t$  onto  $E_p^q(t)$ . For  $a \in A$ , set

$$(N_{pt}, a) = \begin{cases} 1 & \text{if } (p, a, q) \in E_p^q(t) \\ 0 & \text{otherwise} \end{cases}$$

412 Then  $M(\mathcal{A}) = ND$  and  $M(\mathcal{B}) = DN$ . □



**Figure 6.** An in-split from  $\mathcal{A}$  to  $\mathcal{B}$ .

**Example 3.2.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be the automata represented on Figure 6. Here  $Q = \{1, 2\}$  and  $R = \{3, 4, 5\}$ . One has  $M(\mathcal{A}) = ND$  and  $M(\mathcal{B}) = DN$  with

$$N = \begin{bmatrix} a+c & 0 & b \\ 0 & a & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

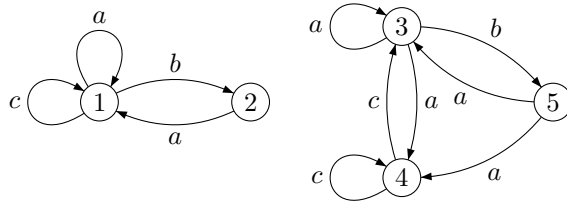
413 A *labeled out-merge* from  $\mathcal{B}$  onto  $\mathcal{A}$  is an out-merge  $(h, k)$  from  $H$  onto  $G$  such that  
 414 for each  $f \in F$  the labels of  $f$  and  $h(f)$  are equal.

415 We say that  $\mathcal{B}$  is a *labeled out-split* of  $\mathcal{A}$ , or that  $\mathcal{A}$  is a *labeled in-merge* of  $\mathcal{B}$ .

416 Thus if  $\mathcal{B}$  is a labeled out-split of  $\mathcal{A}$ , there is a labeled conjugacy from  $X_{\mathcal{B}}$  onto  $X_{\mathcal{A}}$ .

**Proposition 3.6.** *The automaton  $\mathcal{B} = (R, F)$  is a labeled out-split of the automaton  $\mathcal{A} = (Q, E)$  if and only if there is a  $Q \times R$ -row division matrix  $D$  and an alphabetic  $R \times Q$ -matrix  $N$  such that*

$$M(\mathcal{A}) = DN, \quad M(\mathcal{B}) = ND. \quad (3.2)$$



**Figure 7.** An out-split from  $\mathcal{A}$  to  $\mathcal{B}$ .

**Example 3.3.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be the automata represented on Figure 7. Here  $Q = \{1, 2\}$  and  $R = \{3, 4, 5\}$ . One has  $M(\mathcal{A}) = ND$  and  $M(\mathcal{B}) = DN$  with

$$N = \begin{bmatrix} a & b \\ c & 0 \\ a & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let  $\mathcal{A} = (Q, E)$  be an automaton. For a pair of integers  $m, n \geq 0$ , denote by  $\mathcal{A}^{[m,n]}$  the following automaton called the  $(m, n)$ -th *extension* of  $\mathcal{A}$ . The underlying graph of  $\mathcal{A}^{[m,n]}$  is the higher edge graph  $G^{[k]}$  for  $k = m + n + 1$ . The label of an edge

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_m} p_m \xrightarrow{a_{m+1}} p_{m+1} \xrightarrow{a_{m+2}} \cdots \xrightarrow{a_{m+n}} p_{m+n} \xrightarrow{a_{m+n+1}} p_{m+n+1}$$

417 is the letter  $a_{m+1}$ . Observe that  $\mathcal{A}^{[0,0]} = \mathcal{A}$ . By this construction, each graph  $G^{[k]}$   
 418 produces  $k$  extensions according to the choice of the labeling.

419 **Proposition 3.7.** *For  $m \geq 1, n \geq 0$ , the automaton  $\mathcal{A}^{[m-1,n]}$  is a labeled in-merge of the*  
 420 *automaton  $\mathcal{A}^{[m,n]}$  and for  $m \geq 0, n \geq 1$ , the automaton  $\mathcal{A}^{[m,n-1]}$  is a labeled out-merge*  
 421 *of the automaton  $\mathcal{A}^{[m,n]}$ .*

422 *Proof.* Suppose that  $m \geq 1, n \geq 0$ . Let  $k$  be the map from the paths of length  $m + n$   
 423 in  $\mathcal{A}$  onto the paths of length  $m + n - 1$  which erases the first edge of the path. Let  
 424  $h$  be the map from the set of edges of  $\mathcal{A}^{[m,n]}$  to the set of edges of  $\mathcal{A}^{[m-1,n]}$  defined  
 425 by  $h(\pi, a, \rho) = (k(\pi), a, k(\rho))$ . Then  $(h, k)$  is a labeled in-merge from  $\mathcal{A}^{[m,n]}$  onto

426  $\mathcal{A}^{[m-1,n]}$ . The proof that, for  $m \geq 0, n \geq 1$ , the automaton  $\mathcal{A}^{[m,n-1]}$  is an out-merge of  
 427 the automaton  $\mathcal{A}^{[m,n]}$  is symmetrical.  $\square$

428 The following result is the analogue, for automata, of the Decomposition Theorem.

429 **Theorem 3.8.** *Every conjugacy of automata is a composition of labeled splits and merges.*

*Proof.* Let  $\mathcal{A}$  and  $\mathcal{B}$  be two conjugate automata. Let  $\varphi$  be a labeled conjugacy from  $\mathcal{A}$  onto  $\mathcal{B}$ . Let  $G_0$  and  $H_0$  be the underlying graphs of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. By the Decomposition Theorem 2.12, there are sequences  $(G_1, \dots, G_n)$  and  $(H_1, \dots, H_m)$  of graphs with  $G_n = H_m$  and such that  $G_{i+1}$  is a split of  $G_i$  for  $0 \leq i < n$  and  $H_{j+1}$  is a split of  $H_j$  for  $0 \leq j < m$ . Moreover,  $\varphi$  is the composition of the sequence of edge splitting maps from  $G_i$  onto  $G_{i+1}$  followed by the sequence of edge merging maps from  $H_{j+1}$  onto  $H_j$ . Let  $(h_i, k_i)$ , for  $1 \leq i \leq n$ , be a merge from  $G_i$  onto  $G_{i-1}$  and  $(u_j, v_j)$ , for  $1 \leq j \leq m$  be a merge from  $H_j$  onto  $H_{j-1}$ . Then we may define labels on the edges of  $G_1, \dots, G_n$  in such a way that  $G_i$  becomes the underlying graph of an automaton  $\mathcal{A}_i$  and  $(h_i, k_i)$  is a labeled merge from  $\mathcal{A}_i$  onto  $\mathcal{A}_{i-1}$ . In the same way, we may define labels on the edges of  $H_j$  in such a way that  $H_j$  becomes the underlying graph of an automaton  $\mathcal{B}_j$  and  $(u_j, v_j)$  is a labeled merge from  $\mathcal{B}_j$  onto  $\mathcal{B}_{j-1}$ .

$$G_0 \xleftarrow{(h_1, k_1)} G_1 \cdots \xleftarrow{(h_n, k_n)} G_n = H_m \xrightarrow{(u_m, v_m)} \cdots H_1 \xrightarrow{(u_1, v_1)} H_0.$$

430 Let  $h = h_1 \cdots h_n$  and  $u = u_1 u_2 \cdots u_m$ . Since  $\varphi = u_\infty h_\infty^{-1}$ , and  $\varphi$  is a labeled conjugacy,  
 431 we have  $\lambda_{\mathcal{A}} h_\infty = \lambda_{\mathcal{B}} u_\infty$ . This shows that the automata  $\mathcal{A}_n$  and  $\mathcal{B}_m$  are equal. Thus there  
 432 is a sequence of labeled splitting maps followed by a sequence of labeled merging maps  
 433 which is equal to  $\varphi$ .  $\square$

Let  $M$  and  $M'$  be two alphabetic square matrices over the same alphabet  $A$ . We say that  $M$  and  $M'$  are *elementary equivalent* if there exists a nonnegative integral matrix  $D$  and an alphabetic matrix  $N$  such that

$$M = DN, \quad M' = ND \quad \text{or vice-versa.}$$

434 By Proposition 3.5, if  $\mathcal{B}$  is an in-split of  $\mathcal{A}$ , then  $M(\mathcal{B})$  and  $M(\mathcal{A})$  are elemen-  
 435 tary equivalent. We say that  $M, M'$  are *strong shift equivalent* if there is a sequence  
 436  $(M_0, M_1, \dots, M_n)$  such that  $M_i$  and  $M_{i+1}$  are elementary equivalent for  $0 \leq i < n$   
 437 with  $M_0 = M$  and  $M_n = M'$ . The following result is the version, for automata, of the  
 438 Classification Theorem.

439 **Theorem 3.9.** *Two automata are conjugate if and only if their adjacency matrices are*  
 440 *strong shift equivalent.*

441 Note that when  $D$  is a column division matrix, the statement results from Proposi-  
 442 tions 3.4 and 2.9. The following statement proves the theorem in one direction.

443 **Proposition 3.10.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two automata. If  $M(\mathcal{A})$  is elementary equivalent to*  
 444  *$M(\mathcal{B})$ , then  $\mathcal{A}$  and  $\mathcal{B}$  are conjugate.*

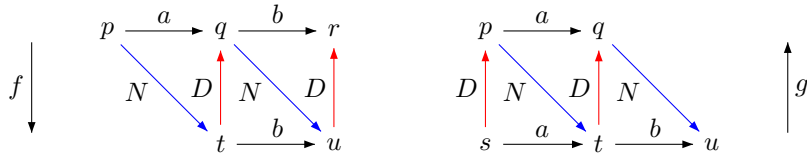
*Proof.* Let  $\mathcal{A} = (Q, E)$  and  $\mathcal{B} = (R, F)$ . Let  $D$  be an  $R \times Q$  nonnegative integral matrix and let  $N$  be an alphabetic  $Q \times R$  matrix such that

$$M(\mathcal{A}) = ND, \quad M(\mathcal{B}) = DN.$$

Consider the map  $f$  from the set of paths of length 2 in  $\mathcal{A}$  into  $F$  defined as follows (see Figure 8 on the left). Let  $p \xrightarrow{a} q \xrightarrow{b} r$  be a path of length 2 in  $\mathcal{A}$ . Since  $(M(\mathcal{A})_{pq}, a) = 1$  and  $M(\mathcal{A}) = ND$  there is a unique  $t \in R$  such that  $(N_{pt}, a) = D_{tq} = 1$ . In the same way, since  $(M(\mathcal{A})_{qr}, b) = 1$ , there is a unique  $u \in R$  such that  $(N_{qu}, b) = D_{ur} = 1$ . Since  $M(\mathcal{B}) = DN$ , we have  $(M(\mathcal{B})_{tu}, b) = D_{tq} = (N_{qu}, b) = 1$  and thus  $(t, u, b)$  is an edge of  $\mathcal{B}$ . We set

$$f(p \xrightarrow{a} q \xrightarrow{b} r) = t \xrightarrow{b} u$$

Similarly, we may define a map  $g$  from the set of paths of length 2 in  $\mathcal{B}$  into  $E$  by



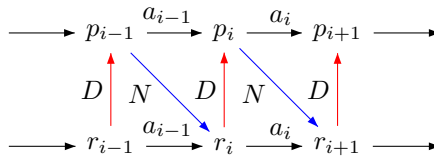
**Figure 8.** The maps  $f$  and  $g$ .

$$g(s \xrightarrow{a} t \xrightarrow{b} u) = p \xrightarrow{a} q$$

if  $D_{sp} = (N_{pt}, a) = D_{tq} = 1$ . Let  $\varphi = f_{\infty}^{[1,0]}$  and  $\gamma = g_{\infty}^{[0,1]}$  (see Figure 8 on the right). We verify that

$$\varphi\gamma = \text{Id}_F, \quad \gamma\varphi = \text{Id}_E$$

445 where  $\text{Id}_E$  and  $\text{Id}_F$  are the identities on  $E^{\mathbb{Z}}$  and  $F^{\mathbb{Z}}$ . Let indeed  $\pi$  be a path in  $X_{\mathcal{A}}$   
 446 and let  $\rho = \varphi(\pi)$ . Set  $\pi_i = (p_i, a_i, p_{i+1})$  and  $\rho_i = (r_i, b_i, r_{i+1})$  (see Figure 9). Then, by  
 447 definition of  $\varphi$ , we have for all  $i \in \mathbb{Z}$ ,  $b_i = a_i$  and  $(N_{p_i r_{i+1}}, a_i) = D_{r_i p_i} = 1$ . Let  
 448  $\sigma = \gamma(\rho)$  and  $\sigma = (s_i, c_i, s_{i+1})$ . By definition of  $\gamma$ , we have  $c_i = b_i$  and  $D_{r_i s_i} =$   
 449  $(N_{s_i r_{i+1}}, b_i) = 1$ . Thus we have simultaneously  $D_{r_i p_i} = (N_{p_i r_{i+1}}, a_i) = 1$  and  $D_{r_i s_i} =$   
 450  $(N_{s_i r_{i+1}}, a_i) = 1$ . Since  $M(\mathcal{A}) = DN$ , this forces  $p_i = s_i$ . Thus  $\sigma = \pi$  and this shows  
 451 that  $\gamma\varphi = \text{Id}_E$ . The fact that  $\varphi\gamma = \text{Id}_F$  is proved in the same way.



**Figure 9.** Conjugacy of automata.

453 *Proof of Theorem 3.9.* In one direction, the above statement is a direct consequence of  
 454 the Decomposition Theorem 2.12. Indeed, if  $\mathcal{A}$  and  $\mathcal{B}$  are conjugate, there is a sequence  
 455  $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$  of automata such that  $\mathcal{A}_i$  is a split or a merge of  $\mathcal{A}_{i+1}$  for  $0 \leq i < n$   
 456 with  $\mathcal{A}_0 = \mathcal{A}$  and  $\mathcal{A}_n = \mathcal{B}$ . The other direction follows from Proposition 3.10.  $\square$

## 457 4 Minimal automata

458 In this section, we define two notions of minimal automaton for sofic shifts: the Krieger  
 459 automaton and the Fischer automaton. The first is defined for any sofic shift, and the  
 460 second for irreducible ones.

461 The main result is that the Fischer automaton has the minimal number of states among  
 462 all deterministic automata recognizing a given sofic shift (Proposition 4.6).

463 We then define the syntactic semigroup of a sofic shift, as an ordered semigroup.  
 464 We show that this semigroup is isomorphic to the transition semigroup of the Krieger  
 465 automaton and, for irreducible shifts, to the transition semigroup of the Fischer automaton  
 466 (Proposition 4.8).

467 **Minimal automata of sets of finite words.** Recall that an automaton  $\mathcal{A} = (Q, E)$  recog-  
 468 nizes a shift  $X$  if  $X = L_{\mathcal{A}}$ . There should be no confusion with the notion of acceptance  
 469 for sets of finite words in the usual sense: if  $\mathcal{A}$  has an initial state  $i$  and a set of terminal  
 470 states  $T$ , the set of finite words recognized by  $\mathcal{A}$  is the set of labels of finite paths from  $i$   
 471 to a terminal state  $t$  in  $T$ . In this chapter<sup>3</sup>, an automaton is called *deterministic* if, for each  
 472 state  $p$  and each letter  $a$ , there is at most one edge starting in  $p$  and carrying the label  $a$ . We  
 473 write, as usual,  $p \cdot u$  for the unique end state, provided it exists, of a path starting in  $p$  and  
 474 labeled  $u$ . For a set  $W$  of  $A^*$ , there exists a unique deterministic minimal automaton (this  
 475 time with a unique initial state) recognizing  $W$ . Its states are the nonempty sets  $u^{-1}W$  for  
 476  $u \in A^*$ , called the *right contexts* of  $u$ , and the edges are the triples  $(u^{-1}W, a, (ua)^{-1}W)$ ,  
 477 for  $a \in A$  (see the chapter of J.-É. Pin).

478 Let  $\mathcal{A} = (Q, E)$  be a finite automaton. For a state  $p \in Q$ , we denote by  $L_p(\mathcal{A})$  or  
 479 simply  $L_p$  the set of labels of finite paths starting from  $p$ . The automaton  $\mathcal{A}$  is said to be  
 480 *reduced* if  $p \neq q$  implies  $L_p \neq L_q$ .

481 A word  $w$  is *synchronizing* for a deterministic automaton  $\mathcal{A}$  if the set of paths labeled  $w$   
 482 is nonempty and all paths labeled  $w$  end in the same state. An automaton is *synchronized*  
 483 if there is a synchronizing word. The following result holds because all states are terminal.

484 **Proposition 4.1.** *A reduced deterministic automaton is synchronized.*

485 *Proof.* Let  $\mathcal{A} = (Q, E)$  be a reduced deterministic automaton. Given any word  $x$ , we  
 486 denote by  $Q \cdot X$  the set  $Q \cdot x = \{q \cdot x \mid q \in Q\}$ .

487 Let  $x$  be a word such that  $Q \cdot x$  has minimal nonzero cardinality. Let  $p, q$  be two  
 488 elements of the set  $Q \cdot x$ . If  $u$  is a word such that  $p \cdot u$  is nonempty, then  $q \cdot u$  is also  
 489 nonempty since otherwise  $Q \cdot xu$  would be of nonzero cardinality less than  $Q \cdot x$ . This  
 490 implies that  $L_p = L_q$  and thus  $p = q$  since  $\mathcal{A}$  is reduced. Thus  $x$  is synchronizing.  $\square$

<sup>3</sup>This contrasts the more traditional definition which assumes in addition that there is a unique initial state.

491

## 4.1 Krieger automata and Fischer automata

492 **Krieger automata.** We denote by  $A^{-\mathbb{N}}$  the set of left infinite words  $x = \cdots x_{-1}x_0$ . For  
 493  $y = \cdots y_{-1}y_0 \in A^{-\mathbb{N}}$  and  $z = z_0z_1 \cdots \in A^{\mathbb{N}}$ , we denote by  $y \cdot z = (w_i)_{i \in \mathbb{Z}}$  the biinfinite  
 494 word defined by  $w_i = y_{i+1}$  for  $i < 0$  and  $w_i = z_i$  for  $i \geq 0$ . Let  $X$  be a shift space. For  
 495  $y \in A^{-\mathbb{N}}$ , the set of *right contexts* of  $y$  is the set  $C_X(y) = \{z \in A^{\mathbb{N}} \mid y \cdot z \in X\}$ . For  
 496  $u \in A^+$ , we denote  $u^\omega = uu \cdots$ .

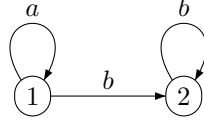
497 The *Krieger automaton* of a shift space  $X$  is the deterministic automaton whose states  
 498 are the nonempty sets of the form  $C_X(y)$  for  $y \in A^{-\mathbb{N}}$ , and whose edges are the triples  
 499  $(p, a, q)$  where  $p = C_X(y)$  for some left infinite word,  $a \in A$  and  $q = C_X(ya)$ .

The definition of the Krieger automaton uses infinite words. One could use instead of  
 the sets  $C_X(y)$  for  $y \in A^{-\mathbb{N}}$ , the sets

$$D_X(y) = \{u \in A^* \mid \exists z \in A^{\mathbb{N}} : yuz \in X\}.$$

500 Indeed  $C_X(y) = C_X(y')$  if and only if  $D_X(y) = D_X(y')$ . However, one cannot dispense  
 501 completely with infinite words (see Proposition 4.2).

502 **Example 4.1.** Let  $A = \{a, b\}$ , and let  $X = X^{(ba)}$ . The Krieger automaton of  $X$  is  
 503 represented in Figure 10. The states are the sets  $1 = C_X(\cdots aaa) = a^\omega \cup a^*b^\omega$  and  
 504  $2 = C_X(\cdots aaab) = b^\omega$ .



**Figure 10.** The Krieger automaton of  $X^{(ba)}$ .

505 **Proposition 4.2.** *The Krieger automaton of a shift space  $X$  is reduced and recognizes  $X$ .*  
 506 *It is finite if and only if  $X$  is sofic.*

507 *Proof.* Let  $\mathcal{A} = (Q, E)$  be the Krieger automaton of  $X$ . Let  $p, q \in Q$  and let  $y, z \in A^{-\mathbb{N}}$   
 508 be such that  $p = C_X(y)$ ,  $q = C_X(z)$ . If  $L_p = L_q$ , then the labels of infinite paths starting  
 509 from  $p$  and  $q$  are the same. Thus  $p = q$ . This shows that  $\mathcal{A}$  is reduced. If  $\mathcal{A}$  finite,  
 510 then  $X$  is sofic by Proposition 3.3. Conversely, if  $X$  is sofic, let  $\mathcal{A}$  be a finite automaton  
 511 recognizing  $X$ . The set of right contexts of a left infinite word  $y$  only depends on the set  
 512 of states  $p$  such that there is a path in the automaton  $\mathcal{A}$  labeled  $y$  ending in state  $p$ . Thus  
 513 the family of sets of right contexts is finite.  $\square$

514 We say that a deterministic automaton  $\mathcal{A} = (Q, E)$  over the alphabet  $A$  is a *subau-*  
 515 *tomaton* of a deterministic automaton  $\mathcal{A}' = (Q', E')$  if  $Q \subset Q'$  and if for each edge  
 516  $(p, a, q) \in E$  such that  $p \in Q$  one has  $q \in Q$  and  $(p, a, q) \in E'$ .

517 The following proposition appears in [22] and in [11] where an algorithm to compute  
 518 the states of the minimal automaton which are in the Krieger automaton is described.

519 **Proposition 4.3.** *The Krieger automaton of a sofic shift  $X$  is, up to an isomorphism, a*  
 520 *subautomaton of the minimal automaton of the set of blocks of  $X$ .*

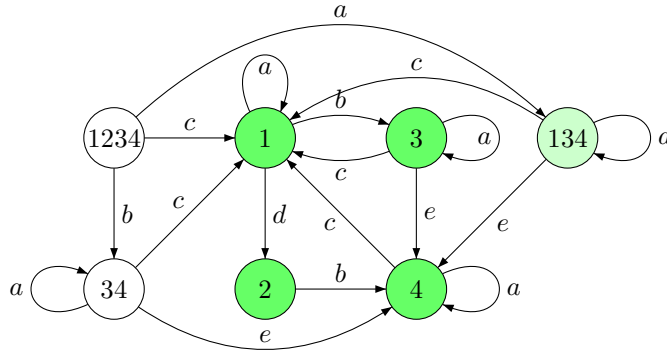
*Proof.* Let  $X$  be a sofic shift. Let  $y \in A^{-\mathbb{N}}$  and set  $y = \cdots y_{-1}y_0$  with  $y_i \in A$  for  $i \leq 0$ .  
 Set  $u_i = y_{-i} \cdots y_0$  and  $U_i = u_i^{-1}\mathcal{B}(X)$ . Since  $\mathcal{B}(X)$  is regular, the chain

$$\dots \subset U_i \subset \dots \subset U_1 \subset U_0$$

521 is stationary. Thus there is an integer  $n \geq 0$  such that  $U_{n+i} = U_n$  for all  $i \geq 0$ . We define  
 522  $s(y) = U_n$ .

523 We show that the map  $C_X(y) \mapsto s(y)$  is well-defined and injective. Suppose first that  
 524  $C_X(y) = C_X(y')$  for some  $y, y' \in A^{-\mathbb{N}}$ . Let  $u \in A^*$  be such that  $y_{-m} \cdots y_0 u \in \mathcal{B}(X)$   
 525 for all  $m \geq n$ . By compactness, there exists a  $z \in A^{\mathbb{N}}$  such that  $yuz \in X$ . Then  
 526  $y' \cdot uz \in X$  implies  $u \in s(y')$ . Symmetrically  $u \in s(y')$  implies  $u \in s(y)$ . This shows  
 527 that the map is well-defined.

528 To show that it is injective, consider  $y, y' \in A^{-\mathbb{N}}$  such that  $s(y) = s(y')$ . Let  $z \in$   
 529  $C_X(y)$ . For each integer  $m \geq 0$ , we have  $z_0 \cdots z_m \in s(y)$  and thus  $z_0 \cdots z_m \in s(y')$ .  
 530 Since  $X$  is closed, this implies that  $y' \cdot z \in X$  and thus  $z \in C_X(y')$ . The converse  
 531 implication is proved in the same way.  $\square$



**Figure 11.** An example of Krieger automaton.

532 **Example 4.2.** Consider the automaton on 7 states given in Figure 11. It is obtained,  
 533 starting with the subautomaton over the states 1, 2, 3, 4, using the subset construction  
 534 computing the accessible nonempty sets of states, starting from the set  $\{1, 2, 3, 4\}$ .

535 The subautomaton with dark shaded states 1, 2, 3, 4 is strongly connected and recog-  
 536 nizes an irreducible sofic shift denoted by  $X$ . The whole automaton is the minimal  
 537 automaton (with initial state  $\{1, 2, 3, 4\}$ ) of the set of blocks of  $X$ . The Krieger automa-  
 538 ton of  $X$  is the automaton on the five shaded states. Indeed, with the notation of the proof,  
 539 there is no left infinite word  $y$  such that  $s(y) = \{1, 2, 3, 4\}$  or  $s(y) = \{3, 4\}$ .

540 **Fischer automata of irreducible shift spaces.** A shift space  $X \subset A^{\mathbb{Z}}$  is called *irre-*  
 541 *ducible* if for any  $u, v \in \mathcal{B}(X)$  there exists a  $w \in \mathcal{B}(X)$  such that  $uwv \in \mathcal{B}(X)$ .

542 An automaton is said to be strongly connected if its underlying graph is strongly con-  
 543 nected. Clearly a shift recognized by a strongly connected automaton is irreducible.

544 A strongly connected component of an automaton  $\mathcal{A}$  is *minimal* if all successors of  
 545 vertices of the component are themselves in the component. One may verify that a mini-  
 546 mal strongly connected component is the same as a strongly connected subautomaton.

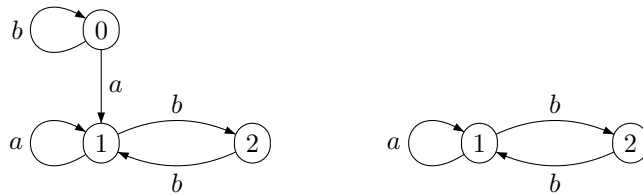
547 The following result is due to Fischer [13] (see also [19, Section 3]). It implies in  
 548 particular that an irreducible sofic shift can be recognized by a strongly connected au-  
 549 tomaton.

550 **Proposition 4.4.** *The Krieger automaton of an irreducible sofic shift  $X$  is synchronized*  
 551 *and has a unique minimal strongly connected component.*

552 *Proof.* Let  $\mathcal{A} = (Q, E)$  be the Krieger automaton of  $X$ . By Proposition 4.2,  $\mathcal{A}$  is reduced  
 553 and by Proposition 4.1, it follows that it is synchronized.

554 Let  $x$  be a synchronizing word. Let  $R$  be the set of states reachable from the state  
 555  $q = Q \cdot x$ . The set  $R$  is a minimal strongly connected component of  $\mathcal{A}$ . Indeed, for  
 556 any  $r \in R$  there is a path  $q \xrightarrow{y} r$ . Since  $X$  is irreducible there is a word  $z$  such that  
 557  $yzx \in \mathcal{B}(X)$ . Since  $q \cdot yzx = q$ ,  $r$  belongs to the same strongly connected component  
 558 as  $q$ . Next, if  $p$  belongs to a minimal strongly connected component  $S$  of  $\mathcal{A}$ , since  $X$   
 559 is irreducible, there is a word  $y$  such that  $p \cdot yx$  is not empty. Thus  $q$  is in  $S$ , which implies  
 560  $S = R$ . Thus  $R$  is the only minimal strongly component of  $\mathcal{A}$ .  $\square$

561 **Example 4.3.** Let  $X$  be the even shift. The Krieger and Fischer automata of  $X$  are  
 562 represented on Figure 12. The word  $a$  is synchronizing.



**Figure 12.** The Krieger and Fischer automata of  $X$ .

563 **Example 4.4.** The Fischer automaton of the irreducible shift of Example 4.2 is the sub-  
 564 automaton on states 1, 2, 3, 4 represented with dark shaded states in Figure 11.

565 Let  $X$  be an irreducible sofic shift  $X$ . The minimal strongly connected component of  
 566 the Krieger automaton of  $X$  is called its *Fischer automaton*.

567 **Proposition 4.5.** *The Fischer automaton of an irreducible sofic shift  $X$  recognizes  $X$ .*

568 *Proof.* The Fischer automaton  $\mathcal{F}$  of  $X$  is a subautomaton of the Krieger automaton of  $X$   
 569 which in turn is a subautomaton of the minimal automaton  $\mathcal{A}$  of the set  $\mathcal{B}(X)$ . Let  $i$  be  
 570 the initial state of  $\mathcal{A}$ . Since  $\mathcal{A}$  is trim, there is a word  $w$  such that  $i \cdot w$  is a state of  $\mathcal{F}$ . Let

571  $v$  be any block of  $X$ . Since  $X$  is irreducible, there is a word  $u$  such that  $wuv$  is a block of  
 572  $X$ . This shows that  $v$  is a label of a path in  $\mathcal{F}$ . Thus every block of  $X$  is a label of a path  
 573 in  $\mathcal{F}$  and conversely. In view of Proposition 3.2, the automaton  $\mathcal{F}$  recognizes  $X$ .  $\square$

574 Let  $\mathcal{A} = (Q, E)$  and  $\mathcal{B} = (R, F)$  be two deterministic automata. A *reduction* from  $\mathcal{A}$   
 575 onto  $\mathcal{B}$  is a map  $h$  from  $Q$  onto  $R$  such that for any letter  $a \in A$ , one has  $(p, a, q) \in E$  if  
 576 and only if  $(h(p), a, h(q)) \in F$ . Thus any labeled in or out-merge is a reduction. However  
 577 the converse is not true since a reduction is not, in general, a conjugacy.

578 For any automaton  $\mathcal{A} = (Q, E)$ , there is reduction from  $\mathcal{A}$  onto a reduced automaton  
 579  $\mathcal{B}$ . It is obtained by identifying the pairs of states  $p, q \in Q$  such that  $L_p = L_q$ .

580 The following statement is Corollary 3.3.20 of [19].

581 **Proposition 4.6.** *Let  $X$  be an irreducible shift space. For any strongly connected de-*  
 582 *terministic automaton  $\mathcal{A}$  recognizing  $X$  there is a reduction from  $\mathcal{A}$  onto the Fischer*  
 583 *automaton of  $X$ .*

584 *Proof.* Let  $\mathcal{A} = (Q, E)$  be a strongly connected automaton recognizing  $X$ . Let  $\mathcal{B} =$   
 585  $(R, F)$  be the reduced automaton obtained from  $\mathcal{A}$  identifying the pairs  $p, q \in Q$  such  
 586 that  $L_p = L_q$ . By Proposition 4.1,  $\mathcal{B}$  is synchronized.

587 We now show that  $\mathcal{B}$  can be identified with the Fischer automaton of  $X$ . Let  $w$  be a  
 588 synchronizing word for  $\mathcal{B}$ . Set  $s = Q \cdot w$ . Let  $r$  be a state such that  $r \cdot w = s$ , and let  
 589  $y \in A^{-\mathbb{N}}$  be the label of a left infinite path ending in the state  $s$ . For any state  $t$  in  $R$ , let  
 590  $u$  be a word such that  $s \cdot u = t$ . The set  $C_X(ywu)$  depends only on the state  $t$ , and not on  
 591 the word  $u$  such that  $s \cdot u = t$ . Indeed, for each right infinite word  $z$ , one has  $ywuz$  in  $X$   
 592 if and only if there is a path labeled  $z$  starting at  $t$ . This holds because  $w$  is synchronizing.

593 Thus the map  $t \mapsto C_X(ywu)$  is well-defined and defines a reduction from  $\mathcal{B}$  onto the  
 594 Fischer automaton of  $X$ .  $\square$

595 This statement shows that the Fischer automaton of an irreducible shift  $X$  is minimal  
 596 in the sense that it has the minimal number of states among all deterministic strongly  
 597 connected automata recognizing  $X$ .

598 The statement also gives the following practical method to compute the Fischer au-  
 599 tomaton of an irreducible shift. We start with a strongly connected deterministic automa-  
 600 ton recognizing  $X$  and merge the pairs of states  $p, q$  such that  $L_p = L_q$ . By the above  
 601 result, the resulting automaton is the Fischer automaton of  $X$ .

## 602 4.2 Syntactic semigroup

603 Recall that a *preorder* on a set is a relation which is reflexive and transitive. The equiva-  
 604 lence associated to a preorder is the equivalence relation defined by  $u \equiv v$  if and only if  
 605  $u \leq v$  and  $v \leq u$ .

606 Let  $S$  be a semigroup. A preorder on  $S$  is said to be *stable* if  $s \leq s'$  implies  $us \leq us'$   
 607 and  $su \leq s'u$  for all  $s, s', u \in S$ . An *ordered semigroup*  $S$  is a semigroup equipped with  
 608 a stable preorder. Any semigroup can be considered as an ordered semigroup equipped  
 609 with the equality order.

610 A *congruence* in an ordered semigroup  $S$  is the equivalence associated to a stable  
 611 preorder which is coarser than the preorder of  $S$ . The quotient of an ordered semigroup  
 612 by a congruence is the ordered semigroup formed by the classes of the congruence.

613 The *set of contexts* of a word  $u$  with respect to a set  $W \subset A^+$  is the set  $\Gamma_W(u)$  of  
 614 pairs of words defined by  $\Gamma_W(u) = \{(\ell, r) \in A^* \times A^* \mid \ell ur \in W\}$ . The preorder on  
 615  $A^+$  defined by  $u \leq_W v$  if  $\Gamma_W(u) \subset \Gamma_W(v)$  is stable and thus defines a congruence of  
 616 the semigroup  $A^+$  equipped with the equality order called the *syntactic congruence*. The  
 617 *syntactic semigroup* of a set  $W \subset A^*$  is the quotient of the semigroup  $A^+$  by the syntactic  
 618 congruence.

619 Let  $\mathcal{A} = (Q, E)$  be a deterministic automaton on the alphabet  $A$ . Recall that for  
 620  $p \in Q$  and  $u \in A^+$ , there is at most one path  $\pi$  labeled  $u$  starting in  $p$ . We set  $p \cdot u = q$  if  
 621  $q$  is the end of  $\pi$  and  $p \cdot u = \emptyset$  if  $\pi$  does not exist. The preorder defined on  $A^+$  by  $u \leq_{\mathcal{A}} v$   
 622 if  $p \cdot u \subset p \cdot v$  for all  $p \in Q$  is stable. The quotient of  $A^+$  by the congruence associated  
 623 to this preorder is the *transition semigroup* of  $\mathcal{A}$ .

624 The following property is standard, see the chapter of J.-É Pin.

625 **Proposition 4.7.** *The syntactic semigroup of a set  $W \subset A^+$  is isomorphic to the transi-*  
 626 *tion semigroup of the minimal automaton of  $W$ .*

627 The *syntactic semigroup* of a shift space  $X$  is by definition the syntactic semigroup of  
 628  $\mathcal{B}(X)$ .

629 **Proposition 4.8.** *Let  $X$  be a sofic shift and let  $S$  be its syntactic semigroup. The tran-*  
 630 *sition semigroup of the Krieger automaton of  $X$  is isomorphic to  $S$ . Moreover, if  $X$  is*  
 631 *irreducible, then it is isomorphic to the transition semigroup of its Fischer automaton.*

632 *Proof.* Let  $\mathcal{A}$  be the minimal automaton of  $\mathcal{B}(X)$ , and let  $\mathcal{K}$  be the Krieger automaton of  
 633  $X$ . We have to show that for any  $u, v \in A^+$ , one has  $u \leq_{\mathcal{A}} v$  if and only if  $u \leq_{\mathcal{K}} v$ .  
 634 Since, by Proposition 4.3,  $\mathcal{K}$  is isomorphic to a subautomaton of  $\mathcal{A}$ , the direct implication  
 635 is clear. Indeed, if  $p$  is a state of  $\mathcal{K}$ , then  $L_p(\mathcal{K})$  is equal to the set  $L_p(\mathcal{A})$ . Consequently,  
 636 if  $u \leq_{\mathcal{A}} v$  then  $u \leq_{\mathcal{K}} v$ . Conversely, suppose that  $u \leq_{\mathcal{K}} v$ . We prove that  $u \leq_{\mathcal{B}(X)} v$ . For  
 637 this, let  $(\ell, r) \in \Gamma_{\mathcal{B}(X)}(u)$ . Then  $\ell ur \in \mathcal{B}(X)$ . Then  $y \cdot \ell ur z \in X$  for some  $y \in A^{-\mathbb{N}}$   
 638 and  $z \in A^{\mathbb{N}}$ . But since  $C_X(y\ell u) \subset C_X(y\ell v)$ , this implies  $rz \in C_X(y\ell v)$  and thus  
 639  $\ell vr \in \mathcal{B}(X)$ . Thus  $u \leq_{\mathcal{B}(X)} v$  which implies  $u \leq_{\mathcal{A}} v$ .

640 Next, suppose that  $X$  is irreducible. We have to show that  $u \leq_{\mathcal{A}} v$  if and only if  
 641  $u \leq_{\mathcal{F}(X)} v$ . Since  $\mathcal{F}(X)$  is a subautomaton of  $\mathcal{K}(X)$  and  $\mathcal{K}(X)$  is a subautomaton of  
 642  $\mathcal{A}$ , the direct implication is clear. Conversely, assume that  $u \leq_{\mathcal{F}(X)} v$ . Suppose that  
 643  $\ell ur \in \mathcal{B}(X)$ . Let  $i$  be the initial state of  $\mathcal{A}$  and let  $w$  be such that  $i \cdot w$  is a state of  
 644  $\mathcal{F}(X)$ . Since  $X$  is irreducible, there is a word  $s$  such that  $wslur \in \mathcal{B}(X)$ . But then  
 645  $i \cdot wslur \neq \emptyset$  implies  $i \cdot wslvr \neq \emptyset$ . Thus  $\ell vr \in \mathcal{B}(X)$ . This shows that  $u \leq_{\mathcal{B}(X)} v$  and  
 646 thus  $u \leq_{\mathcal{A}} v$ .  $\square$

## 647 5 Symbolic conjugacy

648 This section is concerned with a new notion of conjugacy between automata called sym-  
 649 bolic conjugacy. It extends the notion of labeled conjugacy and captures the fact that

650 the automata may be over different alphabets. The table below summarizes the various  
 651 notions.

object type	isomorphism	elementary transformation
shift spaces	conjugacy	split/merge
edge shifts	conjugacy	edge split/merge
652 integer matrices	strong shift equivalence	elementary equivalence
automata (same alphabet)	labeled conjugacy	labeled split/merge
automata	symbolic conjugacy	split/merge
alphabetic matrices	symbolic strong shift	elementary symbolic

653 There are two main results in this section. Theorem 5.7 due to Nasu is a version of the  
 654 Classification Theorem for sofic shifts. It implies in particular that conjugate sofic shifts  
 655 have symbolic conjugate Krieger or Fisher automata. The proof uses the notion of bipar-  
 656 tite automaton, which corresponds to the symbolic elementary equivalence of adjacency  
 657 matrices. Theorem 5.8 is due to Hamachi and Nasu: it characterizes symbolic conjugate  
 658 automata by means of their adjacency matrices.

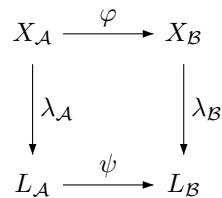
In this section, we will use for convenience automata in which several edges with the  
 same source and target can have the same label. Formally, such an automaton is a pair  
 $\mathcal{A} = (G, \lambda)$  of a graph  $G = (Q, \mathcal{E})$  and a map assigning to each edge  $e \in \mathcal{E}$  of a label  
 $\lambda(e) \in A$ . The adjacency matrix of  $\mathcal{A}$  is the  $Q \times Q$ -matrix  $M(\mathcal{A})$  with elements in  $\mathbb{N}\langle A \rangle$   
 defined by

$$(M(\mathcal{A}))_{pq, a} = \text{Card}\{e \in \mathcal{E} \mid \lambda(e) = a\}. \quad (5.1)$$

659 Note that  $M(\mathcal{A})$  is alphabetic but may have arbitrary nonnegative coefficients. The ad-  
 660 vantage of this version of automata is that for any alphabetic  $Q \times Q$ -matrix  $M$  there is an  
 661 automaton  $\mathcal{A}$  such that  $M(\mathcal{A}) = M$ .

662 We still denote by  $X_{\mathcal{A}}$  the edge shift  $X_G$  and by  $L_{\mathcal{A}}$  the set of labels of infinite paths  
 663 in  $G$ .

664 **Symbolic conjugate automata.** Let  $\mathcal{A}, \mathcal{B}$  be two automata. A *symbolic conjugacy* from  
 665  $\mathcal{A}$  onto  $\mathcal{B}$  is a pair  $(\varphi, \psi)$  of conjugacies  $\varphi : X_{\mathcal{A}} \rightarrow X_{\mathcal{B}}$  and  $\psi : L_{\mathcal{A}} \rightarrow L_{\mathcal{B}}$  such that the  
 following diagram is commutative.



666

### 667 5.1 Splitting and merging maps

668 Let  $A, B$  be two alphabets and let  $f : A \rightarrow B$  be a map from  $A$  onto  $B$ . Let  $X$  be a shift  
 669 space on the alphabet  $A$ . We consider the set of words  $A' = \{f(a_1)a_2 \mid a_1a_2 \in \mathcal{B}_2(X)\}$   
 670 as a new alphabet. Let  $g : \mathcal{B}_2(X) \rightarrow A'$  be the 2-block substitution defined by  $g(a_1a_2) =$   
 671  $f(a_1)a_2$ .

672 The *in-splitting map* defined on  $X$  and relative to  $f$  or to  $g$  is the sliding block map  
 673  $g_{\infty}^{1,0}$  corresponding to  $g$ . It is a conjugacy from  $X$  onto its image by  $X' = g_{\infty}^{1,0}(X)$  since  
 674 its inverse is 1-block. The shift space  $X'$ , is called the *in-splitting* of  $X$ , relative to  $f$  or  
 675  $g$ . The inverse of an in-splitting map is called an *in-merging map*.

676 In addition, any renaming of the alphabet of a shift space is also considered to be an  
 677 in-splitting map (and an in-merging map).

678 **Example 5.1.** Let  $A = B$  and let  $f$  be the identity on  $A$ . The out-splitting of a shift  $X$   
 679 relative to  $f$  is the second higher block shift of  $X$ .

680 The following proposition relates splitting maps to edge splittings as defined in Sec-  
 681 tion 2.2.

682 **Proposition 5.1.** *An in-splitting map on an edge shift is an edge in-splitting map, and*  
 683 *conversely.*

684 *Proof.* Let first  $G = (Q, \mathcal{E})$  be a graph, and let  $f : \mathcal{E} \rightarrow I$  be a map from  $\mathcal{E}$  onto  
 685 a set  $I$ . Set  $\mathcal{E}' = \{f(e_1)e_2 \mid e_1e_2 \in \mathcal{B}_2(X_G)\}$ . Let  $g : \mathcal{B}_2(X_G) \rightarrow \mathcal{E}'$  be the 2-  
 686 block substitution defined by  $g(e_1e_2) = f(e_1)e_2$ . Let  $G' = (Q', \mathcal{E}')$  be the graph on  
 687 the set of states  $Q' = I \times Q$  defined for  $e' = f(e_1)e_2$  by  $i(e') = (f(e_1), i(e_2))$  and  
 688  $t(e') = (f(e_2), t(e_2))$ . Define  $h : \mathcal{E}' \rightarrow \mathcal{E}$  and  $k : Q' \rightarrow Q$  by  $h(f(e_1)e_2) = e_2$  for  
 689  $e_1e_2 \in \mathcal{B}_2(X_G)$  and  $k(i, q) = q$  for  $(i, q) \in I \times Q$ . Then the pair  $(h, k)$  is an in-merge  
 690 from  $G'$  onto  $G$  and  $h_{\infty}$  is the inverse of  $g_{\infty}^{1,0}$ . Indeed, one may verify that  $(h, k)$  is a  
 691 graph morphism from  $G'$  onto  $G$ . Next it is an in-merge because for each  $p, q \in Q$ , the  
 692 partition  $(\mathcal{E}_p^q(t))_{t \in k^{-1}(q)}$  of  $\mathcal{E}_p^q$  is defined by  $\mathcal{E}_p^q(i, q) = E_p^q \cap f^{-1}(i)$ .

693 Conversely, set  $G = (Q, \mathcal{E})$  and  $G' = (Q', \mathcal{E}')$ . Let  $(h, k)$  be an in-merge from  $G'$   
 694 onto  $G$ . Consider the map  $f : \mathcal{E} \rightarrow Q'$  defined by  $f(e) = r$  if  $r$  is the common end of the  
 695 edges in  $h^{-1}(e)$ . The map  $\alpha$  from  $\mathcal{E}'$  to  $Q' \times \mathcal{E}$  defined by  $\alpha(i) = (r, h(i))$  where  $r$  is the  
 696 origin of  $i$  is a bijection by definition of an in-merge.

697 Let us show that, up to the bijection  $\alpha$ , the in-splitting map relative to  $f$  is inverse  
 698 of the map  $h_{\infty}$ . For  $e_1, e_2 \in \mathcal{E}$ , let  $r = f(e_1)$  and  $e' = \alpha^{-1}(r, e_2)$ . Then  $h(e') = e_2$   
 699 and thus  $h_{\infty}$  is the inverse of the map  $g_{\infty}^{1,0}$  corresponding to the 2-block substitution  
 700  $g(e_1e_2) = (r, e_2)$ .

701 □

702 Symmetrically an *out-splitting map* is defined by the substitution  $g(ab) = af(b)$ . Its  
 703 inverse is an out-merging map.

704 We use the term splitting to mean either a in-splitting or out-splitting. The same  
 705 convention holds for a merging.

706 The following result, from [21], is a generalization of the Decomposition Theorem  
 707 (Theorem 2.12) to arbitrary shift spaces.

708 **Theorem 5.2.** *Any conjugacy between shift spaces is a composition of splitting and merg-*  
 709 *ing maps.*

710 The proof is similar to the proof of Theorem 2.12. It relies on the following lemma,  
 711 similar to Lemma 2.13.

712 **Lemma 5.3.** Let  $\varphi : X \rightarrow Y$  be a 1-block conjugacy whose inverse has memory  $m \geq 1$   
 713 and anticipation  $n \geq 0$ . There are in-splitting maps from  $X, Y$  to  $\tilde{X}, \tilde{Y}$  respectively such  
 714 that the 1-block conjugacy  $\tilde{\varphi}$  making the diagram below commutative has an inverse with  
 memory  $m - 1$  and anticipation  $n$ .

$$\begin{array}{ccc} X & \longrightarrow & \tilde{X} \\ \downarrow \varphi & & \downarrow \tilde{\varphi} \\ Y & \longrightarrow & \tilde{Y} \end{array}$$

715

716 *Proof.* Let  $A, B$  the alphabets of  $X$  and  $Y$  respectively. Let  $h : A \rightarrow B$  be the 1-  
 717 block substitution such that  $\varphi = h_\infty$ . Let  $\tilde{X}$  be the in-splitting of  $X$  relative to the  
 718 map  $h$ . Set  $A' = \{h(a_1)a_2 \mid a_1a_2 \in \mathcal{B}_2(X)\}$ . Let  $\tilde{Y} = Y^{[2]}$  be the second higher  
 719 block shift of  $Y$  and let  $B' = \mathcal{B}_2(Y)$ . Let  $\tilde{h} : A' \rightarrow B'$  be the 1-block substitution  
 720 defined by  $\tilde{h}(h(a_1)a_2) = h(a_1)h(a_2)$ . Then the 1-block map  $\tilde{\varphi} = \tilde{h}_\infty$  has the required  
 721 properties.  $\square$

722 Lemma 5.3 has a dual where  $\varphi$  is a 1-block map whose inverse has memory  $m \geq 0$   
 723 and anticipation  $n \geq 1$  and where in-splits are replaced by out-splits.

724 *Proof of Theorem 5.2.* Let  $\varphi : X \rightarrow Y$  be a conjugacy from  $X$  onto  $Y$ . Replacing  $X$  by a  
 725 higher block shift, we may assume that  $\varphi$  is a 1-block map. Using iteratively Lemma 5.3,  
 726 we can replace  $\varphi$  by a 1-block map whose inverse has memory 0. Using then iteratively  
 727 the dual of Lemma 5.3, we finally obtain a 1-block map whose inverse is also 1-block and  
 728 is thus just a renaming of the symbols.  $\square$

**Symbolic strong shift equivalence.** Let  $M$  and  $M'$  be two alphabetic  $Q \times Q$ -matrices  
 over the alphabets  $A$  and  $B$ , respectively. We say that  $M$  and  $M'$  are *similar* if they  
 are equal up to a bijection of  $A$  onto  $B$ . We write  $M \leftrightarrow M'$  when  $M$  and  $M'$  are  
 similar. We say that two alphabetic square matrices  $M$  and  $M'$  over the alphabets  $A$  and  
 $B$  respectively are *symbolic elementary equivalent* if there exist two alphabetic matrices  
 $R, S$  over the alphabets  $C$  and  $D$  respectively such that

$$M \leftrightarrow RS, \quad M' \leftrightarrow SR.$$

729 In this definition, the sets  $CD$  and  $DC$  of two letter words are identified with alphabets  
 730 in bijection with  $A$  and  $B$ , respectively.

731 We say that two matrices  $M, M'$  are *symbolic strong shift equivalent* if there is a  
 732 sequence  $(M_0, M_1, \dots, M_n)$  of alphabetic matrices such that  $M_i$  and  $M_{i+1}$  are symbolic  
 733 elementary equivalent for  $0 \leq i < n$  with  $M_0 = M$  and  $M_n = M'$ .

734 We introduce the following notion. An automaton  $\mathcal{A}$  on the alphabet  $A$  is said to be  
 735 *bipartite* if there are partitions  $Q = Q_1 \cup Q_2$  of the set of states and  $A = A_1 \cup A_2$  of the  
 736 alphabet such that all edges labeled in  $A_1$  go from  $Q_1$  to  $Q_2$  and all edges labeled in  $A_2$   
 737 go from  $Q_2$  to  $Q_1$ .

Let  $\mathcal{A}$  be a bipartite automaton. Its adjacency matrix has the form

$$M(\mathcal{A}) = \begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix}$$

738 where  $M_1$  is a  $Q_1 \times Q_2$ -matrix with elements in  $\mathbb{N}\langle A_1 \rangle$  and  $M_2$  is a  $Q_2 \times Q_1$ -matrix with  
 739 elements in  $\mathbb{N}\langle A_2 \rangle$ . The automata  $\mathcal{A}_1$  and  $\mathcal{A}_2$  which have  $M_1 M_2$  and  $M_2 M_1$  respectively  
 740 as adjacency matrix are called the *components* of  $\mathcal{A}$  and the pair  $\mathcal{A}_1, \mathcal{A}_2$  is a *decomposi-*  
 741 *tion* of  $\mathcal{A}$ . We denote  $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$  a bipartite automaton  $\mathcal{A}$  with components  $\mathcal{A}_1, \mathcal{A}_2$ .  
 742 Note that  $\mathcal{A}_1, \mathcal{A}_2$  are automata on the alphabets  $A_1 A_2$  and  $A_2 A_1$  respectively.

743 **Proposition 5.4.** *Let  $\mathcal{A} = (Q, E)$  be a bipartite deterministic essential automaton. Its*  
 744 *components  $\mathcal{A}_1, \mathcal{A}_2$  are deterministic essential automata which are symbolic conjugate.*  
 745 *If moreover  $\mathcal{A}$  is strongly connected (resp. reduced, resp. synchronized), then  $\mathcal{A}_1, \mathcal{A}_2$  are*  
 746 *strongly connected (resp. reduced, resp. synchronized).*

747 *Proof.* Let  $Q = Q_1 \cup Q_2$  and  $A = A_1 \cup A_2$  be the partitions of the set  $Q$  and the  
 748 alphabet  $A$  corresponding to the decomposition  $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ . It is clear that  $\mathcal{A}_1, \mathcal{A}_2$  are  
 749 deterministic and that they are strongly connected if  $\mathcal{A}$  is strongly connected.

750 Let  $\varphi : X_{\mathcal{A}_1} \rightarrow X_{\mathcal{A}_2}$  be the conjugacy defined as follows. For any  $y = (y_n)_{n \in \mathbb{Z}}$   
 751 in  $X_{\mathcal{A}_1}$  there is an  $x = (x_n)_{n \in \mathbb{Z}}$  in  $X_{\mathcal{A}}$  such that  $y_n = x_{2n} x_{2n+1}$ . Then  $z = (z_n)_{n \in \mathbb{Z}}$   
 752 with  $z_n = x_{2n+1} x_{2n}$  is an element of  $X_{\mathcal{A}_2}$ . We define  $\varphi(y) = z$ . The analogous map  
 753  $\psi : L_{\mathcal{A}_1} \rightarrow L_{\mathcal{A}_2}$  is such that  $(\varphi, \psi)$  is a symbolic conjugacy from  $\mathcal{A}_1$  onto  $\mathcal{A}_2$ .

754 Assume that  $\mathcal{A}$  is reduced. For  $p, q \in Q_1$ , there is a word  $w$  such that  $w \in L_p(\mathcal{A})$   
 755 and  $w \notin L_q(\mathcal{A})$  (or conversely). Set  $w = a_1 a_2 \cdots a_n$  with  $a_i \in A$ . If  $n$  is even,  
 756 then  $(a_1 a_2) \cdots (a_{n-1} a_n)$  is in  $L_p(\mathcal{A}_1)$  but not in  $L_q(\mathcal{A}_1)$ . Otherwise, since  $\mathcal{A}$  is essen-  
 757 tial, there is a letter  $a_{n+1}$  such that  $w a_{n+1}$  is in  $L_p(\mathcal{A})$ . Then  $(a_1 a_2) \cdots (a_n a_{n+1})$  is in  
 758  $L_p(\mathcal{A}_1)$  but not in  $L_q(\mathcal{A}_1)$ . Thus  $\mathcal{A}_1$  is reduced. One proves in the same way that  $\mathcal{A}_2$  is  
 759 reduced.

760 Suppose finally that  $\mathcal{A}$  is synchronized. Let  $x$  be a synchronizing word and set  
 761  $x = a_1 a_2 \cdots a_n$  with  $a_i \in A$ . Suppose that all paths labeled  $x$  end in  $q \in Q_1$ . Let  
 762  $a_{n+1}$  be a letter such that  $q \cdot a_{n+1} \neq \emptyset$  and let  $a_0$  be a letter such that  $a_0 x$  is the la-  
 763 bel of at least one path. If  $n$  is even, then  $(a_1 a_2) \cdots (a_{n-1} a_n)$  is synchronizing for  $\mathcal{A}_1$   
 764 and  $(a_0 a_1) \cdots (a_n a_{n+1})$  is synchronizing for  $\mathcal{A}_2$ . Otherwise,  $(a_0 a_1) \cdots (a_{n-1} a_n)$  is syn-  
 765 chronizing for  $\mathcal{A}_1$  and  $(a_1 a_2) \cdots (a_n a_{n+1})$  is synchronizing for  $\mathcal{A}_2$ .  $\square$

766 **Proposition 5.5.** *Let  $\mathcal{A}, \mathcal{B}$  be two automata such that  $M(\mathcal{A})$  and  $M(\mathcal{B})$  are symbolic*  
 767 *elementary equivalent. Then there is a bipartite automaton  $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$  such that*  
 768  *$M(\mathcal{C}_1), M(\mathcal{C}_2)$  are similar to  $M(\mathcal{A}), M(\mathcal{B})$  respectively.*

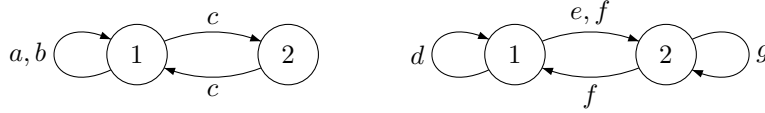
*Proof.* Let  $R, S$  be alphabetic matrices over alphabets  $C$  and  $D$  respectively such that  
 $M(\mathcal{A}) \leftrightarrow RS$  and  $M(\mathcal{B}) \leftrightarrow SR$ . Let  $\mathcal{C}$  be the bipartite automaton on the alphabet  $C \cup D$   
 which is defined by the adjacency matrix

$$M(\mathcal{C}) = \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}$$

769 Then  $M(\mathcal{A})$  is similar to  $M(\mathcal{C}_1)$  and  $M(\mathcal{B})$  is similar to  $M(\mathcal{C}_2)$ .  $\square$

770 **Proposition 5.6.** *If the adjacency matrices of two automata are symbolic strong shift*  
 771 *equivalent, the automata are symbolic conjugate.*

772 *Proof.* Since a composition of conjugacies is a conjugacy, it is enough to consider the case  
 773 where the adjacency matrices are symbolic elementary equivalent. Let  $\mathcal{A}, \mathcal{B}$  be such that  
 774  $M(\mathcal{A}), M(\mathcal{B})$  are symbolic elementary equivalent. By Proposition 5.5, there is a bipartite  
 775 automaton  $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$  such that  $M(\mathcal{C}_1), M(\mathcal{C}_2)$  are similar to  $M(\mathcal{A})$  and  $M(\mathcal{B})$  respec-  
 776 tively. By Proposition 5.4, the automata  $\mathcal{C}_1, \mathcal{C}_2$  are symbolic conjugate. Since automata  
 777 with similar adjacency matrices are obviously symbolic conjugate, the result follows.  $\square$



**Figure 13.** Two symbolic conjugate automata.

**Example 5.2.** Let  $\mathcal{A}, \mathcal{B}$  be the automata represented on Figure 13. The matrices  $M(\mathcal{A})$   
 and  $M(\mathcal{B})$  are symbolic elementary equivalent. Indeed, we have  $M(\mathcal{A}) \leftrightarrow RS$  and  
 $M(\mathcal{B}) \leftrightarrow SR$  for

$$R = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}, \quad S = \begin{bmatrix} z & t \\ t & 0 \end{bmatrix}.$$

Indeed, one has

$$RS = \begin{bmatrix} xz + yt & xt \\ xt & 0 \end{bmatrix}, \quad SR = \begin{bmatrix} zx & zy + tx \\ tx & ty \end{bmatrix}.$$

Thus the following tables give two bijections between the alphabets.

$a$	$b$	$c$
$xz$	$yt$	$xt$

$d$	$e$	$f$	$g$
$zx$	$zy$	$tx$	$ty$

778 The following result is due to Nasu [21]. The equivalence between conditions (i) and  
 779 (ii) is a version, for sofic shifts, of the Classification Theorem (Theorem 7.2.12 in [19]).  
 780 The equivalence between conditions (i) and (iii) is due to Krieger [18].

781 **Theorem 5.7.** *Let  $X, X'$  be two sofic shifts (resp. irreducible sofic shifts) and let  $\mathcal{A}, \mathcal{A}'$*   
 782 *be their Krieger (resp. Fischer) automata. The following conditions are equivalent.*

- 783 (i)  $X, X'$  are conjugate.  
 784 (ii) The adjacency matrices of  $\mathcal{A}, \mathcal{A}'$  are symbolic strong shift equivalent.  
 785 (iii)  $\mathcal{A}, \mathcal{A}'$  are symbolic conjugate.

786 *Proof.* We prove the result for irreducible shifts. The proof of the general case is in [21].

787 Assume that  $X, X'$  are conjugate. By the Decomposition Theorem (Theorem 5.2), it  
 788 is enough to consider the case where  $X'$  is an in-splitting of  $X$ . Let  $f : A \rightarrow B$  be a  
 789 map and let  $A' = \{f(a_1)a_2 \mid a_1a_2 \in \mathcal{B}_2(X)\}$  in such a way that  $X'$  is the in-splitting of

790  $X$  relative to  $f$ . Let  $C = A \cup B$  and let  $Z$  be the shift space composed of all biinfinite  
 791 sequences  $\cdots a_i f(a_i) a_{i+1} f(a_{i+1}) \cdots$  such that  $\cdots a_i a_{i+1} \cdots$  is in  $X$ . Then  $Z$  is an  
 792 irreducible sofic shift. Let  $\mathcal{A}$  be the Fischer automaton of  $Z$ . Then  $\mathcal{A}$  is bipartite and  
 793 its components recognize, up to a bijection of the alphabets,  $X$  and  $X'$  respectively. By  
 794 Proposition 5.4 the components are the Fischer automata of  $X$  and  $X'$  respectively. Since  
 795 the components of a bipartite automaton have symbolic elementary equivalent adjacency  
 796 matrices, this proves that (i) implies (ii).

797 That (ii) implies (iii) is Proposition 5.6. Finally, (iii) implies (i) by definition of sym-  
 798 bolic conjugacy.  $\square$

## 799 5.2 Symbolic conjugate automata

800 The following result is due to Hamachi and Nasu [16]. It shows that, in Theorem 5.7, the  
 801 equivalence between conditions (ii) and (iii) holds for automata which are not reduced.

802 **Theorem 5.8.** *Two essential automata are symbolic conjugate if and only if their adja-  
 803 cency matrices are symbolic strong shift equivalent.*

804 The first element of the proof is a version of the Decomposition Theorem for automata.

805 Let  $\mathcal{A}, \mathcal{A}'$  be two automata. An *in-split* from  $\mathcal{A}$  onto  $\mathcal{A}'$  is a symbolic conjugacy  
 806  $(\varphi, \psi)$  such that  $\varphi : X_{\mathcal{A}} \rightarrow X_{\mathcal{A}'}$  and  $\psi : L_{\mathcal{A}} \rightarrow L_{\mathcal{A}'}$  are in-splitting maps. A similar  
 807 definition holds for out-splits.

808 **Theorem 5.9.** *Any symbolic conjugacy between automata is a composition of splits and  
 809 merges.*

810 The proof relies on the following variant of Lemma 5.3.

811 **Lemma 5.10.** *Let  $\alpha, \beta$  be 1-block maps and  $\varphi, \psi$  be 1-block conjugacies such such that  
 812 the diagram below on the left is commutative.*

813 *If the inverses of  $\varphi, \psi$  have memory  $m \geq 1$  and anticipation  $n \geq 0$ , there exist in-splits  
 814  $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{T}$  of  $X, Y, Z, T$  respectively and 1-block maps  $\tilde{\alpha} : \tilde{X} \rightarrow \tilde{Z}, \tilde{\beta} : \tilde{Y} \rightarrow \tilde{T}$  such  
 815 that the 1-block conjugacies  $\tilde{\varphi}, \tilde{\psi}$  making the diagram below on the right commutative  
 816 have inverses with memory  $m - 1$  and anticipation  $n$ .*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow \alpha & & \downarrow \beta \\ Z & \xrightarrow{\psi} & T \end{array}$$

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & & & Y \\ & \searrow & & & \nearrow \\ & \tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{Y} & \\ & \downarrow \tilde{\alpha} & & \downarrow \tilde{\beta} & \\ & \tilde{Z} & \xrightarrow{\tilde{\psi}} & \tilde{T} & \\ \downarrow \alpha & & & & \downarrow \beta \\ Z & \xrightarrow{\psi} & & & T \\ & \nearrow & & & \searrow \end{array}$$

*Proof.* Let  $A, B, C, D$  be the alphabets of  $X, Y, Z$  and  $T$  respectively. Let  $h : A \rightarrow B$  and  $k : C \rightarrow D$  be the 1-block substitutions such that  $\varphi = h_\infty$  and  $\psi = k_\infty$ . Set  $\tilde{A} = \{h(a_1)a_2 \mid a_1a_2 \in \mathcal{B}_2(X)\}$  and  $\tilde{C} = \{k(c_1)c_2 \mid c_1c_2 \in \mathcal{B}_2(Z)\}$ . Let  $\tilde{X}$  (resp.  $\tilde{Z}$ ) be the image of  $X$  (resp. of  $Z$ ) under the in-splitting map relative to  $h$  (resp.  $k$ ). Set  $\tilde{Y} = Y^{[2]}$ ,  $\tilde{B} = \mathcal{B}_2(Y)$ ,  $\tilde{T} = T^{[2]}$  and  $\tilde{D} = \mathcal{B}_2(T)$ . Define  $\tilde{\alpha}$  and  $\tilde{\beta}$  by

$$\tilde{\alpha}(h(a_1)a_2) = k\alpha(a_1)\alpha(a_2), \quad \tilde{\beta}(b_1b_2) = \beta(b_1)\beta(b_2)$$

and  $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$ ,  $\tilde{k} : \tilde{C} \rightarrow \tilde{D}$  by

$$\tilde{h}(h(a_1)a_2) = h(a_1)h(a_2), \quad \tilde{k}(k(c_1)c_2) = k(c_1)k(c_2)$$

817 Then the 1-block conjugacies  $\tilde{\varphi} = \tilde{h}_\infty$  and  $\tilde{\psi} = \tilde{k}_\infty$  satisfy the conditions of the state-  
818 ment.  $\square$

819 *Proof of Theorem 5.9.* Let  $\mathcal{A} = (G, \lambda)$  and  $\mathcal{A}' = (G', \lambda')$  be two automata with  $G =$   
820  $(Q, \mathcal{E})$  and  $G' = (Q', \mathcal{E}')$ . Let  $(\varphi, \psi)$  be a symbolic conjugacy from  $\mathcal{A}$  onto  $\mathcal{A}'$ . Replacing  
821  $\mathcal{A}$  and  $\mathcal{B}$  by some extension  $\mathcal{A}^{[m,n]}$  and  $\mathcal{B}^{[m,n]}$  we may reduce to the case where  $\varphi, \psi$  are  
822 1-block conjugacies. By using repeatedly Lemma 5.10, we may reduce to the case where  
823 the inverses of  $\varphi, \psi$  have memory 0. Using repeatedly the dual version of Lemma 5.10,  
824 we are reduced to the case where  $\varphi, \psi$  are renaming of the alphabets.  $\square$

825 The second step for the proof of Theorem 5.8 is the following statement.

826 **Proposition 5.11.** *Let  $\mathcal{A}, \mathcal{A}'$  be two essential automata. If  $\mathcal{A}'$  is an in-split of  $\mathcal{A}$ , the*  
827 *matrices  $M(\mathcal{A})$  and  $M(\mathcal{A}')$  are symbolic elementary equivalent.*

*Proof.* Set  $\mathcal{A} = (G, \lambda)$  and  $\mathcal{A}' = (G', \lambda')$ . Let  $\mathcal{A}' = \{f(a)b \mid ab \in \mathcal{B}_2(L_{\mathcal{A}})\}$  be the  
alphabet of  $\mathcal{A}'$  for a map  $f : A \rightarrow B$ . By Proposition 5.1, the symbolic in-splitting map  
from  $X_G$  onto  $X_{G'}$  is also an in-splitting map. Thus there is an in-merge  $(h, k)$  from  
 $G'$  onto  $G$  such that the in-split from  $\mathcal{A}$  onto  $\mathcal{A}'$  has the form  $(h_\infty^{-1}, \psi)$ . We define an  
alphabetic  $Q' \times Q$ -matrix  $R$  and a  $Q \times Q'$ -matrix  $S$  as follows. Let  $r, t \in Q'$  and let  
 $p = k(r)$ ,  $q = k(t)$ . Let  $e$  be an edge of  $\mathcal{A}'$  ending in  $r$ , and set  $a = \lambda(h(e))$ . Then the  
label of any edge going out of  $r$  is of the form  $f(a)b$  for some  $b \in A$ . Thus  $f(a)$  does not  
depend on  $e$  but only on  $r$ . We define a map  $\pi : Q' \rightarrow B$  by  $\pi(r) = f(a)$ . Then, we set

$$R_{rp} = \begin{cases} \pi(r) & \text{if } k(r) = p \\ 0 & \text{otherwise} \end{cases}, \quad S_{pt} = M(\mathcal{A})_{pq}$$

Let us verify that  $M(\mathcal{A}') = RS$  and  $M(\mathcal{A}) \leftrightarrow SR$ . We first have for  $r, t \in Q'$

$$(RS)_{rt} = \sum_{p \in Q} R_{rp}S_{pt} = \pi(r)M_{k(r)k(q)} = M(\mathcal{A}')_{rt}$$

and thus  $RS = M(\mathcal{A}')$ . Next, for  $p, q \in Q$

$$(SR)_{pq} = \sum_{p \in Q} R_{rp}S_{pt} = \sum_{t \in k^{-1}(q)} M(\mathcal{A})_{pq}\pi(t) = \sum_{a \in A} (M(\mathcal{A})_{pq}, a)af(a)$$

828 and thus  $SR \leftrightarrow M(\mathcal{A})$  using the bijection  $a \rightarrow af(a)$  between  $A$  and  $AB$ .  $\square$

829 *Proof of Theorem 5.8.* The condition is sufficient by Proposition 5.6. Conversely, let  
 830  $\mathcal{A}, \mathcal{A}'$  be two symbolic conjugate essential automata. By Theorem 5.9, we may assume  
 831 that  $\mathcal{A}'$  is a split of  $\mathcal{A}$ . We assume that  $\mathcal{A}'$  is an in-split of  $\mathcal{A}$ . By Proposition 5.11, the  
 832 adjacency matrices of  $\mathcal{A}$  and  $\mathcal{A}'$  are symbolic elementary equivalent.  $\square$

## 833 6 Special families of automata

834 In this section, we consider two particular families of automata: local automata and au-  
 835 tomata with finite delay. Local automata are closely related to shifts of finite type. The  
 836 main result is an embedding theorem (Theorem 6.4) related to Nasu's Masking Lemma  
 837 (Proposition 6.5). Automata with finite left and right delay are related to a class of shifts  
 838 called shifts of almost finite type (Proposition 6.10).

### 839 6.1 Local automata

840 Let  $m, n \geq 0$ . An automaton  $\mathcal{A} = (Q, E)$  is said to be  $(m, n)$ -local if whenever  $p \xrightarrow{u}$   
 841  $q \xrightarrow{v} r$  and  $p' \xrightarrow{u} q' \xrightarrow{v} r'$  are two paths with  $|u| = m$  and  $|v| = n$ , then  $q = q'$ . It is local  
 842 if it is  $(m, n)$ -local for some  $m, n$ .

843 **Example 6.1.** The automaton represented in Figure 14 is  $(3, 0)$ -local. Indeed, a simple  
 844 inspection shows that each of the six words of length 3 which are labels of paths uniquely  
 845 determines its terminal vertex. It is also  $(0, 3)$ -local. It is not  $(2, 0)$ -local (check the word  
 846  $ab$ ), but it is  $(2, 1)$ -local and also  $(1, 2)$ -local.

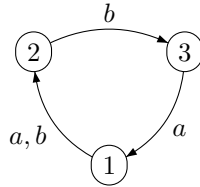


Figure 14. A local automaton.

847 We say that an automaton  $\mathcal{A} = (Q, E)$  is *contained* in an automaton  $\mathcal{A}' = (Q', E')$  if  
 848  $Q \subset Q'$  and  $E \subset E'$ . We note that if  $\mathcal{A}$  is contained in  $\mathcal{A}'$  and if  $\mathcal{A}'$  is local, then  $\mathcal{A}$  is  
 849 local.

850 **Proposition 6.1.** *An essential automaton  $\mathcal{A}$  is local if and only if the map  $\lambda_{\mathcal{A}} : X_{\mathcal{A}} \rightarrow L_{\mathcal{A}}$   
 851 is a conjugacy from  $X_{\mathcal{A}}$  onto  $L_{\mathcal{A}}$ .*

852 *Proof.* Suppose first that  $\mathcal{A}$  is  $(m, n)$ -local. Consider an  $m+1+n$ -block  $w = uav$  of  $L_{\mathcal{A}}$ ,  
 853 with  $|u| = m$ ,  $|v| = n$ . All finite paths of  $\mathcal{A}$  labeled  $w$  have the form  $r \xrightarrow{u} p \xrightarrow{a} q \xrightarrow{v} s$   
 854 and share the same edge  $p \xrightarrow{a} q$ . This shows that  $\lambda_{\mathcal{A}}$  is injective and that  $\lambda_{\mathcal{A}}^{-1}$  is a map  
 855 with memory  $m$  and anticipation  $n$ .

Conversely, assume that  $\lambda_{\mathcal{A}}^{-1}$  exists, and that it has memory  $m$  and anticipation  $n$ . We show that  $\mathcal{A}$  is  $(m + 1, n)$ -local. Let

$$r \xrightarrow{u} p \xrightarrow{a} q \xrightarrow{v} s \quad \text{and} \quad r' \xrightarrow{u} p' \xrightarrow{a} q' \xrightarrow{v} s'$$

856 and be two paths of length  $m + 1 + n$ , with  $|u| = m$ ,  $|v| = n$  and  $a$  a letter. Since  $\mathcal{A}$   
 857 is essential, there exist two biinfinite paths which contain these finite paths, respectively.  
 858 Since  $\lambda_{\mathcal{A}}^{-1}$  has memory  $m$  and anticipation  $n$ , the blocks  $uav$  of the biinfinite words  
 859 carried by these paths are mapped by  $\lambda_{\mathcal{A}}^{-1}$  onto the edges  $p \xrightarrow{a} q$  and  $p' \xrightarrow{a} q'$  respectively.  
 860 This shows that  $p = p'$  and  $q = q'$ .  $\square$

861 The next statement is Proposition 10.3.10 in [4].

862 **Proposition 6.2.** *The following conditions are equivalent for a strongly connected finite*  
 863 *automaton  $\mathcal{A}$ .*

- 864 (i)  $\mathcal{A}$  is local;  
 865 (ii) distinct cycles have distinct labels.

866 Two cycles in this statement are considered to be distinct if, viewed as paths, they are  
 867 distinct.

868 The following result shows the strong connection between shifts of finite type and  
 869 local automata. It gives an effective method to verify whether or not a shift space is of  
 870 finite type.

871 **Proposition 6.3.** *A shift space (resp. an irreducible shift space) is of finite type if and*  
 872 *only if its Krieger automaton (resp. its Fischer automaton) is local.*

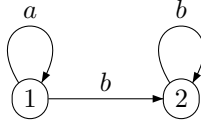
873 *Proof.* Let  $X = X^{(W)}$  for a finite set  $W \subset A^*$ . We may assume that all words of  $W$   
 874 have the same length  $n$ . Let  $\mathcal{A} = (Q, i, Q)$  be the  $(n, 0)$ -local deterministic automaton  
 875 defined as follows. The set of states is  $Q = A^n \setminus W$  and there is an edge  $(u, a, v)$  for  
 876 every  $u, v \in Q$  and  $a \in A$  such that  $ua \in Av$ . Then  $\mathcal{A}$  recognizes the set  $\mathcal{B}(X)$ . Since the  
 877 reduction of a local automaton is local, the minimal automaton of  $\mathcal{B}(X)$  is local. Since  
 878 the Krieger automaton of  $X$  is contained in the minimal automaton of  $\mathcal{B}(X)$ , it is local.  
 879 If  $X$  is irreducible, then its Fischer automaton is also local since it is contained in the  
 880 Krieger automaton.

881 Conversely, Proposition 6.1 implies that a shift space recognized by a local automaton  
 882 is conjugate to a shift of finite type and thus is of finite type.  $\square$

883 **Example 6.2.** Let  $X$  be the shift of finite type on the alphabet  $A = \{a, b\}$  defined by  
 884 the forbidden factor  $ba$ . The Krieger automaton of  $X$  is represented on Figure 15. It is  
 885  $(1, 0)$ -local.

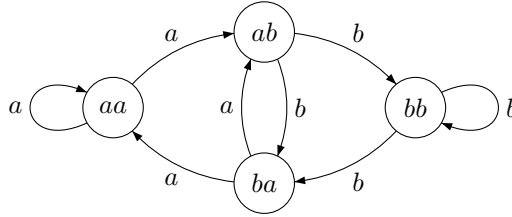
886 For  $m, n \geq 0$ , the *standard*  $(m, n)$ -local automaton is the automaton with states the  
 887 set of words of length  $m + n$  and edges the triples  $(uv, a, u'v')$  for  $u, u' \in A^m$ ,  $a \in A$   
 888 and  $v, v' \in A^n$  such that for some letters  $b, c \in A$ , one has  $uvc = bu'v'$  and  $a$  is the first  
 889 letter of  $vc$ .

890 The standard  $(m, 0)$ -local automaton is also called the De Bruijn automaton of order  
 891  $m$ .



**Figure 15.** The Krieger automaton of a reducible shift of finite type.

892 **Example 6.3.** The standard  $(1, 1)$ -local automaton on the alphabet  $\{a, b\}$  is represented  
on Figure 16.



**Figure 16.** The standard  $(1, 1)$ -local automaton.

893

894 **Complete automata.** An automaton  $\mathcal{A}$  on the alphabet  $A$  is called *complete* if any word  
895 on  $A$  is the label of some path in  $\mathcal{A}$ . As an example, the standard  $(m, n)$ -local automaton  
896 is complete.

897 The following result is from [3].

898 **Theorem 6.4.** Any local automaton is contained in a complete local automaton.

899 The proof relies on the following version of the masking lemma.

900 **Proposition 6.5 (Masking lemma).** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two automata and assume that  $M(\mathcal{A})$   
901 and  $M(\mathcal{B})$  are elementary equivalent. If  $\mathcal{B}$  is contained in an automaton  $\mathcal{B}'$ , then  $\mathcal{A}$  is  
902 contained in some automaton  $\mathcal{A}'$  which is conjugate to  $\mathcal{B}'$ .

*Proof.* Let  $\mathcal{A} = (Q, E)$ ,  $\mathcal{B} = (R, F)$  and  $\mathcal{B}' = (R', F')$ . Let  $D$  be an  $R \times Q$  nonnegative  
integral matrix and  $N$  be an alphabetic  $Q \times R$  matrix such that  $M(\mathcal{A}) = ND$  and  $M(\mathcal{B}) =$   
 $DN$ . Set  $Q' = Q \cup (F' \setminus F)$ . Let  $D'$  be the  $R' \times Q'$  nonnegative integral matrix defined  
for  $r \in R'$  and  $u \in Q'$  by

$$D'_{ru} = \begin{cases} D_{ru} & \text{if } r \in R, u \in Q \\ 1 & \text{if } u \in F' \setminus F \text{ and } u \text{ starts in } r \\ 0 & \text{otherwise} \end{cases}$$

Let  $N'$  be the alphabetic  $Q' \times R'$  matrix defined for  $a \in A$  for  $u \in Q'$  and  $s \in R'$  by

$$(N'_{us}, a) = \begin{cases} (N_{us}, a) & \text{if } u \in Q, s \in R \\ 1 & \text{if } u \in F' \setminus F \text{ and } u \text{ is labeled with } a \text{ and ends in } s, \\ 0 & \text{otherwise.} \end{cases}$$

903 Then  $N'D'$  is the adjacency matrix of an automaton  $\mathcal{A}'$ . By definition,  $\mathcal{A}'$  contains  $\mathcal{A}$  and  
 904 it is conjugate to  $\mathcal{B}'$  by Proposition 3.10.  $\square$

905 We illustrate the proof of Proposition 6.5 by the following example.

**Example 6.4.** Consider the automata  $\mathcal{A}$  and  $\mathcal{B}$  given of Figure 17. The automaton  $\mathcal{A}$  is the local automaton of Example 6.1. The automaton  $\mathcal{B}$  is an in-split of  $\mathcal{A}$ . Indeed, we have  $M(\mathcal{A}) = ND$ ,  $M(\mathcal{B}) = DN$  with

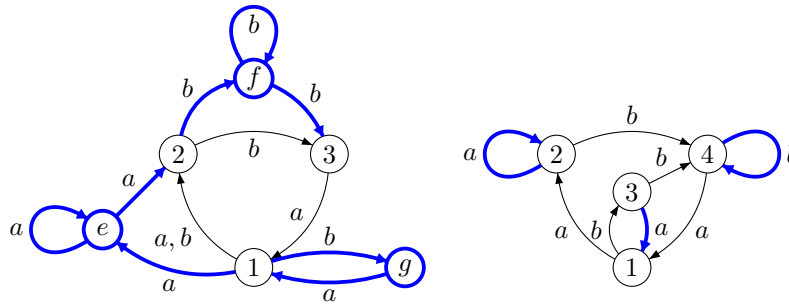
$$N = \begin{bmatrix} 0 & a & b & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



**Figure 17.** The automaton  $\mathcal{B}$  on the right is an in-split of the local automaton  $\mathcal{A}$  on the left.

906

We have represented on the right of Figure 18 the completion of  $\mathcal{B}$  as a complete local automaton with the same number of states. On the left, the construction of the proof of Proposition 6.5 has been carried on to produce a local automaton containing  $\mathcal{A}$ . In terms



**Figure 18.** The automata  $\mathcal{A}'$  and  $\mathcal{B}'$ . Additional edges are drawn thick.

of adjacency matrices, we have  $M(\mathcal{A}') = N'D'$ ,  $M(\mathcal{B}') = D'N'$  with

$$N' = \begin{bmatrix} 0 & a & b & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix}, \quad D' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

907 *Proof of Theorem 6.4.* Since  $\mathcal{A}$  is local, the map  $\lambda_{\mathcal{A}}$  is a conjugacy from  $X_{\mathcal{A}}$  to  $L_{\mathcal{A}}$ . Let  
 908  $(m, n)$  be the memory and anticipation of  $\lambda_{\mathcal{A}}^{-1}$ . There is a sequence  $(\mathcal{A}_0, \dots, \mathcal{A}_{m+n})$   
 909 of automata such that  $\mathcal{A}_0 = \mathcal{A}$ , each  $\mathcal{A}_i$  is a split or a merge of  $\mathcal{A}_{i-1}$  and  $\mathcal{A}_{n+m}$  is  
 910 contained in the standard  $(n+m)$ -local automaton. Applying iteratively Proposition 6.5,  
 911 we obtain that  $\mathcal{A}$  is contained in an automaton which is conjugate to the standard  $(m, n)$ -  
 912 local automaton and which is thus complete.  $\square$

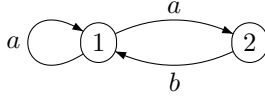
## 913 6.2 Automata with finite delay

An automaton is said to have *right delay*  $d \geq 0$  if for any pair of paths

$$p \xrightarrow{a} q \xrightarrow{z} r, \quad p \xrightarrow{a} q' \xrightarrow{z} r'$$

914 with  $a \in A$ , if  $|z| = d$ , then  $q = q'$ . Thus a deterministic automaton has right delay  
 915 0. An automaton has *finite right delay* if it has right delay  $d$  for some (finite) integer  $d$ .  
 916 Otherwise, it is said to have *infinite right delay*.

917 **Example 6.5.** The automaton represented on Figure 19 has right delay 1.



**Figure 19.** A automaton with right delay 1

918 **Proposition 6.6.** *An automaton has infinite right delay if and only if there exist paths*  
 919  $p \xrightarrow{v} q \xrightarrow{u} q$  and  $p \xrightarrow{v} q' \xrightarrow{u} q'$  with  $q \neq q'$  and  $|u| > 0$ .

920 The following statement is Proposition 5.1.11 in [19].

921 **Proposition 6.7.** *An automaton has finite right delay if and only if it is conjugate to a*  
 922 *deterministic automaton.*

923 In the same way the automaton is said to have *left delay*  $d \geq 0$  if for any pair of paths  
 924  $p \xrightarrow{z} q \xrightarrow{a} r$  and  $p' \xrightarrow{z} q' \xrightarrow{a} r$  with  $a \in A$ , if  $|z| = d$ , then  $q = q'$ .

925 **Corollary 6.8.** *If two automata are conjugate, and if one has finite right (left) delay, then*  
 926 *the other also has.*

927 **Proposition 6.9.** *An essential  $(m, n)$ -local automaton has right delay  $n$  and left delay*  
 928  *$m$ .*

929 *Proof.* Let  $p \xrightarrow{a} q \xrightarrow{z} r$  and  $p \xrightarrow{a} q' \xrightarrow{z} r'$  be two paths with  $a \in A$  and  $|z| = n$ . Since  
 930  $\mathcal{A}$  is essential there is a path  $u \xrightarrow{y} p$  of length  $m$  in  $\mathcal{A}$ . Since  $\mathcal{A}$  is  $(m, n)$ -local, we have  
 931  $q = q'$ . Thus  $\mathcal{A}$  has right delay  $n$ . The proof for the left delay  $m$  is symmetrical.  $\square$

932 A shift space is said to have *almost finite type* if it can be recognized by a strongly  
 933 connected automaton with both finite left and finite right delay.

934 An irreducible shift of finite type is also of almost finite type since a local automaton  
 935 has finite right and left delay by Proposition 6.9.

936 **Example 6.6.** The even shift has almost finite type. Indeed, the automaton of Figure 5 on  
 937 the right has right and left delay 0.

938 The following result is from [20].

939 **Proposition 6.10.** *An irreducible shift space is of almost finite type if and only if its*  
 940 *Fischer automaton has finite left delay.*

941 *Proof.* The condition is obviously sufficient. Conversely, let  $X$  be a shift of almost finite  
 942 type. Assume the Fischer automaton  $\mathcal{A} = (Q, E)$  of  $X$  does not have finite left delay.  
 943 Let, in view of Proposition 6.6  $u, v \in A^*$  and  $p, q, q' \in Q$  with  $q \neq q'$  be such that  
 944  $q \cdot u = q, q' \cdot u = q'$  and  $p = q \cdot v = q' \cdot v$ . Since  $\mathcal{A}$  is strongly connected, there is a word  
 945  $w$  such that  $p \cdot w = q$ .

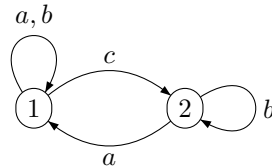
Let  $\mathcal{B} = (R, F)$  be an automaton with finite right and left delay which recognizes  
 $X$ . By Proposition 6.7, we may assume that  $\mathcal{B}$  is deterministic. Let  $\varphi : R \rightarrow Q$  be a  
 reduction from  $\mathcal{B}$  onto  $\mathcal{A}$ . Since  $R$  is finite, there is an  $x \in u^+$  such that  $r \cdot x = r \cdot x^2$   
 for all  $r \in R$  (this means that the map  $r \mapsto r \cdot x$  is idempotent; such a word exists since  
 each element in the finite transition semigroup of the automaton  $\mathcal{B}$  has a power which is  
 an idempotent). Set

$$S = R \cdot x, \quad T = \varphi^{-1}(q) \cap S, \quad T' = \varphi^{-1}(q') \cap S$$

946 Since  $q \neq q'$ , we have  $T \cap T' = \emptyset$ . For any  $t \in T$ , we have  $\varphi(t \cdot vw) = q$  and thus  
 947  $t \cdot vwx \in T$ . For  $t, t' \in T$  with  $t \neq t'$ , we cannot have  $t \cdot vwx = t' \cdot vwx$  since otherwise  
 948  $\mathcal{B}$  would have infinite left delay. Thus the map  $t \mapsto t \cdot vwx$  is a bijection of  $T$ .

949 Let  $t' \in T'$ . Since  $\varphi(t' \cdot vw) = q$ , we have  $t' \cdot vwx \in T$ . Since the action of  
 950  $vwx$  induces a permutation on  $T$ , there exists  $t \in T$  such that  $t \cdot vwx = t' \cdot vwx$ . This  
 951 contradicts the fact that  $\mathcal{B}$  has finite left delay.  $\square$

952 **Example 6.7.** The deterministic automaton represented on Figure 20 has infinite left  
 953 delay. Indeed, there are paths  $\dots 1 \xrightarrow{b} 1 \xrightarrow{a} 1$  and  $\dots 2 \xrightarrow{b} 2 \xrightarrow{a} 1$ . Since this automaton  
 cannot be reduced,  $X = L_{\mathcal{A}}$  is not of almost finite type.



**Figure 20.** An automaton with infinite left delay

## 7 Syntactic invariants

955

956 We introduce in this section the syntactic graph of an automaton. It uses the Green rela-  
 957 tions in the transition semigroup of the automaton. We show that the syntactic graph is an  
 958 invariant for symbolic conjugacy (Theorem 7.4). The proof uses bipartite automata.

959 The final subsection considers the characterization of sofic shifts with respect to the  
 960 families of ordered semigroups known as pseudovarieties.

961

### 7.1 The syntactic graph

962 Let  $\mathcal{A} = (Q, E)$  be a deterministic automaton on the alphabet  $A$ . Each word  $w \in A^*$   
 963 defines a partial map denoted by  $\varphi_{\mathcal{A}}(w)$  from  $Q$  to  $Q$  which maps  $p \in Q$  to  $q \in Q$  if  
 964  $p \cdot w = q$ . The transition semigroup of  $\mathcal{A}$ , already defined in Section 4.2, is the image of  
 965  $A^+$  by the morphism  $\varphi_{\mathcal{A}}$  (in this subsection, we will not use the order on the transition  
 966 semigroup).

967 We give a short summary of *Green relations* in a semigroup (see [17] for example).  
 968 Let  $S$  be a semigroup and let  $S^1 = S \cup 1$  be the monoid obtained by adding an identity  
 969 to  $S$ . Two elements  $s, t$  of  $S$  are  $\mathcal{R}$ -equivalent if  $sS^1 = tS^1$ . They are  $\mathcal{L}$ -equivalent  
 970 if  $S^1s = S^1t$ . It is a classical result (see [17]) that  $\mathcal{LR} = \mathcal{RL}$ . Thus  $\mathcal{LR} = \mathcal{RL}$   
 971 is an equivalence on the semigroup  $S$  called the  $\mathcal{D}$ -equivalence. A class of the  $\mathcal{R}, \mathcal{L}$  or  $\mathcal{D}$ -  
 972 equivalence is called an  $\mathcal{R}, \mathcal{L}$  or  $\mathcal{D}$ -class. An *idempotent* of  $S$  is an element  $e$  such that  
 973  $e^2 = e$ . A  $\mathcal{D}$ -class is *regular* if it contains an idempotent. The equivalence  $\mathcal{H}$  is defined  
 974 as  $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ . It is classical result that the  $\mathcal{H}$ -class of an idempotent is a group. The  
 975  $\mathcal{H}$ -class of idempotents in the same  $\mathcal{D}$ -class are isomorphic groups. The *structure group*  
 976 of a regular  $\mathcal{D}$ -class is any of the  $\mathcal{H}$ -classes of an idempotent of the  $\mathcal{D}$ -class.

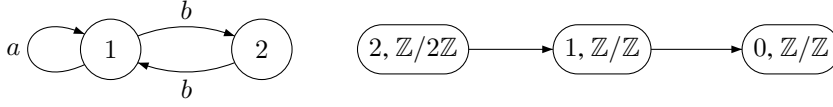
977 When  $S$  is a semigroup of partial maps from a set  $Q$  into itself, each element of  $S$   
 978 has a rank which is the cardinality of its image. The elements of a  $\mathcal{D}$ -class all have the  
 979 same rank, which is called the *rank* of the  $\mathcal{D}$ -class. There is at most one element of rank  
 980 0 which is the *zero* of the semigroup  $S$  and is denoted 0.

981 A *fixpoint* of a partial map  $s$  from  $Q$  into itself is an element  $q$  such that the image of  
 982  $q$  by  $s$  is  $q$ . The rank of an idempotent is equal to the number of its fixpoints. Indeed, in  
 983 this case, every element in the image is a fixpoint.

984 The preorder  $\leq_{\mathcal{J}}$  on  $S$  is defined by  $s \leq_{\mathcal{J}} t$  if  $S^1sS^1 \subset S^1tS^1$ . Two elements  
 985  $s, t \in S$  are  $\mathcal{J}$ -equivalent if  $S^1sS^1 = S^1tS^1$ . One has  $\mathcal{D} \subset \mathcal{J}$  and it is a classical  
 986 result that in a finite semigroup  $\mathcal{D} = \mathcal{J}$ . The preorder  $\leq_{\mathcal{J}}$  induces a partial order on the  
 987  $\mathcal{D}$ -classes, still denoted  $\leq_{\mathcal{J}}$ .

988 We associate with  $\mathcal{A}$  a labeled graph  $G(\mathcal{A})$  called its *syntactic graph*. The vertices of  
 989  $G(\mathcal{A})$  are the regular  $\mathcal{D}$ -classes of the transition semigroup of  $\mathcal{A}$ . Each vertex is labeled by  
 990 the rank of the  $\mathcal{D}$ -class and its structure group. There is an edge from the vertex associated  
 991 with a  $\mathcal{D}$ -class  $D$  to the vertex associated to a  $\mathcal{D}$ -class  $D'$  if and only if  $D \geq_{\mathcal{J}} D'$ .

992 **Example 7.1.** The automaton  $\mathcal{A}$  of Figure 21 on the left is the Fischer automaton of  
 993 the even shift (Example 4.3). The semigroup of transitions of  $\mathcal{A}$  has 3 regular  $\mathcal{D}$ -classes  
 994 of ranks 2 (containing  $\varphi_{\mathcal{A}}(b)$ ), 1 (containing  $\varphi_{\mathcal{A}}(a)$ ), and 0 (containing  $\varphi_{\mathcal{A}}(aba)$ ). Its  
 995 syntactic graph is represented on the right.



**Figure 21.** The syntactic graph of the even shift

996 The following result shows that one may reduce to the case of essential automata.

997 **Proposition 7.1.** *The syntactic graphs of an automaton and of its essential part are iso-*  
 998 *morphic.*

999 *Proof.* Let  $\mathcal{A} = (Q, E)$  be a deterministic automaton on the alphabet  $A$  and let  $\mathcal{A}' =$   
 1000  $(Q', E')$  be its essential part. Let  $w \in A^+$  be such that  $e = \varphi_{\mathcal{A}}(w)$  is an idempotent.  
 1001 Then any fixpoint of  $e$  is in  $Q'$  and thus  $e' = \varphi_{\mathcal{A}'}(w)$  an idempotent of the same rank as  
 1002  $e$ . This shows that  $G(\mathcal{A})$  and  $G(\mathcal{A}')$  are isomorphic.  $\square$

1003 The following result shows that the syntactic graph characterizes irreducible shifts of  
 1004 finite type.

1005 **Proposition 7.2.** *A sofic shift (resp. an irreducible sofic shift) is of finite type if and only*  
 1006 *if the syntactic graph of its Krieger automaton (resp. its Fischer automaton) has nodes of*  
 1007 *rank at most 1.*

1008 In the proof, we use the following classical property of finite semigroups.

1009 **Proposition 7.3.** *Let  $S$  be a finite semigroup and let  $J$  be an ideal of  $S$ . The following*  
 1010 *conditions are equivalent.*

- 1011 (i) *All idempotents of  $S$  are in  $J$ .*  
 1012 (ii) *There exists an integer  $n \geq 1$  such that  $S^n \subset J$ .*

1013 *Proof.* Assume that (i) holds. Let  $n = \text{Card}(S) + 1$  and let  $s = s_1 s_2 \cdots s_n$  with  $s_i \in S$ .  
 1014 Then there exist  $i, j$  with  $1 \leq i < j \leq n$  such that  $s_1 s_2 \cdots s_i = s_1 s_2 \cdots s_i \cdots s_j$ . Let  
 1015  $t, u \in S^1$  be defined by  $t = s_1 \cdots s_i$  and  $u = s_{i+1} \cdots s_j$ . Since  $tu = t$ , we have  $tu^k = t$   
 1016 for all  $k \geq 1$ . Since  $S$  is finite, there is a  $k \geq 1$  such that  $u^k$  is idempotent and thus  
 1017  $u^k \in J$ . This implies that  $t \in J$  and thus  $s \in J$ . Thus (ii) holds.

1018 It is clear that (ii) implies (i).  $\square$

1019 *Proof of Proposition 7.2.* Let  $X$  be a shift space (resp. an irreducible shift space), let  $\mathcal{A}$  be  
 1020 its Krieger automaton (resp. its Fischer automaton) and let  $S$  be the transition semigroup  
 1021 of  $\mathcal{A}$ .

1022 If  $X$  is of finite type, by Proposition 6.3, the automaton  $\mathcal{A}$  is local. Any idempotent in  
 1023  $S$  has rank 1 and thus the condition is satisfied.

1024 Conversely, assume that the graph  $G(\mathcal{A})$  has nodes of rank at most 1. Let  $J$  be the  
 1025 ideal of  $S$  formed of the elements of rank at most 1. Since all idempotents of  $S$  belong  
 1026 to  $J$ , by Proposition 7.3, the semigroup  $S$  satisfies  $S^n = J$  for some  $n \geq 1$ . This  
 1027 shows that for any sufficiently long word  $x$ , the map  $\varphi_{\mathcal{A}}(x)$  has rank at most 1. Thus for  
 1028  $p, q, r, s \in Q$ , if  $p \cdot x = r$  and  $q \cdot x = s$  then  $r = s$ . This implies that  $\mathcal{A}$  is  $(n, 0)$ -local.  $\square$

1029 The following result is from [2].

1030 **Theorem 7.4.** *Two symbolic conjugate automata have isomorphic syntactic graphs.*

1031 We use the following intermediary result.

1032 **Proposition 7.5.** *Let  $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$  be a bipartite automaton. The syntactic graphs of*  
1033  *$\mathcal{A}, \mathcal{A}_1$  and  $\mathcal{A}_2$  are isomorphic.*

1034 *Proof.* Let  $Q = Q_1 \cup Q_2$  and  $A = A_1 \cup A_2$  be the partitions of the set of of states  
1035 and of the alphabet of  $\mathcal{A}$  corresponding to the decomposition  $(\mathcal{A}_1, \mathcal{A}_2)$ . Set  $B_1 = A_1A_2$   
1036 and  $B_2 = A_2A_1$ . The semigroups  $S_1 = \varphi_{\mathcal{A}_1}(B_1^+)$  and  $S_2 = \varphi_{\mathcal{A}_2}(B_2^+)$  are included  
1037 in the semigroup  $S = \varphi_{\mathcal{A}}(A^+)$ . Thus the Green relations of  $S$  are refinements of the  
1038 corresponding Green relations in  $S_1$  or in  $S_2$ . Any idempotent  $e$  of  $S$  belongs either to  
1039  $S_1$  or to  $S_2$ . Indeed, if  $e = 0$  then  $e$  is in  $S_1 \cap S_2$ . Otherwise, it has at least one fixpoint  
1040  $p \in Q_1 \cup Q_2$ . If  $p \in Q_1$ , then  $e$  is in  $\varphi_{\mathcal{A}}(B_1^+)$  and thus  $e \in S_1$ . Similarly if  $p \in Q_2$  then  
1041  $e \in S_2$ .

Let  $e$  be an idempotent in  $S_1$  and let  $e = \varphi_{\mathcal{A}}(u)$ . Since  $u \in B_1^+$ , we have  $u = au'$   
with  $a \in A_1$  and  $u' \in B_2^*A_2$ . Let  $v = u'a$ . Then  $f = \varphi_{\mathcal{A}}(v)^2$  is idempotent. Indeed, we  
have

$$\varphi_{\mathcal{A}}(v^3) = \varphi_{\mathcal{A}}(u'au'au'a) = \varphi_{\mathcal{A}}(u'uua) = \varphi_{\mathcal{A}}(u'ua) = \varphi_{\mathcal{A}}(v^2)$$

1042 Moreover  $e, f$  belong the same  $\mathcal{D}$ -class. Similarly, if  $e \in S_2$ , there is an idempotent in  
1043  $S_1$  which is  $\mathcal{D}$  equivalent to  $e$ . This shows that a regular  $\mathcal{D}$ -class of  $\varphi_{\mathcal{A}}(A^+)$  contains  
1044 idempotents in  $S_1$  and in  $S_2$ .

Finally, two elements of  $S_1$  which are  $\mathcal{D}$ -equivalent in  $S$  are also  $\mathcal{D}$ -equivalent in  $S_1$ .  
Indeed, let  $s, t \in S_1$  be such that  $s\mathcal{R}Lt$ . Let  $u, u', v, v' \in S$  be such that

$$suu' = s, \quad v'vt = t, \quad su = tv$$

1045 in such a way that  $s\mathcal{R}su$  and  $vt\mathcal{L}t$ . Then  $su = tv$  implies that  $u, v$  are both in  $S_1$ .  
1046 Similarly  $suu' = s$  and  $v'vt = t$  imply that  $u'v' \in S_1$ . Thus  $s\mathcal{D}t$  in  $S_1$ . This shows  
1047 that a regular  $\mathcal{D}$  class  $D$  of  $S$  contains exactly one  $\mathcal{D}$ -class  $D_1$  of  $S_1$  (resp.  $D_2$  of  $S_2$ ).  
1048 Moreover, an  $\mathcal{H}$ -class of  $D_1$  is also an  $\mathcal{H}$ -class of  $D$ .

1049 Thus the three syntactic graphs are isomorphic.  $\square$

1050 *Proof of Theorem 7.4.* Let  $\mathcal{A} = (Q, E)$  and  $\mathcal{B} = (R, F)$  be two symbolic conjugate  
1051 automata on the alphabets  $A$  and  $B$ , respectively. By the Decomposition Theorem (The-  
1052 orem 5.9), we may assume that the symbolic conjugacy is a split or a merge. Assume that  
1053  $\mathcal{A}'$  is an in-split of  $\mathcal{A}$ . By Proposition 7.1, we may assume that  $\mathcal{A}$  and  $\mathcal{A}'$  are essential. By  
1054 Proposition 5.11, the adjacency matrices of  $\mathcal{A}$  and  $\mathcal{A}'$  are symbolic elementary equivalent.

1055 By Proposition 5.5, there is a bipartite automaton  $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$  such that  $M(\mathcal{C}_1), M(\mathcal{C}_2)$   
1056 are similar to  $M(\mathcal{A}), M(\mathcal{B})$  respectively. By Proposition 7.5, the syntactic graphs of  
1057  $\mathcal{C}_1, \mathcal{C}_2$  are isomorphic. Since automata with similar adjacency matrices have obviously  
1058 isomorphic syntactic graphs, the result follows.  $\square$

1059 A refinement of the syntactic graph which is also invariant by flow equivalence has  
1060 been introduced in [9]. The vertices of the graph are the *idempotent-bound  $\mathcal{D}$*  classes,

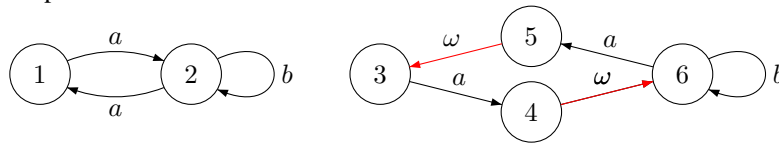
1061 where an element  $s$  of a semigroup  $S$  is called idempotent-bound if there exist idempoten-  
 1062 ts  $e, f \in S$  such that  $s = esf$ . The elements of a regular  $\mathcal{D}$ -class are idempotent-  
 1063 bound.

1064 **Flow equivalent automata.** Let  $\mathcal{A}$  be an automaton on the alphabet  $A$  and let  $G$  be its  
 1065 underlying graph. An *expansion* of  $\mathcal{A}$  is a pair  $(\varphi, \psi)$  of a graph expansion of  $G$  and a  
 symbol expansion of  $L_{\mathcal{A}}$  such that the diagram below is commutative. The inverse of an

$$\begin{array}{ccc} X_{\mathcal{A}} & \xrightarrow{\varphi} & X_{\mathcal{B}} \\ \downarrow \lambda_{\mathcal{A}} & & \downarrow \lambda_{\mathcal{B}} \\ L_{\mathcal{A}} & \xrightarrow{\psi} & L_{\mathcal{B}} \end{array}$$

1066  
 1067 automaton expansion is called a contraction.

1068 **Example 7.2.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be the automata represented on Figure 22. The second au-  
 tomaton is an expansion of the first one.



**Figure 22.** An automaton expansion

1069

1070 The *flow equivalence* of automata is the equivalence generated by symbolic conjuga-  
 1071 ties, expansions and contractions.

1072 Theorem 7.4 has been generalized by Costa and Steinberg [12] to flow equivalence.

1073 **Theorem 7.6.** *Two flow equivalent automata have isomorphic syntactic graphs.*

1074 **Example 7.3.** The syntactic graphs of the automata  $\mathcal{A}, \mathcal{B}$  of Example 5.2 are isomorphic  
 1075 to the syntactic graph of the Fischer automaton  $\mathcal{C}$  of the even shift. Note that the automata  
 1076  $\mathcal{A}, \mathcal{B}$  are not flow equivalent to  $\mathcal{C}$ . Indeed, the edge shifts  $X_{\mathcal{A}}, X_{\mathcal{B}}$  on the underlying  
 1077 graphs of the automata  $\mathcal{A}, \mathcal{B}$  are flow equivalent to the full shift on 3 symbols while the  
 1078 edge shift  $X_{\mathcal{C}}$  is flow equivalent to the full shift on 2 symbols. Thus the converse of  
 1079 Theorem 7.6 is false.

1080

## 7.2 Pseudovarieties

1081 In this subsection, we will see how one can formulate characterizations of some classes  
 1082 of sofic shifts by means of properties of their syntactic semigroup. In order to formulate  
 1083 these syntactic characterizations of sofic shifts, we introduce the notion of pseudovariety

1084 of ordered semigroups. For a systematic exposition, see the original articles [25], [27], or  
 1085 the surveys in [26] or [24].

1086 A morphism of ordered semigroups  $\varphi$  from  $S$  into  $T$  is an order compatible semigroup  
 1087 morphism, that is such that  $s \leq s'$  implies  $\varphi(s) \leq \varphi(s')$ . An ordered subsemigroup of  $S$   
 1088 is a subsemigroup equipped with the restriction of the preorder.

1089 A *pseudovariety* of finite ordered semigroups is a class of ordered semigroups closed  
 1090 under taking ordered subsemigroups, finite direct products and image under morphisms  
 1091 of ordered semigroups.

1092 Let  $V$  be a pseudovariety of ordered semigroups. We say that a semigroup  $S$  is *locally*  
 1093 *in*  $V$  if all the submonoids of  $S$  are in  $V$ . The class of these semigroups is a pseudovariety  
 1094 of ordered semigroups.

1095 The following result is due to Costa [10].

1096 **Theorem 7.7.** *Let  $V$  be a pseudovariety of finite ordered semigroups containing the class*  
 1097 *of commutative ordered monoids such that every element is idempotent and greater than*  
 1098 *the identity. The class of shifts whose syntactic semigroup is locally in  $V$  is invariant*  
 1099 *under conjugacy.*

1100 The following statements give examples of pseudovarieties satisfying the above con-  
 1101 dition.

1102 **Proposition 7.8.** *An irreducible shift space is of finite type if and only if its syntactic*  
 1103 *semigroup is locally commutative.*

1104 An *inverse semigroup* is a semigroup which can be represented as a semigroup of  
 1105 partial one-to-one maps from a finite set  $Q$  into itself. The family of inverse semigroups  
 1106 does not form a variety (it is not closed under homomorphic image. However, according  
 1107 to Ash's theorem [1], the variety generated by inverse semigroups is characterized by the  
 1108 property that the idempotents commute. Using this result, the following result is proved  
 1109 in [10].

1110 **Theorem 7.9.** *An irreducible shift space is of almost finite type if and only if its syntactic*  
 1111 *semigroup is locally in the pseudovariety generated by inverse semigroups.*

1112 The fact that shifts of almost finite type satisfy this condition was proved in [2]. The  
 1113 converse was conjectured in the same paper.

1114 In [12] it is shown that this result implies that the class of shifts of almost finite type is  
 1115 invariant under flow equivalence. This is originally from [15].

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<sup>1172</sup> **Abstract.** This chapter presents some of the links between automata theory and symbolic dynamics.  
<sup>1173</sup> The emphasis is on two particular points. The first one is the interplay between some particular  
<sup>1174</sup> classes of automata, such as local automata and results on embeddings of shifts of finite type. The  
<sup>1175</sup> second one is the connection between syntactic semigroups and the classification of sofic shifts up  
<sup>1176</sup> to conjugacy.

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