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Recognizability in S-adic shifts

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Abstract

We investigate questions related to the notion of recognizability of sequences of morphisms, a generalization of Mossé's Theorem. We consider the most general class of morphisms including ones with erasable letters. The main result states that a sequence of morphisms with finite alphabet rank is eventually recognizable for aperiodic points, improving and simplifying a result of Berthé et al. (2019). This also provides a new simple proof for the recognizability of a single morphism on its shift space. The main ingredient of the proof is elementary morphisms.

1 Introduction

Given a bi-infinite sequence in $A^{\mathbb{Z}}$ and a morphism (also called a substitution) $\sigma\colon A^* \to A^*$, recognizability is a form of injectivity of σ that allows one to uniquely desubstitute y to another sequence x, i.e., to express y as a concatenation of substitution words dictated by the letters in x. The sequences y and x are traditionally required to be in the shift space $X(\sigma)$, which is the set of bi-infinite sequences (also called points) whose finite factors are factors of $\sigma^n(a)$ for some integer n and some letter a in A.

By Mossé's Theorem [12, 13], every aperiodic primitive morphism σ is recognizable in the shift $X(\sigma)$; see the precise definitions in Section 3. This surprising result was initially formulated (in an incorrect way) by [11]; see [9] on the genesis of the theorem and its possible variants. It was further generalized by Bezuglyi, Kwiatkowski, and Medynets [3], who proved that every aperiodic non-erasing morphism σ is recognizable in $X(\sigma)$. Next, it was proved by Berthé, Thuswaldner, Yassawi, and the fourth author [2] that every non-erasing morphism σ is recognizable in $X(\sigma)$ for aperiodic points, and the first three authors proved in [1] that every morphism σ is recognizable in $X(\sigma)$ for aperiodic points.

There is a strong link between recognizability and automata theory due to a translation of the property of recognizability in terms of finite monoids. For instance, there is a quadratic-time algorithm to check whether an injective morphism is recognizable in the full shift for aperiodic points [1].

In this paper, we investigate recognizability in the context of sequences of morphisms $\sigma = (\sigma_n \colon A_{n+1}^* \to A_n^*)_{n\geq 0}$. Such a sequence defines an S-adic shift, generated by iterations of the form $\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_n$; see Section 6 for the definition. In fact, σ defines a sequence $X^{(n)}(\sigma)$ of shift spaces and here, by recognizability of σ_n, we mean that any sequence in $X^{(n)}(\sigma)$ can be desubstituted in at most one (and usually exactly one) way as sequences in $X^{(n+1)}(\sigma)$ using σ_n. We distinguish between recognizability of σ, where each σ is recognizable, and eventual recognizability, where all but finitely many morphisms σ_n are recognizable. We consider the most general class of sequences of morphisms, including ones with morphisms with erasable letters.

Recognizability of S-adic shifts has been studied in [2], where it is proved that a morphism σ is eventually recognizable for aperiodic points in its shift spaces under some mild conditions: the morphisms σ_n are non-erasing, the sequence $\mathrm{Card}(A_n)$ contains a bounded subsequence, and σ is everywhere growing (or the points in each $X^{(n)}(\sigma)$ generate a bounded number of different languages). This means that, for large enough n, every aperiodic point in $X^{(n)}(\sigma)$ has a unique centered σ_n-representation as a shift of the image by σ_n of some $x \in X^{(n+1)}(\sigma)$. This recognizability property implies a natural representation of an S-adic shift as a Bratteli–Vershik system [2]. A weaker notion, called quasi-recognizability, is studied in [4].

The main result of this paper is a generalization to possibly erasing sequences of morphisms of the result of [2] concerning the recognizability of a sequence of morphisms in its shift spaces for aperiodic points.

When a morphism σ_n erases a letter, it is possible that a sequence in $X^{(n)}(\sigma)$ cannot be desubstituted as a sequence in $X^{(n+1)}(\sigma)$ using σ_n. We are therefore not only concerned with recognizability but also with representability, which means that $X^{(n)}(\sigma)$ is the shift closure of the image of $X^{(n+1)}(\sigma)$ by σ_n; see Section 5 for details.

We prove the following result, where the alphabet rank of a sequence of morphisms $(\sigma_n \colon A_{n+1}^* \to A_n^*)_{n\geq 0}$ is $\liminf_{n\to\infty} \mathrm{Card}(A_n)$; a more precise statement is given in Section 7.

Main Theorem *Any sequence of morphisms with finite alphabet rank is eventually recognizable for aperiodic points and eventually representable.*

Our proof is much simpler than that of [2], and we we do not require that $\mathrm{Card}(\{\mathcal{L}_x \mid x \in X^{(n)}(\sigma)\})$ is bounded, where $\mathcal{L}_x$ is the set of factors of a point x. Moreover, our proof gives a bound equal to the alphabet rank minus 2 on the number of levels at which σ is not recognizable for aperiodic points, improving the bound of order $K(K + L\log K)$ obtained in [2] for alphabets of size at most K and $\mathrm{Card}(\{\mathcal{L}_x \mid x \in X^{(n)}(\sigma)\}) \leq L$. We also show that this bound is tight.

Our result allows one also to get a new simpler proof of the recognizability of a (possibly erasable) morphism σ on $X(\sigma)$ for aperiodic points obtained in [1].

As in [1], our proof relies on the notion of elementary morphism, due to Ehrenfeucht and Rozenberg [6]. By a result of Karhumäki, Maňuch and Plandowski [8], every elementary morphism is recognizable for aperiodic points; see also [2]. We use this result to prove eventual recognizability.

The paper is organized as follows. After an introductory section on basic notions of symbolic dynamics, we formulate the precise definition of a morphism recognizable on a shift space and prove some elementary properties of recognizable morphisms. In Section 4, we introduce elementary morphisms and recall that every elementary morphism is recognizable for aperiodic points (Proposition 4.3). The main results are proved in Section 6.

2 Symbolic dynamics

We briefly recall some basic definitions of symbolic dynamics. For a more complete presentation, see [10] or the recent [5].

2.1 Words

Let A be a finite alphabet. We let A^* denote the free monoid on A, i.e., the set of finite words over the alphabet A. The empty word is denoted by ε. We let $|u|$ denote the *length* of the word u.

A word $s \in A^*$ is a *factor* of $w \in A^*$ if $w = rst$; the word r is called a *prefix* of w, and it is *proper* if $r \neq w$.

2.2 Shift spaces

We consider the set $A^{\mathbb{Z}}$ of two-sided infinite sequences (also called points) on A. For $x = (x_n)_{n\in\mathbb{Z}}$, and $i \leq j$, we let $x_{[i,j)}$ denote the word $x_i x_{i+1} \cdots x_{j-1}$, where $x_{[i,i)}$ is the empty word; the word $x_{[i,j)}$ is called a *factor* of x.

The set $A^{\mathbb{Z}}$ is a compact metric space for the distance defined for $x \neq y$ by $d(x,y) = 2^{-\min\{|n| \mid n\in\mathbb{Z}, x_n\neq y_n\}}$. The *shift transformation* $T\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $T((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. A *shift space* X on a finite alphabet A is a closed and shift-invariant subset of $A^{\mathbb{Z}}$.

A point $x \in A^{\mathbb{Z}}$ is *periodic* if there is an $n \geq 1$ such that $T^n(x) = x$. Otherwise, it is *aperiodic*. A periodic point has the form $w^\infty = \cdots ww \cdot ww \cdots$ (the letter of index 0 of w^∞ is the first letter of w).

2.3 Morphisms

A morphism $\sigma\colon A^* \to B^*$ is a monoid morphism from A^* to B^*. It is *erasing* if $\sigma(a)$ is the empty word for some $a \in A$, *non-erasing* otherwise.

The morphism σ is extended to a map from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$ by

$$\sigma(\cdots x_{-2}x_{-1} \cdot x_0 x_1 \cdots) = \cdots \sigma(x_{-2})\sigma(x_{-1}) \cdot \sigma(x_0)\sigma(x_1) \cdots,$$

i.e., $\sigma((x_n)_{n\in\mathbb{Z}}) = (y_n)_{n\in\mathbb{Z}}$ with $y_{[|\sigma(x_{[0,n)})|,|\sigma(x_{[0,n+1)})|)} = \sigma(x_n)$ for all $n \geq 0$ and $y_{[|\sigma(x_{[n,0)})|,|\sigma(x_{[n+1,0)})|)} = \sigma(x_n)$ for all $n < 0$; this map is defined only for points in $A^{\mathbb{Z}}$ containing infinitely many letters on the left and infinitely many letters on the right that are not erased.

Let $\sigma\colon A^* \to A^*$ be a morphism from A^* to itself. For $n \geq 0$, we let σ^n denote the morphism obtained with n iterations of σ. The *language of* σ, denoted $\mathcal{L}(\sigma)$, is the set of factors of the words $\sigma^n(a)$ for some $n \geq 0$ and $a \in A$. The *shift defined by* σ, denoted by $X(\sigma)$, is the set of sequences with all their factors in $\mathcal{L}(\sigma)$. The morphism σ is *primitive* if there exists $n \geq 1$ such that the word $\sigma^n(a)$ contains the letter b for all $a, b \in A$.

2.4 S-adic shifts

Let $\sigma = (\sigma_n)_{n\geq 0}$ be a sequence of morphisms $\sigma_n\colon A_{n+1}^* \to A_n^*$, where A_n are finite alphabets:

$$A_0^* \xleftarrow{\sigma_0} A_1^* \xleftarrow{\sigma_1} A_2^* \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_{n-1}} A_n^* \xleftarrow{\sigma_n} A_{n+1}^* \xleftarrow{\sigma_{n+1}} \cdots .$$

For $0 \leq n \leq m$, we define the morphism $\sigma_{[n,m)}\colon A_m^* \to A_n^*$ by

$$\sigma_{[n,m)} = \sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{m-1},$$

where $\sigma_{[n,n)}$ is the identity. For $n \geq 0$, the language of $\mathcal{L}^{(n)}(\sigma)$ is the subset of A_n^* of factors of the words $\sigma_{[n,m)}(a)$, $a \in A_m$, $m \geq n$, and the shift $X^{(n)}(\sigma)$ is the set of sequences with all their factors in $\mathcal{L}^{(n)}(\sigma)$. The *$S$-adic shift* defined by σ is $X^{(0)}(\sigma)$.

A sequence of morphisms $\sigma = (\sigma_n)_{n\geq 0}$ is *non-erasing* if all σ_n are non-erasing. It is primitive if for each $n \geq 0$ there exists $m > n$ such that the word $\sigma_{[n,m)}(a)$ contains the letter b for all $a \in A_m$, $b \in A_n$.

3 Recognizable morphisms

Let $\sigma\colon A^* \to B^*$ be a morphism. A *σ-representation* of $y \in B^{\mathbb{Z}}$ is a pair (x, k) of a sequence $x \in A^{\mathbb{Z}}$ and an integer k such that

$$y = T^k(\sigma(x)), \tag{3.1}$$

where T is the shift transformation. The σ-representation (x, k) is *centered* if $0 \leq k < |\sigma(x_0)|$. In particular, a centered σ-representation (x, k) satisfies $\sigma(x_0) \neq \varepsilon$. We say that the σ-representation (x, k) is *in* X if $x \in X$.

Note that, if y has a σ-representation (x, k), then it has a centered σ-representation (x', k') with x' a shift of x.

Definition 3.1 Let X be a shift space on A. A morphism $\sigma\colon A^* \to B^*$ is *recognizable in* X (respectively recognizable in X for aperiodic points) if every point in $B^{\mathbb{Z}}$ (respectively every aperiodic point in $B^{\mathbb{Z}}$) has at most one centered σ-representation in X. A morphism $\sigma\colon A^* \to B^*$ is *fully recognizable* (respectively fully recognizable for aperiodic points) if it is recognizable in $A^{\mathbb{Z}}$ (respectively recognizable in $A^{\mathbb{Z}}$ for aperiodic points).

Note that an equivalent definition of recognizability in X is that, for every $x, x' \in X$ and $0 \le k < ||\sigma(x'_0)| - |\sigma(x_0)||$ such that $\sigma(x) = T^k(\sigma(x'))$, one has $k = 0$ and $x = x'$.

Example 3.2 The Fibonacci morphism $\sigma\colon a \mapsto ab, b \mapsto a$ is fully recognizable.

Example 3.3 The Thue-Morse morphism $\sigma\colon a \mapsto ab, b \mapsto ba$ is not fully recognizable since $(ab)^\infty$ can be obtained as $\sigma(a^\infty)$ and as $T(\sigma(b^\infty))$. However, it is fully recognizable for aperiodic points since any sequence containing aa or bb has at most one factorization in $\{ab, ba\}$.

Example 3.4 The morphism $\sigma\colon a \to aa, b \mapsto ab, c \mapsto ba$ is not fully recognizable for aperiodic points. Indeed, every sequence without occurrence of bb has two factorizations in words of $\{aa, ab, ba\}$.

By [2, 1], the family of morphisms recognizable for aperiodic points is closed under composition.

4 Elementary morphisms

Definition 4.1 A morphism $\sigma\colon A^* \to C^*$ is *elementary* if for every alphabet B and every pair of morphisms $\alpha\colon B^* \to C^*$ and $\beta\colon A^* \to B^*$ such that $\sigma = \alpha \circ \beta$, one has $\mathrm{Card}(B) \ge \mathrm{Card}(A)$.

If $\sigma\colon A^* \to C^*$ is elementary, one has in particular $\mathrm{Card}(C) \ge \mathrm{Card}(A)$ and moreover σ is non-erasing.

Example 4.2 The Thue-Morse morphism $\sigma\colon a \mapsto ab, b \mapsto ba$ is elementary.

The notion of elementary morphism appears for the first time in [6]. The following result is from [8]. It also appears in [2] with the stronger hypothesis that $\sigma\colon A^* \to B^*$ is such that the incidence matrix of σ has rank $\mathrm{Card}(A)$. An independent proof is given in [1].

Proposition 4.3 *Any elementary morphism is fully recognizable for aperiodic points.*

5 Representable S-adic shifts

Contrary to sequences of non-erasing morphisms σ, a point in $X^{(n)}(\sigma)$ need not have a σ_n-representation in $X^{(n+1)}(\sigma)$ when σ_n is erasing, as the following example shows.

Example 5.1 Let the sequence of morphisms σ be defined by

$$\begin{aligned}
&\sigma_0: a \mapsto a, b \mapsto \varepsilon,\\
&\sigma_1: a \mapsto a, b \mapsto bb, c \mapsto ab,\\
&\sigma_n: a \mapsto a, b \mapsto bb, c \mapsto cab, \quad \text{for all } n \ge 2.
\end{aligned}$$

Since $\sigma_{[1,n)}(a) = a$, $\sigma_{[1,n)}(b) = b^{2^{n-1}}$, $\sigma_{[1,n)}(c) = abab^2 \cdots ab^{2^{n-2}}$ for all $n \geq 2$, we have $X^{(0)}(\sigma) = \{a^\infty\}$ and $X^{(1)}(\sigma)$ consists of the points in $\{a,b\}^{\mathbb{Z}}$ containing at most one a, hence a^∞ has no σ_0-representation in $X^{(1)}(\sigma)$.

We say that a sequence of morphisms σ is *representable at level* n if every point in $X^{(n)}(\sigma)$ has at least one σ_n-representation in $X^{(n+1)}(\sigma)$. It is *representable* if it is representable at each level. We say that a sequence of morphisms is *eventually representable* if there is an integer M such that it is representable at each level at least equal to M.

Note that $X^{(n)}(\sigma)$ is the shift-closure of $\sigma_n(X^{(n+1)}(\sigma))$ if and only if σ is representable at level n.

The following lemma is proved in [2, Lemma 4.2] for sequences of non-erasing morphisms. We recall its proof to make clear where the non-erasing property is used.

Lemma 5.2 *Let $\sigma = (\sigma_n)_{n\geq 0}$ with $\sigma_n \colon A_{n+1}^* \to A_n^*$ be a sequence of morphisms. If $\sigma_{[n,m)}$ is non-erasing, $0 \leq n < m$, then every point in $X^{(n)}(\sigma)$ has at least one $\sigma_{[n,m)}$-representation in $X^{(m)}(\sigma)$. In particular, if σ_n is non-erasing, then σ is representable at level n.*

Proof. Let $y \in X^{(n)}(\sigma)$. Then each word $y_{[-\ell,\ell)}$ is a factor of $\sigma_{[n,N)}(a)$ for some $a \in A_N$, $N \geq m$, hence $y_{[-\ell+i,\ell-j)} = \sigma_{[n,m)}(w)$ for some $w \in \mathcal{L}^{(m)}(\sigma)$, $0 \leq i,j < \max_{a\in A_m} |\sigma_{[n,m)}(a)|$. Since $|w| \to \infty$ as $\ell \to \infty$, a Cantor diagonal argument gives a word $x \in X^{(m)}(\sigma)$ and $0 \leq k < |\sigma_{[n,m)}(x_0)|$ such that $\sigma_{[n,m)}(x_{[-\ell,\ell)}) = y_{[|\sigma_{[n,m)}(x_{[-\ell,0)})|-k,|\sigma_{[n,m)}(x_{[0,\ell)})|-k)}$ for all $\ell \geq 1$. Since $\sigma_{[n,m)}$ is non-erasing, (k,x) is a $\sigma_{[n,m)}$-representation of y. ∎

Lemma 5.3 *If σ is not representable at level n, then $\sigma_{[n,m)}$ is erasing for all $m > n$.*

Proof. If $\sigma_{[n,m)}$ is non-erasing, then, by Lemma 5.2, each $y \in X^{(n)}(\sigma)$ has $\sigma_{[n,m)}$-representation in $X^{(m)}(\sigma)$, thus it also has a σ_n-representation in $X^{(n+1)}(\sigma)$, i.e., σ is representable at level n. ∎

6 Recognizable S-adic shifts

A sequence of morphisms $\sigma = (\sigma_n)_{n\geq 0}$ with $\sigma_n \colon A_{n+1}^* \to A_n^*$ is *recognizable at level* n (respectively *recognizable at level* n *for aperiodic points*) if σ_n is recognizable (respectively recognizable for aperiodic points) in $X^{(n+1)}(\sigma)$. We say that σ is *recognizable* (respectively *recognizable for aperiodic points*) if it is recognizable (respectively recognizable for aperiodic points) at each nonnegative level n, and σ is *eventually recognizable* (respectively *eventually recognizable for aperiodic points*) if there is a nonnegative integer M such that σ is recognizable (respectively recognizable for aperiodic points) at level n for each $n \geq M$.

We show that non-recognizability at level n and representability between levels $n+1$ and m implies non-recognizability between levels n and m.

Lemma 6.1 *If σ is not recognizable at level n and each point in $X^{(n+1)}(\sigma)$ has a $\sigma_{[n+1,m)}$-representation in $X^{(m)}(\sigma)$, then $\sigma_{[n,m)}$ is not recognizable in $X^{(m)}(\sigma)$. The same statement holds for recognizability for aperiodic points.*

Proof. This is proved in [2, Lemma 3.5] for non-erasing morphisms; we recall the proof. If σ is not recognizable at level n, then there exists $z \in X^{(n)}(\sigma)$ with two centered σ_n-representations $(y, \ell) \neq (y, \ell')$ in $X^{(n+1)}(\sigma)$. Let (x, k) and (x', k') be centered $\sigma_{[n+1,m)}$-representations in $X^{(m)}(\sigma)$ of y and y', respectively. Then $(x, |\sigma_n(y_{[-k,0)})|+\ell)$ and $(x', |\sigma_n(y'_{[-k',0)})|+\ell')$ are centered $\sigma_{[n,m)}$-representations of z. To see that the two representations are different, note that $y_{[-k,0)}$, $z_{[-\ell,0)}$, and $\sigma_n(y_{[-k,0)})z_{[-\ell,0)}$ are proper prefixes of $\sigma_{[n+1,m)}(x_0)$, $\sigma_n(y_0)$, and $\sigma_{[n,m)}(x_0)$ respectively. Since each proper prefix of $\sigma_{[n,m)}(x_0)$ has a unique decomposition as $\sigma_n(u)v$ with $u \in A_{n+1}^*$, $v \in A_n^*$, such that ua is a prefix of $\sigma_{[n+1,m)}(x_0)$ and v is a proper prefix of $\sigma_n(a)$ for some $a \in A_{n+1}$, $(x, |\sigma_n(y_{[-k,0)})|+\ell) = (x', |\sigma_n(y'_{[-k',0)})|+\ell')$ would imply that $k = k'$ and $\ell = \ell'$, thus $y = y'$, contradicting that $(y, \ell) \neq (y, \ell')$. Therefore, $\sigma_{[n,m)}$ is not recognizable on $X^{(m)}(\sigma)$.

Taking aperiodic points y, y' proves the statement for aperiodic points. ∎

7 Levels of recognizability and representability

We can now state and prove our main results, which give bounds for the number of levels where a sequence of morphisms can be non-recognizable for aperiodic points or non-representable, in terms of the size of the alphabets.

Proposition 7.1 *Let $\sigma = (\sigma_n)_{n\geq 0}$ with $\sigma_n\colon A_{n+1}^* \to A_n^*$ be a sequence of morphisms. Let $m > n_1 > n_2 > \cdots > n_K \geq 0$, $K \geq 0$, be such that, for each $1 \leq k \leq K$, σ is not recognizable at level n_k for aperiodic points or σ is not representable at level n_k. Then we have $K < \mathrm{Card}(A_m)$. Moreover, $K = \mathrm{Card}(A_m)-1 \geq 1$ implies that $X^{(n_K)}(\sigma)$ has no aperiodic points.*

Proof. Let $m > n_1 > n_2 > \cdots > n_K \geq 0$ be as in the statement of the proposition. Since the proposition is trivial for $K = 0$, we assume that $K \geq 1$.

We define $\alpha_0\colon A_m^* \to A_m^*$ as the identity morphism and set $n_0 = m$, $B_0 = A_m$. For each $1 \leq k < K$, we show inductively that the morphism $\sigma_{[n_k,n_{k-1})} \circ \alpha_{k-1}$ is not elementary and admits therefore a decomposition $\sigma_{[n_k,n_{k-1})} \circ \alpha_{k-1} = \alpha_k \circ \beta_k$ (see Figure 7.1) with morphisms $\alpha_k\colon B_k^* \to A_{n_k}^*$, $\beta_k\colon B_{k-1}^* \to B_k^*$, for some alphabet B_k satisfying $\mathrm{Card}(B_k) < \mathrm{Card}(B_{k-1})$.

Indeed, consider the sequence of morphisms

$$\sigma' = (\sigma_0, \sigma_1, \ldots, \sigma_{n_k}, \ldots, \sigma_{n_{k-1}-1}, \alpha_{k-1}, \beta_{k-1}, \beta_{k-2}, \ldots, \beta_1, \sigma_m, \sigma_{m+1}, \ldots).$$

Since $\alpha_{k-1} \circ \beta_{k-1} \circ \beta_{k-2} \circ \cdots \circ \beta_1 = \sigma_{[n_{k-1},m)}$, we have for all $0 \leq h \leq n_{k-1}$ that the languages $\mathcal{L}^{(h)}(\sigma)$ and $\mathcal{L}^{(h)}(\sigma')$ differ only by a finite set and hence $X^{(h)}(\sigma) = X^{(h)}(\sigma')$. If σ is not representable at level n_k, then σ' is also not representable at level n_k and, by Lemma 5.3, $\sigma_{[n_k,n_{k-1})} \circ \alpha_{k-1}$ is erasing, thus

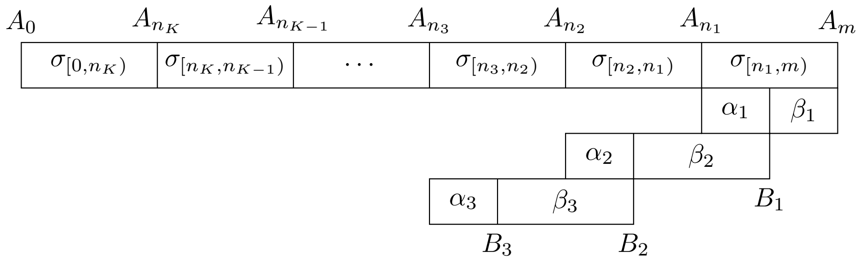

Figure 7.1: Alphabets and morphisms in the proof of Proposition 7.1.

non-elementary. If σ is not recognizable at level n_k for aperiodic points and each point in $X^{(n_k+1)}(\sigma)$ has a $\sigma_{[n_k+1,n_{k-1})} \circ \alpha_{k-1}$-representation in $X^{(n_{k-1}+1)}(\sigma')$, then $\sigma_{[n_k,n_{k-1})} \circ \alpha_{k-1}$ is not recognizable for aperiodic points in $X^{(n_{k-1}+1)}(\sigma')$ by Lemma 6.1, thus it is non-elementary. Finally, if there exists a point in $X^{(n_k+1)}(\sigma)$ without $\sigma_{[n_k+1,n_{k-1})} \circ \alpha_{k-1}$-representation in $X^{(n_{k-1}+1)}(\sigma')$, then $\sigma_{[n_k+1,n_{k-1})} \circ \alpha_{k-1}$ is erasing by Lemma 5.3, hence $\sigma_{[n_k,n_{k-1})} \circ \alpha_{k-1}$ is erasing, thus non-elementary.

We get that $\mathrm{Card}(A_m) > \mathrm{Card}(B_1) > \mathrm{Card}(B_2) > \cdots > \mathrm{Card}(B_K) \geq 1$, thus $K < \mathrm{Card}(A_m)$. If $K = \mathrm{Card}(A_m)-1$, then $\mathrm{Card}(B_K) = 1$, hence $X^{(n_K)}(\sigma)$ consists of a single periodic orbit, thus σ is recognizable at level n_K for aperiodic points (and thus not representable at level n_K by the assumption on n_K). ∎

Let $\sigma = (\sigma_n)_{n\geq 0}$ with $\sigma_n\colon A_{n+1}^* \to A_n^*$ be a sequence of morphisms. The *alphabet rank* of σ is $\liminf_{n\to\infty} \mathrm{Card}(A_n)$.

Theorem 7.2 *Let $\sigma = (\sigma_n)_{n\geq 0}$ with $\sigma_n\colon A_{n+1}^* \to A_n^*$ be a sequence of morphisms with finite alphabet rank. Then σ is eventually recognizable for aperiodic points and eventually representable.*

Moreover, if r is the alphabet rank, the number of levels at which σ is not recognizable for aperiodic points is bounded by $r-2$ and the number of levels at which σ is not representable is bounded by $r-1$.

Proof. Suppose that σ is not representable at $K = r < \infty$ levels, then applying Proposition 7.1 for some m which is larger than these levels and satisfies $\mathrm{Card}(A_m) = K$ gives a contradiction. Similarly, we cannot have $K = r-1$ levels where σ is not recognizable for aperiodic points by Proposition 7.1 because the level n_K in Proposition 7.1 can only be non-representable for $K = r-1$. ∎

Note that the condition of finite alphabet rank holds in particular when the sizes of the alphabets are bounded.

In the particular case of a constant sequence of morphisms $\sigma' = (\sigma, \sigma, \dots)$, all shifts $X^{(n)}(\sigma')$ are equal to the shift space $X(\sigma)$ of the morphism σ, and non-recognizability of σ in $X(\sigma)$ for aperiodic points means that σ' is non-recognizable at aperiodic points at all levels. Since this is not possible by Theorem 7.2, this proves the main result of [1], in a way that is simpler than all the previous proofs of recognizability.

Corollary 7.3 *Any morphism σ is recognizable for aperiodic points in $X(\sigma)$.*

In the same way, we have a simple proof of [1, Proposition 5.1] concerning the representability of σ.

Corollary 7.4 *For any morphism σ, any point in $X(\sigma)$ has a σ-representation in $X(\sigma)$.*

Finally, we consider the tightness of the bounds in Proposition 7.1 and Theorem 7.2. We have already seen in Example 5.1 that a sequence can be non-representable at level 0 with $\mathrm{Card}(A_1) = 2$. An example of a primitive sequence of morphisms with $\mathrm{Card}(A_n) = 3$ for all $n \geq 1$ that is not recognizable for aperiodic points at level 0 is given in [2, Example 4.3]. The following example shows that we can have $r-2$ levels of non-recognizability for aperiodic points, where r is the finite alphabet rank.

Example 7.5 Let $K \geq 1$, $A_n = \{a_0, a_1, \ldots, a_n\}$ for $0 \leq n \leq K$, $A_n = \{a_0, a_1, \ldots, a_{K+1}\}$ for $n > K$, and

$$\begin{aligned} &\sigma_n : a_i \mapsto a_0 a_i a_0 \text{ for all } 0 \leq i \leq n,\ a_{n+1} \mapsto a_n, && \text{for } 0 \leq n \leq K, \\ &\sigma_n : a_i \mapsto a_0 a_i a_0 \text{ for all } 0 \leq i \leq K+1, && \text{for } n > K. \end{aligned}$$

Then, for all $0 \leq n \leq K+1$, $X^{(n)}(\sigma)$ consists of the closure of the shift orbits of the sequences $\cdots a_0 a_0 \cdot a_i a_0 a_0 \cdots$, $0 \leq i \leq n$. For $0 \leq n \leq K$, we have

$$\sigma_n(\cdots a_0 a_0 \cdot a_n a_0 a_0 \cdots) = \cdots a_0 \cdot a_0 a_n a_0 a_0 \cdots = \sigma_n(\cdots a_0 \cdot a_0 a_{n+1} a_0 a_0 \cdots),$$

thus σ is not recognizable at level n for aperiodic points for all $1 \leq n \leq K$ (and not recognizable at level 0).

Example 7.5 can be easily modified to obtain a sequence of morphisms that is not recognizable at any level n for aperiodic points.

Example 7.6 Let $A_n = \{a_0, a_1, \ldots, a_{n+1}\}$ for all $n \geq 0$, and

$$\sigma_n : a_i \mapsto a_0 a_i a_0 \text{ for all } 0 \leq i \leq n+1,\ a_{n+2} \mapsto a_{n+1}, \quad \text{for all } n \geq 0.$$

Then σ is not recognizable at level n for aperiodic points for all $n \geq 0$ because

$$\sigma_n(\cdots a_0 a_0 \cdot a_{n+1} a_0 a_0 \cdots) = \cdots a_0 \cdot a_0 a_{n+1} a_0 a_0 \cdots = \sigma_n(\cdots a_0 \cdot a_0 a_{n+2} a_0 a_0 \cdots).$$

We do not know whether the bound of $r-1$ levels of non-representability is tight, where r is the finite alphabet rank.

A final remark concerns the existence of a recognizable S-adic representation for a given shift space X. Can one always modify the morphisms of a sequence σ in such a way that $X = X^{(0)}(\sigma')$ with σ' recognizable (keeping properties of σ like primitivity, constant length, etc..)? The answer is known in the particular case of automatic shifts, which are S-adic shifts defined by sequences $(\varphi, \sigma, \sigma, \ldots)$, where σ is a morphism of constant length and φ a letter coding (that is, has constant length 1). It has been shown in [14] that, for aperiodic automatic shifts generated by a primitive morphism σ, we can assume w.l.o.g. that φ is injective on $X(\sigma)$, thus $(\varphi, \sigma, \sigma, \ldots)$ is recognizable. It is not known if a similar result holds for general morphic shifts, i.e., when σ is not of constant length. Other results exhibiting recognizable sequences of morphisms can be found in [7].

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