Aldo De Luca
1941-2018

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May 28, 2019

Aldo de Luca passed away on October 16th, 2018. Aldo was born in Naples, Italy, the 30th of November 1941 and he obtained his Degree in Physics cum laude at the University of Naples in 1964.

Aldo's scientific activity started in the 1960's, during the extremely fervent intellectual period in Neapolitan scientific research. He collaborated with Eduardo Caianiello's Cybernetic group at the Institute of Theoretical Physics at the University of Naples. His scientific formation received special impact from the school "Automata Theory" organized by Caianiello in Ravello, Italy, in 1964, with the participation of researchers as Martin Davis, Warren McCulloch,

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Maurice Nivat, Michael Rabin and Marcel-Paul Schützenberger. At this school, which is considered one of the landmarks in the early days of theoretical computer science in Italy, Aldo met for the first time Schützenberger, who would go on to have a strong influence on his scientific activity.

From 1967 to 1981 Aldo was a C.N.R. (National Research Council) researcher at the Institute of Cybernetics of Arco Felice. In 1972/1973, Schützenberger was visiting professor at the Institute of Cybernetics, and Aldo, under his influence, firmly moved his research to the theory of formal languages, codes and combinatorics on words.

In 1980 Aldo became full professor of algebra at the University of Naples "Federico II", then he moved to the University of Rome "La Sapienza", where he was full professor of computer science until 2003. At the same time, he spent some years at the Interdisciplinary center 'B. Segre' of the National Academy of Lincei, Italy. In 2003, he moved back to the University of Naples "Federico II", as a full professor of computer science, then emeritus professor.

Aldo was a pioneer of theoretical computer science in Italy. He was a member of EATCS since its creation, and he was one of the promoters of the Italian Chapter of the association. He was also one of the founders of the international conference WORDS, one of the most important conferences in the area of formal languages and automata theory.

As a colorful anecdote remembering the sense of humor of Aldo and also his close frendship with Marco Schützenberger, let us quote the following one. During a conference in 1975 at the Instituto di Alta Matematica in Rome, Marco Schützenberger said in passing, to make a transition, 'As my friend Aldo de Luca likes to say: *Parlamme e cose cchiú allere ... quant costa na cascia e muorte, dicette o schiattamuorte*' (free translation from Neapolitan: 'Let us talk about something more enjoyable...how much does a coffin cost, said the undertaker').

The heritage of Aldo de Luca

Aldo has left a considerable heritage by his contribution to combinatorics on words, automata theory and semigroup theory. We have chosen, in the following lines to single out two contributions, one in combinatorics on words and the other one on finiteness conditions. They give an idea of the importance of his contribution by the difficulty of the proofs and by the significance of the results. A complete list of publications of Aldo can be found at `http://www.aldodeluca.it`

Combinatorics on Words

Sturmian words are easy to define. They are the infinite words on a binary alphabet $A = \{a, b\}$ having $n + 1$ distinct factors of length n for every $n \geq 0$.

The best known example of Sturmian word is the infinite Fibonacci word

$$x = abaababaabaab \cdots$$

which is the unique fixed point of the Fibonacci morphism $\varphi : a \to ab, b \to a$. The words $F_n = \varphi^n(a)$ for $n \geq 0$ are prefixes of x and are called the (finite) Fibonacci words.

Since there is one more factor of length $n+1$ than of length n, a Sturmian word has for every n exactly one factor u of length n having two left extensions au and bu. Such a factor is called *left-special* (one defines symmetrically the right-special factors).

For example, the Fibonacci words F_n are left-special with respect to the infinite Fibonacci word x. Indeed, this is true for $n = 0$ since aa and ba are factors of x and next, one has

$$aF_{n+1} = \varphi(bF_n), \quad abF_{n+1} = \varphi(aF_n)$$

showing by induction that all F_n are left-special.

The different left-special factors of a Sturmian word are necessarily prefixes of one another. A Sturmian (infinite) word is called *standard* if its left-special factors are its prefixes (or equivalently if all its prefixes are left-special).

For example, the infinite Fibonacci word is standard. Indeed, all finite Fibonacci words are among its prefixes and thus all its prefixes are left-special.

Aldo invented the fundamental notion of *iterated palindromic closure*. It is the following transformation Pal on words. For a word w, denote by $w^{(+)}$ the shortest palindrome having w as a prefix. For example, one has $(aaba)^{(+)} = aabaa$ and $(abaa)^{(+)} = abaaba$.

We then define Pal as the unique map from A^* to A^* such that $\mathrm{Pal}(\varepsilon) = \varepsilon$ and next, for $w \in A^*$ and $a \in A$,

$$\mathrm{Pal}(wa) = (\mathrm{Pal}(w)a)^{(+)}.$$

For example, $\mathrm{Pal}(aba) = abaaba$.

Since $\mathrm{Pal}(w)$ is a prefix of $\mathrm{Pal}(wa)$, the map Pal extends to the set A^ω of infinite words on A. The following result appears in a fundamental paper by Aldo on Sturmian words [4].

Theorem 1 (de Luca, 1997) *Let A be a binary alphabet. The map* Pal *is a bijection from the set of words in A^ω with an infinite number of occurrences of each letter onto the set of standard Sturmian words.*

The word $y \in A^\omega$ is called the *directive word* of $\mathrm{Pal}(y)$. For example, the word $y = (ab)^\omega$ is the directive word of the Fibonacci infinite word. Theorem 1 is closely related to another important statement, known as Justin's Formula stated in (1) below.

For a letter $a \in A$, let ψ_a be the automorphism of the free group defined for every $b \in A$ by

$$\psi_a(b) = \begin{cases} ab & \text{if } b \neq a \\ a & \text{otherwise} \end{cases}$$

and let ψ_u be defined for every word $u \in A^*$ by $\psi_{ua} = \psi_u\psi_a$ for every $u \in A^*$ and $a \in A$. Then for every $u \in A^*$ and $v \in A^\omega$,

$$\mathrm{Pal}(uv) = \psi_u(\mathrm{Pal}(v)) \tag{1}$$

It is easy to deduce from Justin's Formula that $x = \mathrm{Pal}((ab)^\omega)$ is the Fibonacci infinite word. Indeed, since $(ab)^\omega = (ab)(ab)^\omega$, one obtains by (1),

$$x = \psi_{ab}(x).$$

Since $\psi_{ab} = \varphi^2$, where φ is the Fibonacci morphism, we conclude that x is the Fibonacci infinite word.

The beauty of all this is that it carries on to larger alphabets, as shown by Droubay, Justin and Pirillo in [12]. Indeed, a generalization of Sturmian words can be defined on arbitrary finite alphabets as follows. An infinite word x on the alphabet A is called *episturmian* if its set of factors is closed under reversal and if, for each $n \geq 1$, there is exactly one left-special factor u of length n, that is, with more than one extension au for $a \in A$ which is a factor of x.

For example, the Tribonacci word

$$x = abacaba\cdots$$

which is the unique fixed point of the Tribonacci morphism $\tau : a \to ab, b \to ac, c \to a$ is episturmian.

Like for Stumian words, an episturmian word is *standard* if its left-special factors are its prefixes.

The Tribonacci word is standard. As for the Fibonacci word, this is easy to verify by induction since $T_n = \tau^n(a)$ satisfies

$$aT_{n+1} = \tau(cT_n), \quad abT_{n+1} = \tau(aT_n), \quad acT_{n+1} = \tau(bT_n).$$

Then Theorem 1 and Justin's Formula hold for any alphabet, replacing 'Sturmian' by 'episturmian'. The word $y \in A^\omega$ is again called the directive word of $x = \mathrm{Pal}(y)$.

For example, the directive word of the Tribonacci word is $y = (abc)^\omega$. This can be verified again easily using Justin's Formula. Indeed, the word $x = \mathrm{Pal}((abc)^\omega)$ satisfies

$$x = \psi_{abc}(x)$$

whence the conclusion since $\psi_{abc} = \tau^3$ where τ is the Tribonacci morphism.

An episturmian word x is *strict* if the unique left-special factor u of length n of x is for every n such that au is a factor of x for every $a \in A$.

For example, the Tribonacci word is strict. Strict episturmian words are also called 'Arnoux-Rauzy' words after the initial paper of Arnoux and Rauzy [1]. As a complement to Theorem 1, the word $\mathrm{Pal}(y)$ is a strict episturmian word if every letter appears infinitely often in y [12].

The authors of [12] give due credit to Aldo by calling *Al* the following condition for an infinite word x: if v is a prefix of x, then $v^{(+)}$ is also a prefix of x.

It is clear that an infinite word satisfies Al if and only if $x = \mathrm{Pal}(y)$ for some $y \in A^\omega$ and thus if and only if it is episturmian.

A formulation of Theorem 1 in terms of finite words has been given by Aldo by using the notion of *central* word.

Central words are closely related to finite *standard* words, that are the basic bricks for constructing standard Sturmiam words, in the sense that every standard Sturmian word is the limit of a sequence of finite standard words (see [16]). For instance, the finite Fibonacci words are standard. Finite standard words can be defined directly as the words appearing in the pairs (u, v) obtained starting with (a, b) and applying iteratively one of the so-called *Rauzy rules*

$$\begin{tikzcd} & (u,v) \arrow[dl] \arrow[dr] & \\ (u,uv) & & (vu,v) \end{tikzcd}$$

A finite word w over the alphabet $\{a, b\}$ is central if wab (or equivalently wba) is a standard word. In [4], Aldo also proved that *the map* Pal *is a bijection from* $\{a, b\}^*$ *onto the set of central words.*

In relation with this result (and thus also with Theorem 1), a very deep connection between palindromes and *periods* in finite words was discovered by Aldo and Filippo Mignosi in [6]. They proved that *a word w in $\{a, b\}^*$ is central if and only if, for some relatively prime natural numbers p, q , w has two periods p and q and is of length $|w| = p + q - 2$.* This also shows that central words correspond to the extremal case of the famous Fine and Wilf periodicity lemma (see [15]).

Aldo's work on Sturmian words has recently been put in perspective in the very nice book of Christophe Reutenauer [20], where its connections with number theory (in particular with continued fractions) are presented.

Finiteness conditions

One of the important contributions of Aldo de Luca concerns finiteness conditions in semigroups. He published in 1998 a monograph on this subject with his former student Stefano Varricchio [8], which contains many of their contributions in this area (Stefano Varrichio died in 2008, see its obituary by Aldo [5]).

One of the main results that they obtained is the solution of a conjecture formulated many years before by John Brzozowski [2]. Consider, for some $n \geq 1$, the congruence $\sim_n$ on the free monoid generated by the pairs (x^{n+1}, x^n). The conjecture of Brzozowski is that the classes of this congruence are regular.

The monoid $M_n(A) = A^*/\sim_n$ is of interest in automata and semigroup theory because any aperiodic monoid (that is a finite monoid with no nontrivial subgroup) is an image of some $M_n(A)$.

The more general problem of the congruence generated by the pairs (x^{n+m}, x^n) for $n, m \geq 1$ was also studied. The corresponding quotient $M_{n,m}(A)$ is called the *Burnside monoid*. In particular, the Burnside group satisfying the identity $x^m = 1$ on two generators was shown to be infinite for m large enough by Novikov and Adjan (1968).

The case $n = 1$ corresponds to free idempotent monoids. It was solved long ago by Green and Rees (see [15]). In this case, the monoid is finite. On the contrary, the monoid $M_n(A)$ is infinite for $n \geq 2$ on at least two letters. Indeed, if $\mathrm{Card}(A) \geq 3$, this follows from the existence of infinite square-free words on three letters. For $\mathrm{Card}(A) = 2$, the fact that $M_n(A)$ is infinite is shown in [3].

The conjecture of Brzozowski was solved positively in 1992 for $n \geq 5$ by Aldo de Luca and Stefano Varricchio (it was announced at ICALP 1990 [7] and published in [17]). An independent proof for $n \geq 6$ appeared in [18]. The result was extended to $n \geq 4$ by Pereira do Lago (again announced at LATIN 1992 [9] and published in [10]) using the technique of proof of Aldo and Stefano. Finally, Victor Guba, extending McCammond's technique of proof, proved it for $n \geq 3$ [14, 13]. The case $n = 2$ is still open (although partial results are known [19]).

Let us try to give an idea of the problems involved, of their difficulty, and of the path followed by Aldo and Stefano. What follows is strongly inspired by the expositions made by Jean-François Rey [21] and by Alair Pereira do Lago himself with Imre Simon [11].

The first thing to investigate is the behaviour of the rewriting system π with productions $x^{n+1} \to x^n$ for some $n \geq 2$.

$$\begin{tikzcd}
& w \arrow[dl] \arrow[dr] & \\
(abcbcb)^2c = a(bc)^2ba(bc)^3 \arrow[d] & & (abcbcb)^2a(bc)^2 \\
a(bc)^2ba(bc)^2 & &
\end{tikzcd}$$

Figure 1: A critical pair of π.

Consider for $n = 2$ the two possible derivations from the word $w = (abcbcb)^3c = (abcbcb)^2a(bc)^3$ of length 19, leading to a critical pair of irreducible words of words of lengths 13 and 17 (see Figure 1).

In the search for an equivalent confluent system (in which, by definition, every word reduces to a unique irreducible shorter word by productions reducing the length), we will add the production

$$(abcbcb)^2abcbc \to (abcbcb)abcbc$$

which reduces a cube minus one letter to a square minus one letter.

A crucial observation is that the new production is of the form $\ell \to s$ where s is a border of ℓ, that is both a prefix and a suffix of ℓ. Note that this is also true of the productions of π.

The *tour de force* realized by Aldo de Luca and Stefano Varrichio is to replace the system π by an equivalent confluent system Σ whose productions are of the form

$$\sigma : \ell_\sigma \to s_\sigma$$

where s_σ (the *short* part) is a border of ℓ_σ (the *long* part). Denote by Ω the set of productions σ of this form.

We introduce a partial order on Ω defining $\sigma \preceq \tau$ if $l_\tau = ul_\sigma v$ and $s_\tau = us_\sigma v$.

The production $\sigma \in \Omega$ is *stable* if the period $\mathrm{per}(s_\sigma)$ of s_σ is equal to $|l_\sigma| - |s_\sigma|$. In particular, the production $x^{n+1} \to x^n$ is stable if and only if x is primitive. A set $X \subset \Omega$ is stable if every element of X is stable.

The construction of Σ goes through the following steps. The *overlap* of two words u, v, denoted $\mathrm{over}(u, v)$ is the longest suffix of u which is a prefix of v.

Given two productions σ, τ in Ω, such that $|s_\sigma| < |\,\mathrm{over}(s_\tau, \ell_\sigma)|$ and $\mathrm{per}(s_\tau) > \mathrm{per}(s_\sigma)$, we define $\tau/\sigma = s_\sigma^{-1}\,\mathrm{over}(s_\tau, \ell_\sigma)$. (see Figure 2). We then say that the production

$$l_\tau(\tau/\sigma)^{-1} \to s_\tau(\tau/\sigma)^{-1}$$

is the *right cut* of τ by σ. One defines symmetrically the left cut of τ by σ. For

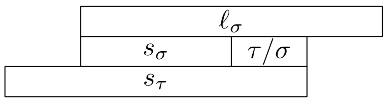

Figure 2: The right cut of τ by σ.

a subset X of Ω, we denote by $\mathcal{S}(X)$ the closure of X by left and right cuts.

We then define recursively the set $\Sigma = \cup_{i \geq 1}\Sigma_i$ as follows. Set $\pi_0 = \pi'_0 = \Sigma_0 = \Sigma' = \emptyset$. For $i \geq 1$, let $\pi_i = \{x^{n+1} \to x^n \mid x \in A^i\}$. Let π'_i be the set of productions $\tau \in \pi_i$ such that τ is stable and for every $\sigma \in \Sigma_{i-1}$, ℓ_σ is not a factor of ℓ_τ. Let $\Sigma'_i = \mathcal{S}(\pi'_i \cup \Sigma_{i-1})$ and let Σ_i be the set of productions in Σ'_i which are minimal for the order $\preceq$.

The following result is called in [7] the Equivalence Theorem.

Theorem 2 (de Luca, Varricchio, 1990) *The systems π and Σ generate the same congruence.*

The importance of the property of being stable has been discovered by Alair Pereira do Lago, who, buiding on the work of Aldo de Luca and Stefano Varricchio, has shown the following result.

A monoid M is finite $\mathcal{J}$-above if for every element $m \in M$ the set of $n \in M$ such that $m \leq_{\mathcal{J}} n$ (that is such that $m \in MnM$) is finite.

Theorem 3 *Assume that Σ is stable. Then it is confluent, the monoid $M_n(A)$ is finite $\mathcal{J}$-above and each congruence class is recognizable.*

The paper of Alair Pereira contains actually a fairly complete description of the monoid $M_n(A)$ and in particular of the structure of its Green classes, extending early unpublished work of Imre Simon (see [11]).

Ackowledgments The authors wish to thank Christian Choffrut, Jean-Eric Pin, Christophe Reutenauer and Jacques Sakarovitch for their comments on a preliminary version of this paper.

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