Embeddings of automata

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Outline

(*Joint work with Marie-Pierre Béal*).
Description of methods allowing to embed an automaton into a complete one, within a given family incuding the following:

- Unambiguous automata
- Weakly deterministic automata (new construction)
- Local automata (new result)

Automata

Let $\mathcal{A} = (Q, E)$ be a finite automaton. We denote by $L(\mathcal{A})$ the set of labels of finite paths in $\mathcal{A}$ and by $M_{\mathcal{A}}$ the adjacency matrix of $\mathcal{A}$. Its coefficient (p, q) is the sum of letters $a \in A$ such that $p \xrightarrow{a} q$.
We say that $\mathcal{A}$ is irreducible if for any $p, q \in Q$ there exists $w \in A^*$ such that $p \xrightarrow{w} q$.
An input merge equivalence is an equivalence on the set Q of states of $\mathcal{A}$ such that for any pair p, p' of equivalent states, any letter a and any state q, one has

(i) $p \xrightarrow{a} q$ if and only if $p' \xrightarrow{a} q$ (equivalent states have the same output),
(ii) $q \xrightarrow{a} p$, $q \xrightarrow{a} p'$ implies $p = p'$ (two distinct equivalent states have disjoint inputs).

Symmetric notion: output merge.

Automata equivalences

We say that two automata $\mathcal{A}$ and $\mathcal{B}$ are equivalent if $L(\mathcal{A}) = L(\mathcal{B})$. We say that they are elementary equivalent if there exist matrices R, S with elements in $\mathfrak{P}(A) \cup 1$ such that $M_{\mathcal{A}} = RS$ and $M_{\mathcal{B}} = SR$. The following result is an element of William's Classification Theorem.

Proposition

If the automaton $\mathcal{A}$ is obtained from the automaton $\mathcal{B}$ by an input (or output) merge, then $\mathcal{A}$ and $\mathcal{B}$ are elementary equivalent.

We say that two automata $\mathcal{A}$ and $\mathcal{B}$ are strongly equivalent if there is a sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ of automata such that $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_n = \mathcal{B}$ and $\mathcal{A}_i$ is elementary equivalent to $\mathcal{A}_{i+1}$ for $1 \leq i \leq n-1$.

Proposition

Two strongly equivalent irreducible automata are equivalent.

Example

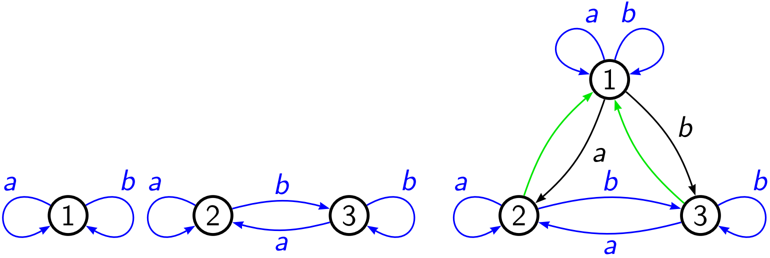


Figure: Two elementary equivalent automata $\mathcal{A}$ (on the left) and $\mathcal{B}$ (middle) and the auxiliary graph (on the right).

$$M_{\mathcal{A}} = [a+b] = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_{\mathcal{B}} = \begin{bmatrix} a & b \\ a & b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}.$$

$\mathcal{A}$ is an output merge of $\mathcal{B}$.

Unambiguous automata

An automaton $\mathcal{A} = (Q, E)$ on the alphabet A is said to be unambiguous if for all $p, q \in Q$ and $w \in A^*$ there is at most one path from p to q labeled w.

Theorem (Ashley, Marcus, Perrin, Tuncel, 1993)

Let $\mathcal{U}$ be an irreducible local automaton. Any irreducible unambiguous automaton $\mathcal{A}$ such that $L(\mathcal{A}) \subseteq L(\mathcal{U})$ is a subautomaton of an irreducible unambiguous automaton $\bar{\mathcal{A}}$ strongly equivalent to $\mathcal{U}$.

In the particular case where $L(\mathcal{U}) = A^*$, we obtain:

Corollary (Ehrenfeucht, Rozenberg, 1983)

Any irreducible unambigous automaton is a subautomaton of a complete one.

Weakly deterministic automata

An automaton $\mathcal{A}$ is said to have delay d if for any pair of paths

$$p \xrightarrow{a} q \xrightarrow{z} r, \quad p \xrightarrow{a} q' \xrightarrow{z} r',$$

with $a \in A$ $z \in A^*$ and $|z| = d$ then $q = q'$. The automaton is weakly deterministic if it has delay d for some integer $d \geq 0$.
An edge automaton is such that distinct edges have distinct labels.

Theorem (Ashley, Marcus, Perrin, Tuncel, 1993)

Let $\mathcal{U}$ be an irreducible edge automaton. Any weakly deterministic irreducible automaton $\mathcal{A}$ such that $L(\mathcal{A}) \subseteq L(\mathcal{U})$ is a subautomaton of a weakly deterministic irreducible automaton $\bar{\mathcal{A}}$ which is equivalent to $\mathcal{U}$ and has the same delay as $\mathcal{A}$.

Corollary

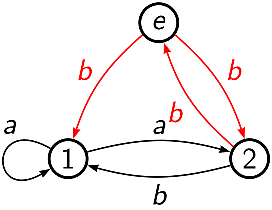
Any weakly deterministic irreducible automaton is a subautomaton of a complete one with the same delay.

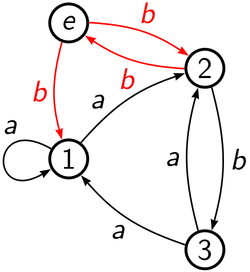
Sketch of proof

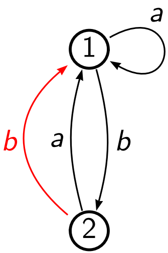
Three steps using the fact that any weakly deterministic automaton is strongly equivalent to a deterministic one and a transfer technique called Nasu's masking lemma.

- Use a sequence of input splits followed by a sequence of output merges to transform $\mathcal{A}$ in a strongly equivalent deterministic automaton $\mathcal{B}$.
- Transform $\mathcal{B}$ in a deterministic automaton by adding edges.
- work backwards to transform each of the automata in complete ones using Nasu's masking lemma.

Example







$$R_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \color{red}{e} \end{array} \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{pmatrix} a & 0 \\ 0 & b \\ a & 0 \\ \color{red}{b} & \color{red}{0} \end{pmatrix} \end{array}, \quad S_2 = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & \color{red}{e} \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & \color{red}{0} \\ 0 & 0 & 1 & \color{red}{1} \end{pmatrix} \end{array}$$

Local automata

An automaton is said to be (m, a)-local if for all pairs of paths $p \xrightarrow{u} q \xrightarrow{v} r$ and $p' \xrightarrow{u} q' \xrightarrow{v} r'$ with $|u| = m$ and $|v| = a$, one has $q = q'$. The automaton is said to be *local* if it is (m, a)-local for some $m, a \geq 0$.

A word w is a constant for an automaton $\mathcal{A}$ if $p \xrightarrow{w} q$ and $r \xrightarrow{w} s$ imply $p \xrightarrow{w} s$ and $r \xrightarrow{w} q$.

Proposition

An irreducible automaton is local if and only if it is unambiguous and there exists an integer k such that any word of length k is a constant.

The least possible integer k is called the order of $\mathcal{A}$.

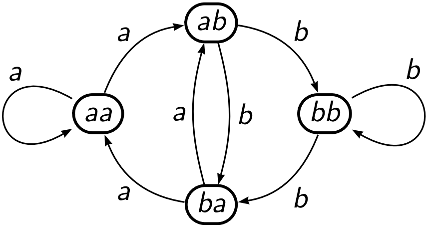


Figure: The free $(1, 1)$-local automaton.

The free $(n, 0)$-local automaton is usually known as the de Bruijn automaton of order n.

Main result

Theorem (Beal, Perrin, 2007)

Let $\mathcal{U}$ be an irreducible local automaton. Any local automaton $\mathcal{A}$ such that $L(\mathcal{A}) \subseteq L(\mathcal{U})$ is a subautomaton of a local automaton $\bar{\mathcal{A}}$ strongly equivalent to $\mathcal{U}$.

The proof uses a result due to Nasu, called the masking lemma. In the particular case where $L(\mathcal{U}) = A^*$, we obtain the following result. It solves a problem raised in Montalbano, 1993 and gives a new proof of the result of Bruyere, 1998 according to which any locally parsable code is contained in a maximal one with the same delay

Corollary

Any local automaton is a subautomaton of a complete one with the same order.

Sketch of proof

Three steps using the fact that any irreducible local automaton is strongly equivalent to a subautomaton of the free local automaton and Nasu's masking lemma.

- Use a sequence of output splits to transform $\mathcal{A}$ in a subautomaton of the free local automaton of the same order.
- Embed in the free local automaton
- Work backwards using Nasu's masking lemma.

Example

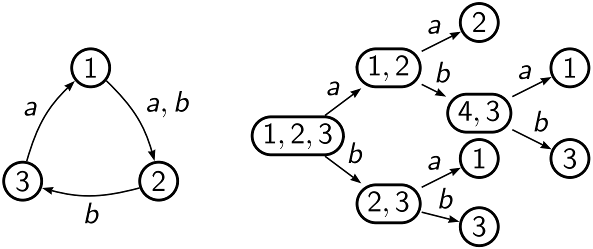


Figure: A local automaton $\mathcal{A}$ with delay 3 and the action on subsets.

Note that $\mathcal{A}$ cannot be completed in a local deterministic automaton. Indeed, we have $2 \in \mathrm{Im}(a) \cap \mathrm{Im}(b)$.

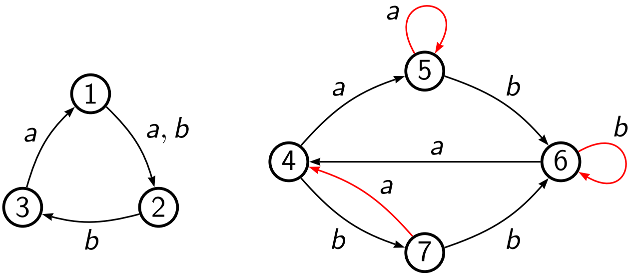


Figure: A split $\mathcal{A}_1$ of $\mathcal{A}$ and its completion with the red edges.

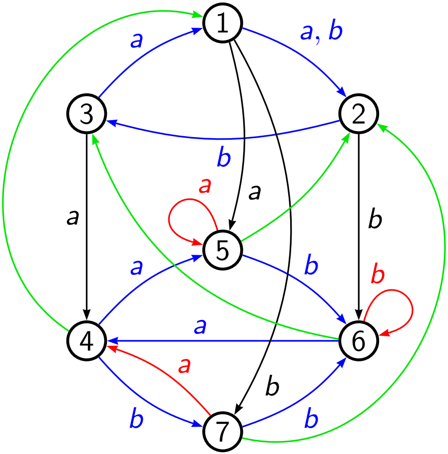


Figure: The auxiliary graph (with black edges). The automata $\mathcal{A}$ and $\mathcal{A}_1$ are represented with blue edges and the additional edges with red lines.

Matrix form

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & a & 0 & b \\ 0 & 0 & b & 0 \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ a & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad S = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & e & f & g \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

The result

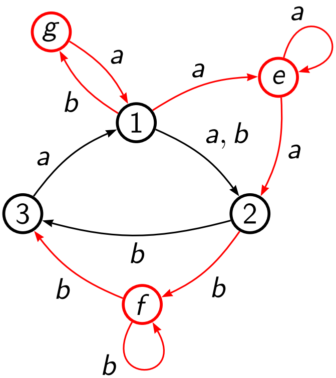


Figure: The result of the embedding.