Minimal subshifts, Schützenberger groups and profinite semigroups

Dominique Perrin

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Minimal symbolic systems are a special family of symbolic dynamical systems. I will recall recent results obtained on these systems involving discrete groups.

Profinite algebra is a theory allowing to define and study objects defined as limits of finite structures. I will show how it allows to give an account of the role played by discrete groups in minimal systems.
Symbolic systems

Consider the set $A^\mathbb{Z}$ of biinfinite sequences $x = (x_n)_{n \in \mathbb{Z}}$ with the shift $\sigma : A^\mathbb{Z} \to A^\mathbb{Z}$ defined by $y = \sigma(x)$ if $y_n = x_{n+1}$.

A symbolic system (or two-sided subshift) is a set $X \subset A^\mathbb{Z}$ of biinfinite sequences which is

1. closed for the product topology,
2. invariant by the shift, that is $\sigma(X) \subset X$.

A set of words on the alphabet $A$ is factorial if it contains $A$ and the factors (or substrings) of its elements. A factorial set $F$ is biextendable if for any $w \in F$ there are letters $a, b \in A$ such that $awb \in F$.

Proposition

The set of words appearing in the sequences of a symbolic system $X$ is a biextendable set and any biextendable set is obtained in this way.
Rotations and Sturmian words

Rotation of angle $\alpha$. $R(z) = z + \alpha \mod 1$. Natural coding: let $s(\alpha) = (s_n)$ be the sequence

$$s_n = \begin{cases} 
  a & \text{if } \lfloor (n + 1)\alpha \rfloor = \lfloor n\alpha \rfloor, \\
  b & \text{otherwise}
\end{cases}$$

For $\alpha = (3 - \sqrt{5})/2$ this gives the Fibonacci sequence which is the fixpoint of $a \mapsto ab$, $b \mapsto a$. 
Minimal systems

The symbolic system $X$ is **minimal** if it does not contain properly another nonempty one.

An infinite factorial set $F$ is said to be **uniformly recurrent** if for any word $w \in F$ there is an integer $n \geq 1$ such that $w$ is a factor of any word of $F$ of length $n$.

Remark that a uniformly recurrent set $F$ is **recurrent**: for every $u, v \in F$, there is some $x$ such that $uxv \in F$.

**Proposition**

A system is minimal if and only if the set of its factors is uniformly recurrent.
A morphism $\varphi : A^* \rightarrow A^*$ is **primitive** if there is an integer $n$ such that any letter $a$ appears in all the $\varphi^n(b)$ for $b \in A$.

**Proposition**

Let $\varphi : A^* \rightarrow A^*$ be a primitive morphism and let $x$ be a fixed point of $\varphi$. 
**The set of factors of $x$ is uniformly recurrent.**
The Fibonacci morphism \( \varphi : a \mapsto ab, b \mapsto a \) is primitive. The set \( F \) of factors of its fixed point

\[ \lim \varphi^n(a) = abaababa \cdots \]

is the Fibonacci set. It is uniformly recurrent. One has

\[ F = \{ a, b, aa, ab, ba, aab, aba, baa, \ldots \}. \]
The Thue-Morse set

The **Thue-Morse morphism** $\tau : a \mapsto ab$, $b \mapsto ba$ is primitive. The set $F$ of factors of its fixed point

$$\lim \tau^n(a) = abbabaab \cdots$$

is the **Thue-Morse set**. It is uniformly recurrent. One has

$$F = \{a, b, aa, ab, ba, bb, aab, aba, abb, baa, bab, bba, \ldots\}.$$
Factor complexity

The factor complexity of a factorial set $F$ on the alphabet $A$ is the sequence $p_n(F) = \text{Card}(F \cap A^n)$. We have $p_0(F) = 1$ and we assume $p_1(F) = \text{Card}(A)$ for any factorial set. The sets of bounded complexity are the factors of eventually periodic sequences. The binary Sturmian sets are, by definition, those of complexity $n + 1$ (like the Fibonacci set).
Return words

Let $F$ be uniformly recurrent. A **return word** to $x \in F$ is a nonempty word $y$ such that $xy$ is in $F$ and ends with $x$ for the first time. For any $x \in F$, the set

$$\mathcal{R}(x) = \{ y \in F \mid xy \in F \cap A^*x \setminus A^+xA^+ \}.$$ 

of return words to $x$ is finite.
Let $F$ be a factorial set. For a given word $w \in F$, set

\[
L(w) = \{ a \in A \mid aw \in F \}, \\
E(w) = \{ (a, b) \in A \times A \mid awb \in F \}, \\
R(w) = \{ b \in A \mid wb \in F \}.
\]

The extension graph of $w$ in $F$ is the graph on the set of vertices which is the disjoint union of $L(w)$ and $R(w)$ and with edges the set $E(w)$. For example, if $A = \{a, b\}$ and $F \cap A^2 = \{aa, ab, ba\}$, the extension graph of $\varepsilon$ is

![Extension graph example](image-url)
A factorial set $F$ is a tree set if for any $w \in F$, the extension graph of $w$ is a tree.

**Proposition**

*The complexity of a tree set $F$ on $k$ letters is $p_n(F) = (k - 1)n + 1$.***

Any Sturmian set is a tree set. Thus the Fibonacci set is a tree set. In contrast, the Thue-Morse set is not a tree set since $p_2(F) = 4$. 
The return theorem

Theorem (BDDLPRR, Monatsh. Math., 2014)

Let $F$ be a uniformly recurrent set. If $F$ is a tree set, the set of return words to any $x \in F$ is a basis of the free group on $A$.

It is not known whether the converse is true.

BDDLPRR: Valérie Berthé, Clelia De Felice, Francesco Dolce, Julien Leroy, Dominique Perrin, Christophe Reutenauer and Giuseppina Rindone.
Example

Let $F$ be the Fibonacci set. Then

$$\mathcal{R}(aba) = \{aba, baaba\}.$$ 

which is a basis of $FG(a, b)$. 
Questions:

- What algebraic structure can be given to a uniformly recurrent set?
- Can such a structure be compatible with the topology on $A^\mathbb{Z}$?

Answers:

- embed $A^*$ in the free profinite monoid $\hat{A}^*$,
- take the closure of $F$ in $\hat{A}^*$. 
The **profinite monoid metric** (or pro-$M$ metric) on a monoid $M$ is defined by

$$d(u, v) = \begin{cases} 2^{-r(u,v)} & \text{if } u \neq v \\ 0 & \text{otherwise} \end{cases}$$

where $r(u, v)$ is the minimal cardinality of a finite monoid $N$ for which there is a morphism $\varphi : M \to N$ such that $\varphi(u) \neq \varphi(v)$. Thus two words are close to each other if a large monoid is needed to separate them.

Replacing monoids by groups, one obtains the **profinite group metric** (or pro-$G$ metric) first introduced by Hall (1950).
The free profinite monoid

The completion of the free monoid for the topology induced by the profinite metric is the free profinite monoid, denoted \( \hat{A}^* \). It is a compact monoid. Its elements are called pseudowords. A sequence \( x_n \) of pseudowords converges if for any morphism into a finite monoid, the image of the sequence is ultimately constant. For any word \( x \), the sequence \( x^n \) converges to a pseudoword denoted \( x^\omega \).
The free profinite group

Likewise, the completion of the free group $FG(A)$ for the pro-$G$ metric is the **free profinite group**, denoted $\hat{FG}(A)$. It is a compact group. In contrast with the monoid case, one has for any $x \in FG(A)$, $\lim x^n = 1$. 
The profinite monoid on one letter is the monoid of \textit{profinite natural integers}, denoted \( \hat{\mathbb{N}} \).

The profinite group on one letter is the group of \textit{profinite integers}, denoted \( \hat{\mathbb{Z}} \).

The latter can be identified with the set of infinite expansions in the factorial number system formed of the

\[(\cdots c_3 c_2 c_1)! = c_1 + c_2 2! + c_3 3! + \ldots\]

with digits \(0 \leq c_i \leq i\).
Rauzy graphs

Minimal sets are not in general recognizable by finite monoids. However, the profinite setup allows one to define them by limits of sequences of finite monoids.

Let indeed $F$ be a factorial set. For each $n$, let $F_n$ be the set of words such that all their factors of length $n$ are in $F$. The set $F$ is the intersection of all $F_n$.

Each $F_n$ is recognized by an automaton $R_n(F)$ called the $n$-Rauzy graph of $F$, corresponding to a morphism into a finite monoid. Thus the closure of $F$ in $\hat{A}^*$ can be seen as defined by the limit of the graphs $R_n(F)$ or the limit of their monoids.
Greene’s relations

For two elements \( x, y \) in a topological monoid \( M \), denote

- \( x \leq_R y \) if \( x \in yM \),
- \( x \leq_L y \) if \( x \in My \),
- \( x \leq_J y \) if \( x \in MyM \).

and by \( R, L, J \) the equivalences associated to these preorders with \( H = R \cap L \). The classes are, in each case, closed subsets of \( M \).
The Schützenberger group of a $\mathcal{J}$-class

Let $J$ be a $\mathcal{J}$-class of a topological monoid $M$ and let $H$ be an $\mathcal{H}$-class contained in $J$. The set $\Gamma(H)$ of translations

$$\rho_x : y \in H \mapsto yx \in H$$

for all $x \in M$ such that $Hx = H$ forms a topological group which depends only of $J$, called the Schützenberger group of $J$. Thus, for $x \in H$, $\Gamma(H)$ is the fundamental group of the Cayley graph of $M$ at $x$. When $H$ is a group, it is isomorphic with $\Gamma(H)$. 
Example

The $\mathcal{J}$-class of $a^\omega$ in $\widehat{A}^*$ is made of one $\mathcal{H}$-class. The $\mathcal{J}$-class of $(ab)^\omega$ has four $\mathcal{H}$-classes.

\[
\begin{array}{|c|c|}
  (ab)^\omega & (ab)^\omega a \\
  b(ab)^\omega & (ba)^\omega \\
\end{array}
\]

In both cases the Schützenberger group is the group of profinite integers.
The fundamental $\mathcal{J}$-class

**Theorem (Almeida 2005)**

Let $F$ be a uniformly recurrent set. The set of infinite pseudowords with all their finite factors in $F$ forms a maximal $\mathcal{J}$-class of $\hat{A}^*$. The Schützenberger group of this $\mathcal{J}$-class is denoted $G(F)$ and called the Schützenberger group of $F$. 
Generators of the Schützenberger group

Let $F$ be a uniformly recurrent set and let $x \in J(F)$. Let $x_n$ be a sequence of elements of $F$ converging to $x$. A return word to $x$ is a limit of a converging sequence of return words to $x_n$.

**Theorem (Almeida, Costa, 2010)**

Let $F$ be a uniformly recurrent set with return sets of bounded cardinality. For any infinite pseudoword $x \in \bar{F}$ the Schützenberger group of the $H$-class of $x$ is generated by the set of return words to $x$. 
Tree sets and free groups

**Theorem (Almeida, Costa, 2015)**

Let $F$ be a uniformly recurrent set. If $F$ is a tree set, the group $G(F)$ is a free profinite group. More precisely, the natural map from $G(F)$ into $\text{FG}(A)$ is an isomorphism.

The proof uses the Return Theorem.
The finite index basis theorem

The previous result is connected with the following result, itself part of the Finite Index Basis Theorem (BDDLPRR, J. Pure Appl. Algebra, 2015)

**Theorem**

Let $F$ be a uniformly recurrent tree set. Any subgroup of finite index of the free group $FG(A)$ has a basis included in $F$.

For example, the subgroup formed by the words of even length has the basis

$$X = \{aa, ab, ba\}$$

included in the Fibonacci set.
Beyond tree sets

For any uniformly recurrent set $F$, there is a natural morphism from $G(F)$ into $\hat{A}^*$ since a maximal subgroup of $J(F)$ is included in $\hat{A}^*$.

**Theorem**

Let $F$ be a uniformly recurrent set with return sets of bounded cardinality. For any morphism $f$ from $FG(A)$ onto a finite group $G$, the following conditions are equivalent.

(i) For every $x \in F$, the image of $R_F(x)$ by $f$ generates $G$.

(ii) The natural morphism from $G(F)$ into $G$ is surjective.
Morphisms of finite order

Let \( f : A^* \to G \) be a morphism from \( A^* \) into a finite group \( G \) and let \( \varphi : A^* \to A^* \) be a morphism. We denote by \( \varphi_G \) the map from \( G^A \) into itself defined for \( h \in G^A \) and \( a \in A \) by \( \varphi_G(h)(a) = h(\varphi(a)) \). We say that \( \varphi \) has finite \( f \)-order if there is an integer \( n \geq 1 \) such that \( \varphi^n_G(f) = f \). The least such integer is called the \( f \)-order of \( \varphi \).

Any substitution \( \varphi \) which is invertible in \( FG(A) \) is of finite \( f \)-order for any morphism \( f \). Thus the Fibonacci morphism has finite \( f \)-order for any \( f \).
Fixed points of morphisms

Theorem (Almeida, Costa, 2013)

Let $\varphi$ be a non-periodic primitive substitution over $A$ and let $f : A^* \to G$ be a morphism onto a finite group. The natural morphism from $G(F)$ into $G$ is surjective if and only if $\varphi$ has finite $f$-order.
Example 1: trivial image

Let $F$ be the Thue-Morse set and let $f : A^* \rightarrow S_3$ be represented below on the left. The $f$-order of $\tau$ is infinite since, $f(\tau(a)) = f(\tau(b)) = (1)$. The intersection of $F$ with the generating set of the submonoid fixing 1 is represented on the right.
The word $aa$ has rank 3 and image $I = \{1, 2, 4\}$. The action on the images accessible from $I$ is given below.

$\begin{align*} &1, 2, 8 \xrightarrow{b} 1, 3, 10 \\
&1, 2, 4 \xrightarrow{b} 1, 3, 6 \xrightarrow{b} 1, 3, 5 \xrightarrow{a} 1, 2, 7 \xrightarrow{b} 1, 3, 9 \xrightarrow{a} 1, 2, 11 \end{align*}$

**Figure**: The action on the minimal images

All words with image $\{1, 2, 4\}$ end with $aa$. The paths returning for the first time to $\{1, 2, 4\}$ are labeled by the set $\mathcal{R}_F(aa) = \{b^2a^2, bab^2aba^2, bab^2a^2, b^2aba^2\}$. Moreover each of the words of $\mathcal{R}_F(aa)$ defines the trivial permutation on the set $\{1, 2, 4\}$. Thus the image of $G(F)$ in $S_3$ is trivial.
Consider again the Thue-Morse substitution $\tau$ and the Thue-Morse set $F$. Let $f$ be the morphism $f : a \mapsto (123), b \mapsto (345)$ from $A^*$ onto the alternating group $A_5$. It can be verified that $\tau$ has $f$-order 6.

Let $Z$ be the generating set of the submonoid stabilizing 1 and let $X = Z \cap F$. 
We represent below the set $X$ keeping only the nodes with two sons.
The image of $\tau^4(b)$ is $\{1, 3, 4, 9, 10\}$ and thus it is minimal. The action on its image is shown below. The return words to $\tau^4(b)$ are $\tau^4(b)$, $\tau^3(a)$ and $\tau^5(ab)$. The permutations on the image of $\tau^4(b)$ are the 3 cycles of length 5 indicated in Figure 2. Since they generate the group $A_5$, the image of $G(F)$ in $A_5$ is $A_5$.

Figure: The action on the minimal images.
Conclusion and perspectives

- Profinite monoids allow to handle simultaneously all possible morphisms from $A^*$ into finite groups.
- They allow to formulate results going beyond tree sets.
- Possible next step: replace iterated morphisms by $S$-adic expansions (see the conference of Valérie Berthé).
- An intriguing perspective: some tree sets are encodings of transformations on compact surfaces (suspension surfaces obtained from interval exchanges or more generally linear involutions, see BDDLPRRR, JETDS 2016). Is there a connexion between a connexion between the fundamental group of these surfaces and the Schützenberger group $G(F)$?