Combinatorics on Sturmian words

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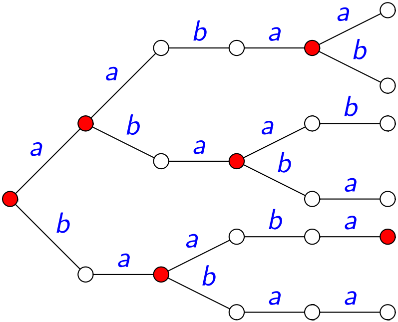
On the occasion of Eric Goles anniversary

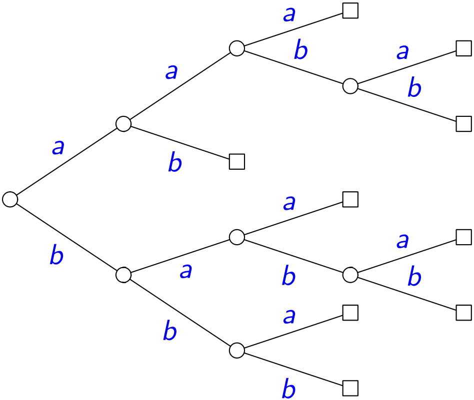
29 novembre 2011



The factors of length ≤ 5 of the Fibonacci word $x = abaababa\ldots$ fixpoint of $a \mapsto ab,\ b \mapsto a$.

A miracle

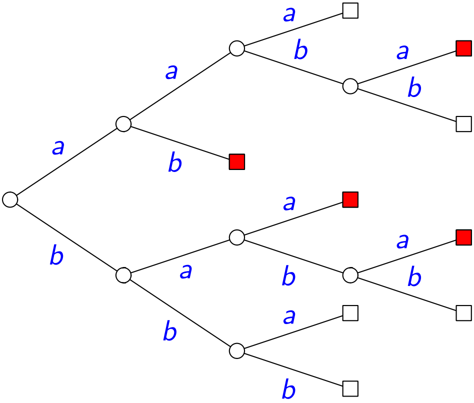
Consider the set X below obtained by a perturbation of the set of binary words of length 3.



There are 4 words which are factors of the Fibonacci word.

A miracle

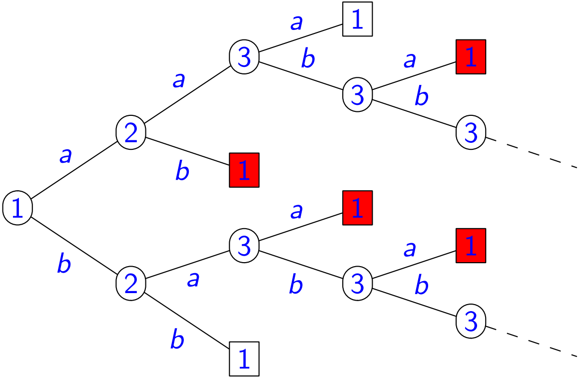
Consider the set X below obtained by a perturbation of the set of binary words of length 3.



There are 4 words which are factors of the Fibonacci word.

Second miracle

Consider the first returns to 1 for the substitution $a \mapsto (123)$, $b \mapsto (12)$.



The intersection with F is the same as the previous one.

Outline

We show that

- in a Sturmian set F, any finite F-maximal bifix code of degree d on k letters has $(k-1)d+1$ elements (Cardinality Theorem).
- if an infinite word x is such that $\mathrm{Card}(F(x) \cap X) \leq d$ for some finite maximal bifix code X of degree d, then x is ultimately periodic (Periodicity Theorem).
- in a Sturmian set, any finite F-maximal bifix code of F-degree d is a basis of a subgroup of index d of the free group on A and conversely (Sturmian Basis Theorem).

Based on Bifix codes and Sturmian words, by Jean Berstel, Clelia De Felice, Dominique Perrin, Christophe Reutenauer, Giuseppina Rindone (BDPRR, 2010).

Sturmian sets

Given a set F of words over an alphabet A, the right order of a word u in F is the number of letters a such that $ua \in F$. A word u is right-special if its right order is at least 2. A right-special word is strict if its right order is equal to $\mathrm{Card}(A)$.

A set of words F is Sturmian if it is the set of factors of an infinite word and if

- it is closed under reversal
- it contains, for each $n \geq 1$, exactly one right-special word u of length n which is moreover strict.

It is easy to see that for a Sturmian set F on an alphabet A with k letters, the set $F \cap A^n$ has $(k-1)n+1$ elements for each n.

Example

Set $A = \{a, b\}$. The Fibonacci set is the set of factors of the inifinite word

$$x = abaababaabaababaababaabaababaabaab\cdots$$

called the Fibonacci word. It is the fixpoint $f^\omega(a)$ of the morphism $f : A^* \to A^*$ defined by $f(a) = ab$ and $f(b) = a$.

Example

Set $A = \{a, b, c\}$. The morphism $f : A^* \to A^*$ defined by $f(a) = ab$, $f(b) = ac$ and $f(c) = a$ has the fixpoint

$$x = abacabaabacababacabaabacabacabaabacab\cdots$$

called the Tribonacci word. The set $F(x)$ is Sturmian.

Bifix codes

A set X of nonempty words is a prefix code if any two distinct elements of X are incomparable for the prefix order.

Example

The set $X = \{a, ba\}$ is a prefix code.

A set X of nonempty words is a bifix code if any two distinct elements of X are incomparable for the prefix order and for the suffix order.

Example

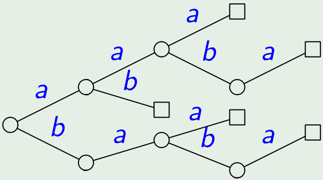
The set $X = \{a, bab\}$ is a bifix code.

Maximal bifix codes

A prefix code (resp. a bifix code) $X \subset F$ is F-maximal if it is not properly contained in any other prefix code (resp. bifix code) $Y \subset F$.

Example

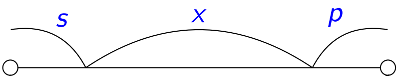
Let $A = \{a, b\}$ and let F be the set of words without factor bb . The set $X = \{aaa, aaba, ab, baa, baba\}$ is a finite F-maximal bifix code.



Parses

A parse of a word w with respect to a set X is a triple (s, x, p) such that $w = sxp$ with

- s has no suffix in X,
- $x \in X^*$
- p has no prefix in X



Example

The set $X = \{a, bab\}$ is a finite bifix code. The word bab has two parses : $(1, bab, 1)$ and (b, a, b).

Degree of a bifix code

Let X be a bifix code. For any word w and any letter $a \in A$
The F-degree, denoted $d_F(X)$, of a bifix code X is the maximum of the number of parses of the words of F.

Theorem (Schützenberger, 1965)

Let F be a recurrent set and let $X \subset F$ be a finite bifix code. Then X is an F-maximal bifix code if and only if its F-degree is finite.

Example

Let F be the Fibonacci set. The set $X = \{a, bab, baab\}$ is an F-maximal bifix code of degree 2. The parses of bab are $(1, bab, 1)$ and (b, a, b).

Example

Let F be the Fibonacci set. The set $X = \{aaba, ab, baa, baba\}$ is an F-maximal bifix code of degree 3. The word $aaba$ has three parses $(1, aaba, 1)$, (a, ab, a) and $(aa, 1, ba)$.

The Cardinality Theorem

The following result generalizes the fact that a Sturmian word has $d+1$ factors of length d.

Theorem (BDPRR, 2010)

Let F be a Sturmian set on an alphabet with k letters. For any finite F-maximal bifix code $X \subset F$, one has
$\mathrm{Card}(X) = (k-1)d_F(X) + 1$.

Let $x = a_0 a_1 \cdots$, with $a_i \in A$, be an infinite word. It is periodic if there is an integer $n \geq 1$ such that $a_{i+n} = a_i$ for all $i \geq 0$.
It is ultimately periodic if the equalities hold for all i large enough. Thus, x is ultimately periodic if there is a word u and a periodic infinite word y such that $x = uy$. The following result, due to Coven and Hedlund, is well-known.

Theorem (Coven and Hedlund, 1973)

Let $x \in A^{\mathbb{N}}$ be an infinite word. If there exists an integer $d \geq 1$ such that x has at most d factors of length d then x is ultimately periodic.

The Periodicity Theorem

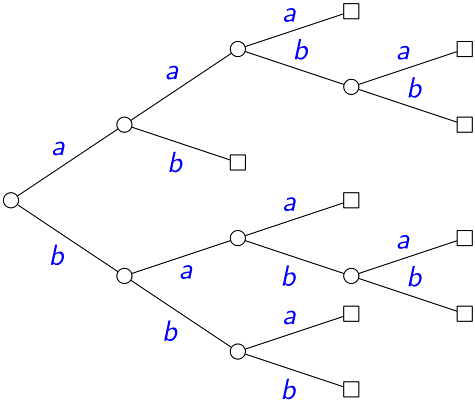
The following statement implies the Coven-Hedlund Theorem since A^d is a maximal bifix code of degree d.

Theorem (BDPRR, 2010)

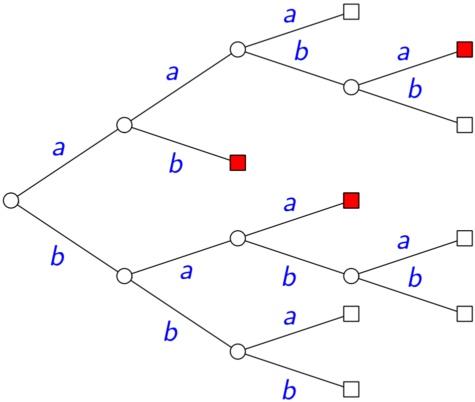
Let $x \in A^{\mathbb{N}}$ be an infinite word. If there exists a finite maximal bifix code X of degree d such that $\mathrm{Card}(X \cap F(x)) \leq d$, then x is ultimately periodic.

The proof uses the Critical Factorization Theorem.

Consider the maximal bifix code of degree 3 below.



Consider the maximal bifix code of degree 3 below.



Assume that $X \cap F(x)$ is the set of red nodes. Then a factor aab can only be followed by a second aab. Thus $x = u(aab)^{\omega}$.

Sturmian Basis Theorem

Theorem (BDPRR, 2010)

Let F be a Sturmian set and let $d \geq 1$ be an integer. A bifix code $X \subset F$ is a basis of a subgroup of index d of A° if and only if it is a finite F-maximal bifix code of F-degree d.

Note that this contains the Cardinality Theorem. Indeed, by Schreier's formula, if H is a subgroup of rank n and index d of a free group of rank k, then

$$n - 1 = d(k - 1)$$

Let X be a F-maximal bifix code of F-degree d. By the above theorem, it is a basis of a subgroup of index d of the free group A° which has rank k. Thus $\mathrm{Card}(X) = (k-1)d + 1$ by Schreier's formula.

Corollary

Let F be a Sturmian set. For any $n \geq 1$, the set $F \cap A^n$ is a basis of the subgroup of A° generated by A^n.

Direct proof : show by descending induction on $i = d, \ldots, 0$ that for any $u \in F \cap A^i$, one has $uA^{d-i} \subset \langle X \rangle$. It is true for $i = d$. Next consider a right-special word $u \in F \cap A^i$. By induction hypothesis, we have $uaA^{d-i-1} \subset \langle X \rangle$ for any $a \in A$. Thus $uA^{d-i} \subset \langle X \rangle$. For another $v \in A^i$, let w be such that $vw \in F \cap A^d$. Then $vt = vw(uw)^{-1}ut$ for any $t \in A^{d-i}$.

Example

Let F be the Fibonacci set. We have $F \cap A^2 = \{aa, ab, ba\}$ and $bb = ba(aa)^{-1}ab$.

The following corollary contains the well-known fact that a subgroup of finite index of a free group has a positive basis.

Corollary

Let F be a Sturmian set. Any subgroup of finite index of the free group on A has a basis contained in F.

Let indeed H be a subgroup of index d of A°. Let Z be the bifix codes which generates the submonoid $H \cap A^*$. Then Z is a maximal bifix code of degree d. The set $X = Z \cap F$ is an F-maximal bifix code of degree $e \leq d$. By the Sturmian Basis Theorem, it is the basis of a subgroup K of index e. But then $K \subset H$ implies that d divides e. Thus $d = e$ and $H = K$.

As a further consequence of the Sturmian Basis Theorem, we have the following result.

Corollary

Let F be a Sturmian set on an alphabet with k letters. The number $N_{d,k}$ of finite F-maximal bifix codes $X \subset F$ of F-degree d satisfies $N_{1,k} = 1$ and

$$N_{d,k} = d(d!)^{k-1} - \sum_{i=1}^{d-1} [(d-i)!]^{k-1} N_{i,k}.$$

The formula results directly from the formula, due to Hall (1949), for the number of subgroups of index d in a free group of rank k. The values for $k = 2$ are given by the recurrence

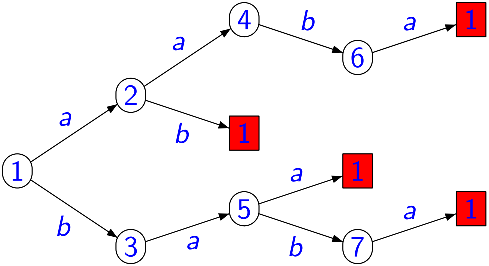
$$N_{d,2} = d\ d! - \sum_{i=1}^{d-1} (d-i)! N_{i,2}.$$

The first values are

d	1	2	3	4	5	6	7	8	9	10
$N_{d,2}$	1	3	13	71	461	3447	29093	273343	2829325	3199890[illegible]

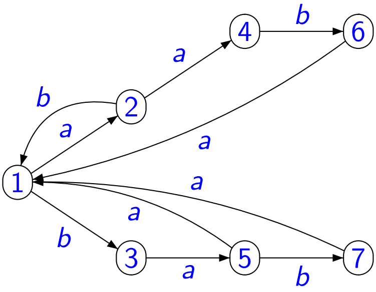
The formula is known to enumerate also the indecomposable permutations on $d+1$ elements (see Dress, Franz 1985, Ossona, Rosenstiehl 2004 and Cori 2009).

Stallings foldings



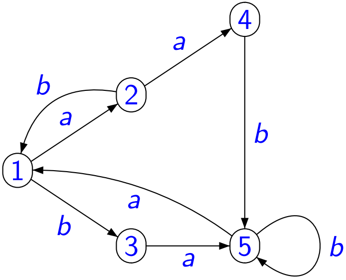
An F-maximal bifix code of F-degree 3.

Stallings foldings



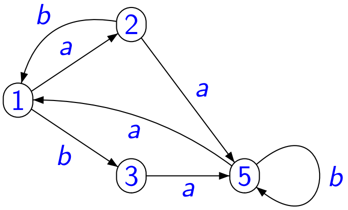
Fusion of 5, 6, 7.

Stallings foldings



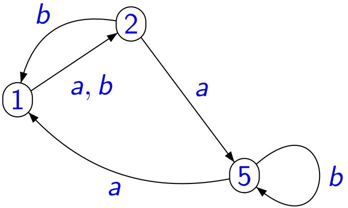
Fusion of $4, 5$.

Stallings foldings



Fusion of 2, 3.

Stallings foldings



$a \mapsto (125),\ b \mapsto (12).$