The road coloring problem
celebration of 10 years of F'SATIE

Dominique Perrin

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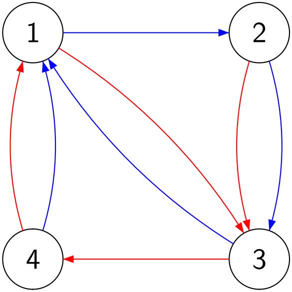
An example

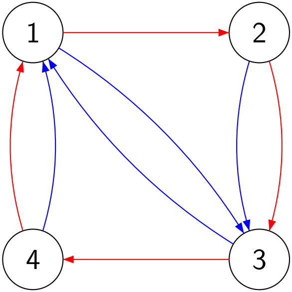
An irreducible directed graph with four vertices and two edges going out of each vertex.



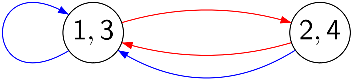
BRB is a homing sequence: starting anywhere, it leads to 1

Another example

With a different coloring:



there is no homing sequence: the sets $\{1,3\}$ and $\{2,4\}$ cannot be collapsed further:



The road coloring problem

A graph is road colorable if there is a coloring of its edges such that

1. the edges going out of a vertex have distinct colors and
2. there is a homing sequence.

aperiodic graph: the gcd of the cycle lengths is 1.

Problem (Adler,Goodwin, Weiss, 1977)

Is any irreducible directed graph which is aperiodic and has constant outdegree road colorable?

Solved by Avram N. Trahtman in 2007.
`http://www.cs.biu.ac.il/ trakht/syn.html`

Example

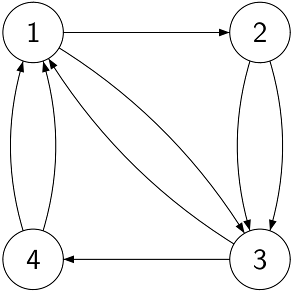


Figure: A road colorable graph

Motivations

- data storage: the coloring defines an encoding of data corresponding to the graph edges in a state-dependant coding. A homing sequence resets the encoder even if errors have occurred (optical or magnetic storage) (Lind, Marcus, 1995).
- automated design: devices accepting parts in any orientation and output them in some predetermined orientation (Eppstein, 1990).
- Communication protocols: test sequences to check whether a protocol conforms to its specification (Aho et al. 1991)

The Franaszek encoder (IBM patent)

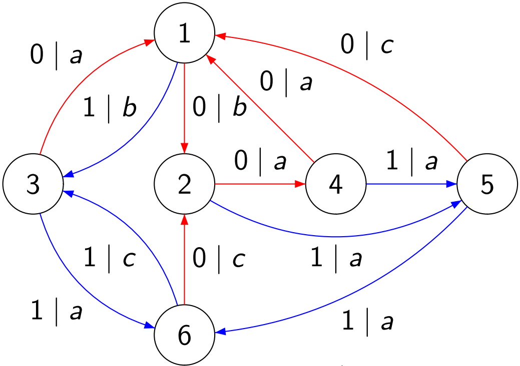


Figure: A binary input is encoded at rate 1/2 by sequences of $a = 00$, $b = 01$ and $c = 10$. The result satisfies the $[2, 7]$ constraint: two 1's are separated by at least 2 and at most 7 0's.

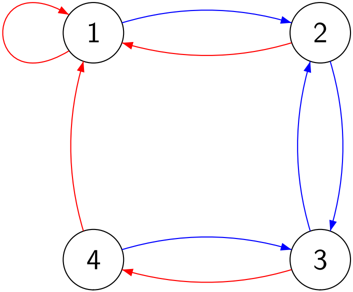
010 is a homing sequence.

Background

1. Statement of the problem (Adler et al., 1977).
2. Graphs with a cycle of prime length (O'Brien, 1981).
3. Colorings and eigenvectors (Friedman, 1990)
4. Synchronized prefix codes (P. and Schützenberger, 1992)
5. Eulerian graphs (Kari, 2001)
6. Solution (Trahtman, 2007)

A simple case

Hypothesis: the graph has a loop.



Solution: color red the loop and the edges of a spanning tree of the reverse graph with root in the loop vertex.

O'Brien's theorem

A directed graph is admissble if it is irreducible, aperiodic and has constant outdegree.

Theorem (O'Brien, 1981)

If an admissible graph without multiple edges has a cycle C of prime length then it is road colorable with all edges in C of the same color.

Proof: extension of the loop case.

Friedman's theorem

Let M be the adjacency matrix of the graph G and let w be a left eigenvector of M for the eigenvalue k (the outdegree of the vertices):

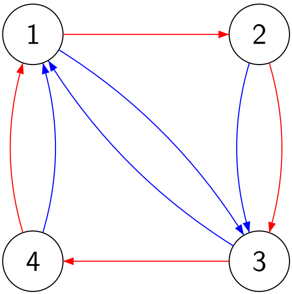
$$wM = km$$

Suppose that w has integer components without common factor.

Theorem (Friedman, 1990)

If an admissible graph G has a cycle C of length prime to $W = \sum w(v)$, then G is road colorable with all edges in C of the same color.

Example



We have

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}, \quad w = \begin{bmatrix} 2 & 1 & 2 & 1 \end{bmatrix}, \quad W = 6$$

The partition $\{1,2\}\{3,4\}$ into maximal collapsible classes is formed of classes of equal weight.

Synchronized prefix codes

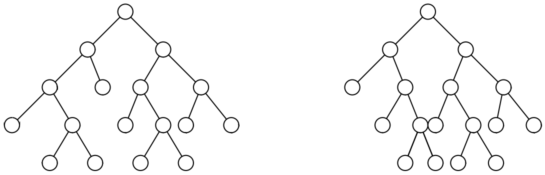


Figure: Two isomorphic binary trees/flipping equivalent prefix codes

synchronized prefix code: the colored graph obtained by merging the leaves of the tree with the root has a homing sequence.

Theorem (P., Schützenberger, 1992)

Any finite maximal prefix code is flipping equivalent to a synchronized one.

Kari's theorem

Consider a colored graph with distinct colors for edges going out of each vertex. Two vertices u, v of a colored graph form a stable pair if for any color sequence p there is a sequence q such that pq leads u and v to the same vertex.

Theorem (Culik, Karhumaki, Kari, 2001)

If the graph obtained by collapsing stable pairs is road colorable, then so is the original one.

Corollary (Kari, 2001)

Any admissible eulerian directed graph is road colorable.

A directed graph is eulerian if the indegree of each vertex is equal to its outdegree.

Trahtman's solution

Theorem (Trahtman, 2007)

For any admissible graph G without stable pairs, there exists a set S of edges such that

1. *from any vertex starts exactly one edge in S,*
2. *all vertices of maximal index enter a cycle by the same vertex.*

Moreover G is road colorable with all vertices of S of the same color.

The index of a vertex is number of S-edges used before entering a S-cycle.

Example

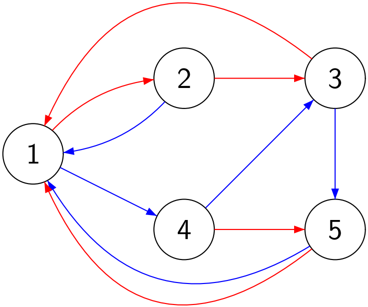


Figure: The red edges form a set for which index(4) = 2 and index(5) = 1

$BRBRB$ is a homing sequence.

A still unsolved problem

Problem (Cerny, 1964)

Suppose that a graph with n vertices is road colorable. Is there a homing sequence of length at most n^2?

True in some particular cases:

- aperiodic colorings (Trahtman, 2002)
- graphs with an n-cycle (Pin, 1980)
- eulerian graphs (Kari, 2001)
- ...