The road coloring problem
celebration of 10 years of F’SATIE

Dominique Perrin

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An example

An irreducible directed graph with four vertices and two edges going out of each vertex.

B\textcolor{red}{R}B is a homing sequence: starting anywhere, it leads to 1
Another example

With a different coloring:

there is no homing sequence: the sets \( \{1, 3\} \) and \( \{2, 4\} \) cannot be collapsed further:
The road coloring problem

A graph is **road colorable** if there is a coloring of its edges such that

1. the edges going out of a vertex have distinct colors and
2. there is a homing sequence.

**aperiodic** graph: the gcd of the cycle lengths is 1.

**Problem (Adler, Goodwin, Weiss, 1977)**

*Is any irreducible directed graph which is aperiodic and has constant outdegree road colorable?*

Solved by Avram N. Trahtman in 2007.

http://www.cs.biu.ac.il/ trakht/syn.html
Figure: A road colorable graph
Motivations

- **data storage**: the coloring defines an encoding of data corresponding to the graph edges in a state-dependant coding. A homing sequence resets the encoder even if errors have occurred (optical or magnetic storage) (Lind, Marcus, 1995).

- **automated design**: devices accepting parts in any orientation and outputting them in some predetermined orientation (Eppstein, 1990).

- **Communication protocols**: test sequences to check whether a protocol conforms to its specification (Aho et al. 1991)
Figure: A binary input is encoded at rate $1/2$ by sequences of $a = 00$, $b = 01$ and $c = 10$. The result satisfies the $[2, 7]$ constraint: two 1’s are separated by at least 2 and at most 7 0’s.

$010$ is a homing sequence.
Background

3. Colorings and eigenvectors (Friedman, 1990).
5. Eulerian graphs (Kari, 2001).
Hypothesis: the graph has a loop.

Solution: color red the loop and the edges of a spanning tree of the reverse graph with root in the loop vertex.
O’Brien’s theorem

A directed graph is **admissible** if it is irreducible, aperiodic and has constant outdegree.

**Theorem (O’Brien, 1981)**

*If an admissible graph without multiple edges has a cycle $C$ of prime length then it is road colorable with all edges in $C$ of the same color.*

Proof: extension of the loop case.
Let $M$ be the adjacency matrix of the graph $G$ and let $w$ be a left eigenvector of $M$ for the eigenvalue $k$ (the outdegree of the vertices):

$$wM = km$$

Suppose that $w$ has integer components without common factor.

**Theorem (Friedman, 1990)**

*If an admissible graph $G$ has a cycle $C$ of length prime to $W = \sum w(v)$, then $G$ is road colorable with all edges in $C$ of the same color.*
We have

\[
M = \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 2 & 0 \\
1 & 0 & 0 & 1 \\
2 & 0 & 0 & 0
\end{bmatrix}, \quad w = [2 \ 1 \ 2 \ 1], \quad W = 6
\]

The partition \( \{1, 2\}\{3, 4\} \) into maximal collapsible classes is formed of classes of equal weight.
Synchronized prefix codes

**Theorem (P., Schützenberger, 1992)**

*Any finite maximal prefix code is flipping equivalent to a synchronized one.*

**Figure:** Two isomorphic binary trees/flipping equivalent prefix codes

synchronized prefix code: the colored graph obtained by merging the leaves of the tree with the root has a homing sequence.
Consider a colored graph with distinct colors for edges going out of each vertex. Two vertices $u, v$ of a colored graph form a stable pair if for any color sequence $p$ there is a sequence $q$ such that $pq$ leads $u$ and $v$ to the same vertex.

**Theorem (Culik, Karhumaki, Kari, 2001)**

*If the graph obtained by collapsing stable pairs is road colorable, then so is the original one.*

**Corollary (Kari, 2001)**

*Any admissible eulerian directed graph is road colorable.*

A directed graph is eulerian if the indegree of each vertex is equal to its outdegree.
Theorem (Trahtman, 2007)

For any admissible graph $G$ without stable pairs, there exists a set $S$ of edges such that

1. from any vertex starts exactly one edge in $S$,
2. all vertices of maximal index enter a cycle by the same vertex.

Moreover $G$ is road colorable with all vertices of $S$ of the same color.

The index of a vertex is number of $S$-edges used before entering a $S$-cycle.
Figure: The red edges form a set for which $\text{index}(4) = 2$ and $\text{index}(5) = 1$

$BRBRBR$ is a homing sequence.
A still unsolved problem

Problem (Cerny, 1964)

Suppose that a graph with $n$ vertices is road colorable. Is there a homing sequence of length at most $n^2$?

True in some particular cases:

- aperiodic colorings (Trahtman, 2002)
- graphs with an $n$-cycle (Pin, 1980)
- eulerian graphs (Kari, 2001)
- ...