Variants of the road coloring problem

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Outline

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- The road coloring theorem
- Unambiguous automata
- Synchronized prefix codes
- Commutative equivalence

Synchronizing words

For a deterministic automaton of A (set of letters=colors) on Q (set of states)

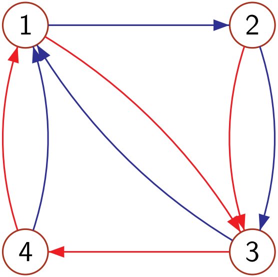
$$Q \times A \to Q$$

a word $w \in A^*$ is synchronizing if

$$\mathrm{Card}(Q \cdot w) = 1$$

Synchronized coloring of the edges of a graph: there exists a synchronizing word.
The degree of a coloring is the minimal nonzero value of $\mathrm{Card}(Q \cdot w)$ for $w \in A^*$. Thus $\mathcal{A}$ is synchronized if it has degree 1.

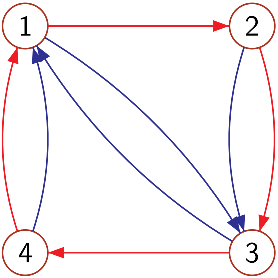
Example



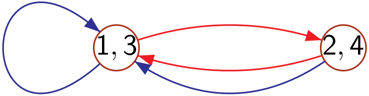
BRB is synchronizing.

Another example

With a different coloring:



there is no synchronizing word : the sets $\{1, 3\}$ and $\{2, 4\}$ cannot be collapsed further:



The road coloring problem

problem: find a coloring a graph which turns it into a synchronized deterministic automaton.

admissible multigraph:

- constant outdegree
- strongly connected
- aperiodic (gcd length of cycles$=1$)

Theorem (Trahtman, 2007)

Any admissible graph has a synchronized coloring.

Sketch of proof

p, q synchronizable: $\exists w \in A^* : p \cdot w = q \cdot w$.
p, q strongly synchronizable: $\forall u \in A^*$, $p \cdot u, q \cdot u$ synchronizable.

Lemma (Culik& al., 2001)

Let $\equiv$ be the equivalence merging strongly synchronizing states. If $G/\equiv$ has a synchronized coloring, then G also does.

height of a vertex: length of red path leading to a red cycle.

Lemma (Trahtman)

If all vertices of maximal heigth belong to the same red tree, there is a strongly synchronizing pair.

Road coloring algorithm

$n =$ number of states.

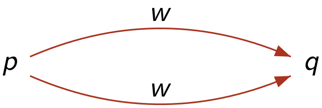
Theorem (M.P. Béal, DP)

There is an $O(n^2)$ algorithm to find a synchronized coloring.

Trahtman's proof: $O(n^3)$.
improvement: linear algorithm to find an equivalent automaton which has a strongly synchronizing pair.

Unambiguous automata

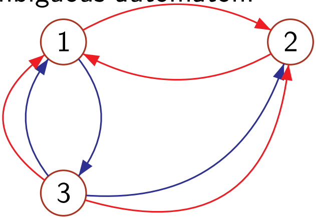
unambiguous: Two paths with the same origin, end and label coincide (No diamonds)



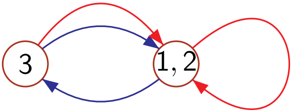
complete: any word $w \in A^*$ is the label of some path.
synchronizing word: $\{(p,q) \in Q \times Q \mid p \xrightarrow{w} q\}$ is a direct product.

Example

A complete unambiguous automaton:



Synchronizing word: RB $= \{2,3\} \times \{3\}$ Determinization:



Two questions

Generalization of the road coloring theorem to unambiguous automata.

Question

Does any complete unambiguous aperiodic automaton have the same underlying graph as a synchronized one?

Question

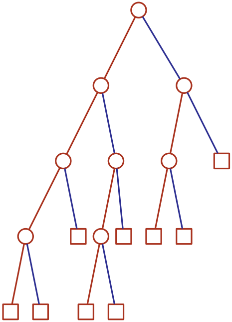
What are the underlying graphs of unambiguous automata on k colors?

necessary condition: The adjacency matrix has spectral radius $\leq k$.

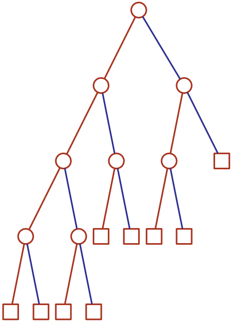
Prefix codes

$X \subset A^+$ finite maximal prefix code.
synchronized by x if $A^*x \subset X^*$.
road coloring theorem $\implies$ any finite maximal prefix code of period 1 has the same tree as a synchronized one.
Consequence: among the optimal prefix codes for a weight distribution (Huffman algorithm), there is a synchronized one (provided the period is 1).

Example



non synchronized



synchronized by $RLLRR$

Alphabet with cost

cost $c : A \longrightarrow \mathbb{N}$,
cost of a word $c(a_1 a_2 \cdots a_n) = c(a_1) + c(a_2) + \ldots + c(a_n)$.
cost of a prefix code $c(X) = \sum_{x \in X} c(x) w(x)$ for given weights w.
Optimal prefix code: $c(X)$ minimal (Polynomial algorithm?).

Theorem

Among the optimal prefix codes for weigths w and cost c, there is a synchronized one (provided the period is 1).

Commutative equivalence

Two sets of words X, Y are commutatively equivalent if one can be obtained from the other by a rearrangement of the letters within its words.

Theorem (P., Schützenberger, 1992)

Any finite maximal prefix code of period 1 *is commutatively equivalent to a synchronized one.*

Sketch of proof

Non-commutative factorization theorem.

Theorem (Reutenauer, 1985)

For any finite maximal prefix code X of degree d on A, there exist non commutative polynomials $L, R, D \in \mathbb{Z}\langle A \rangle$ s.t.

$$X - 1 = L(A-1)R, \quad R = d + D(A-1)$$

Conjecture

One has $L, R \in \mathbb{N}\langle A \rangle$ and D is a disjoint union of $d-1$ prefix-closed sets.