The road coloring problem
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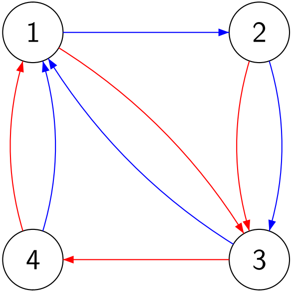
An example

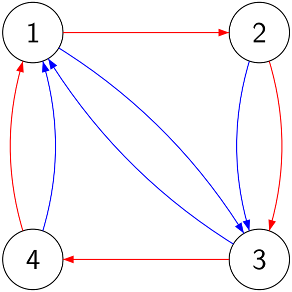
An irreducible directed graph with four vertices and two edges going out of each vertex.



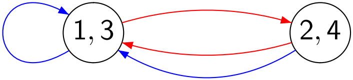
BRB is a homing sequence: starting anywhere, it leads to 1.
Synchronizing word for a deterministic complete automaton.

Another example

With a different coloring:



there is no homing sequence: the sets $\{1, 3\}$ and $\{2, 4\}$ cannot be collapsed further:



The road coloring problem

A graph is road colorable if there is a coloring of its edges such that

1. the edges going out of a vertex have distinct colors and
2. there is a homing sequence.

aperiodic graph: the gcd of the cycle lengths is 1.

Problem (Adler,Goodwin, Weiss, 1977)

Is any irreducible directed graph which is aperiodic and has constant outdegree road colorable?

Solved by Avram N. Trahtman in 2007.
`http://www.cs.biu.ac.il/ trakht/syn.html`

Example

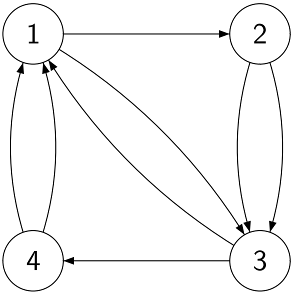


Figure: A road colorable graph

Equivalent automata: the undelying graphs are identical. Any complete deterministic automaton which is aperiodic is equivalent to a synchronized one.

Motivations

- text compression: the optimal prefix code obtained by Huffman algorithm can be chosen synchronized (error resilience).
- data storage: the coloring defines an encoding of data corresponding to the graph edges in a state-dependant coding. A homing sequence resets the encoder even if errors have occurred (optical or magnetic storage) (Lind, Marcus, 1995).
- automated design: devices accepting parts in any orientation and output them in some predetermined orientation (Eppstein, 1990).
- Communication protocols: test sequences to check whether a protocol conforms to its specification (Aho et al. 1991)

The Franaszek encoder (IBM patent)

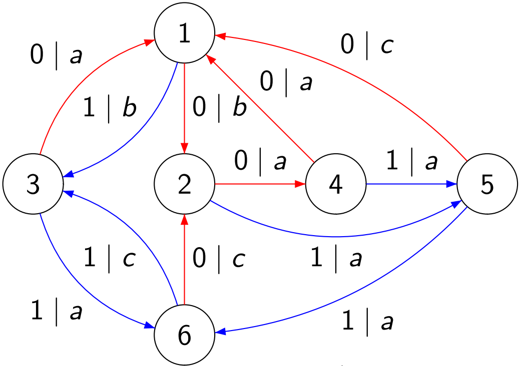


Figure: A binary input is encoded at rate 1/2 by sequences of $a = 00$, $b = 01$ and $c = 10$. The result satisfies the $[2, 7]$ constraint: two 1's are separated by at least 2 and at most 7 0's.

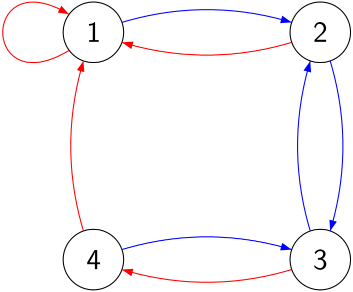
010 is a homing sequence.

Background

1. Statement of the problem (Adler et al., 1977).
2. Graphs with a cycle of prime length (O'Brien, 1981).
3. Colorings and eigenvectors (Friedman, 1990)
4. Synchronized prefix codes (P. and Schützenberger, 1992)
5. Eulerian graphs (Kari, 2001)
6. Solution (Trahtman, 2007)

A simple case

Hypothesis: the graph has a loop.



Solution: color red the loop and the edges of a spanning tree of the reverse graph with root in the loop vertex.

O’Brien’s theorem

A directed graph is admissible if it is irreducible, aperiodic and has constant outdegree.

Theorem (O’Brien, 1981)

If an admissible graph without multiple edges has a cycle C of prime length then it is road colorable with all edges in C of the same color.

Proof: extension of the loop case.

Friedman's theorem

Let M be the adjacency matrix of the graph G and let w be a left eigenvector of M for the eigenvalue k (the outdegree of the vertices):

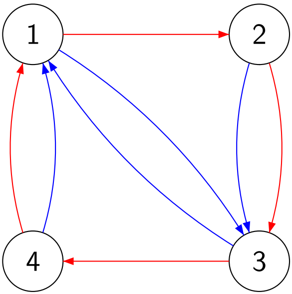
$$wM = kw$$

Suppose that w has integer components without common factor.

Theorem (Friedman, 1990)

If an admissible graph G has a cycle C of length prime to $W = \sum w(v)$, then G is road colorable with all edges in C of the same color.

Example



We have

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}, \quad w = \begin{bmatrix} 2 & 1 & 2 & 1 \end{bmatrix}, \quad W = 6$$

The partition $\{1,2\}\{3,4\}$ into maximal collapsible classes is formed of classes of equal weight.

Synchronized prefix codes

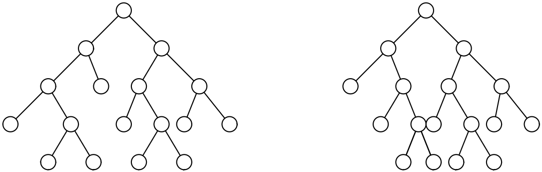


Figure: Two isomorphic binary trees/flipping equivalent prefix codes

synchronized prefix code: the colored graph obtained by merging the leaves of the tree with the root has a homing sequence.

Theorem (P., Schützenberger, 1992)

Any finite maximal prefix code is flipping equivalent (resp. commutatively equivalent) to a synchronized one.

Kari's theorem

Consider a colored graph with distinct colors for edges going out of each vertex. Two vertices p, q of a colored graph form a stable pair if for any color sequence u there is a sequence v such that uv leads p and q to the same vertex.

Theorem (Culik, Karhumaki, Kari, 2001)

If the graph obtained by collapsing stable pairs is road colorable, then so is the original one.

Corollary (Kari, 2001)

Any admissible eulerian directed graph is road colorable.

A directed graph is eulerian if the indegree of each vertex is equal to its outdegree.

Trahtman's solution

Theorem (Trahtman, 2007)

For any admissible graph G without stable pairs, there exists a set S of edges such that

1. *from any vertex starts exactly one edge in S,*
2. *all vertices of maximal index enter a cycle by the same vertex.*

Moreover G is road colorable with all vertices of S of the same color.

The index of a vertex is number of S-edges used before entering a S-cycle.

Example

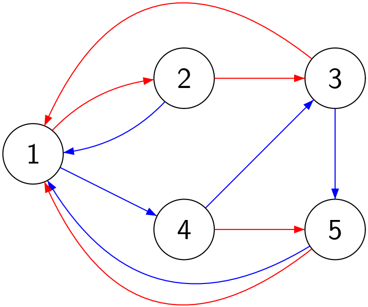


Figure: The red edges form a set for which index(4) = 2 and index(5) = 1

BRBRB is a homing sequence.

Proof: first part

We choose a coloring for which the number of states on the red cycles is maximal.

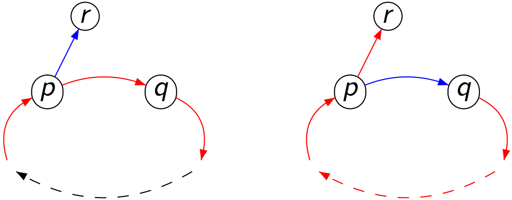


Figure: Case 1: all states have index 0

Case 2: Let p be of maximal index $\ell \geq 1$.

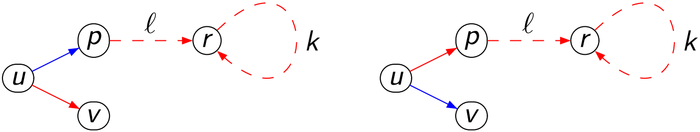


Figure: Case 2: u is not on the red cycle

After the exchange, the vertices of maximal index are ancestors of u.

Case 3. u is on the cycle.

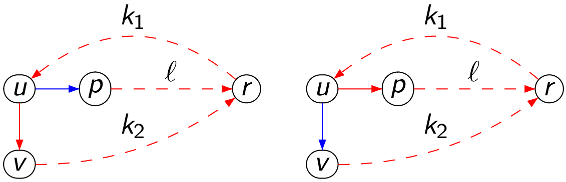


Figure: Case 3. $k_2 > \ell$

After the exchange, the vertices of maximal index are ancestors of v.

Case 4. $k_2 = \ell$ ($k_2 \leq \ell$ implies $k_2 = \ell$ since otherwise, flipping the edges out of u would increase the total length of red cycles)

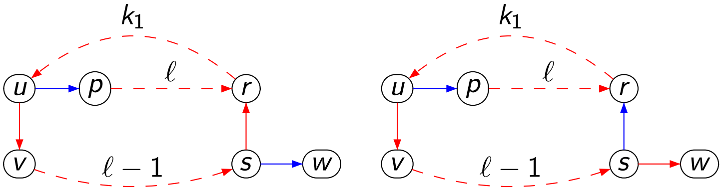


Figure: Case 4. There is an edge from s to $w \neq r$.

After the exchange, r is not anymore on a red cycle and the index of p has increased.

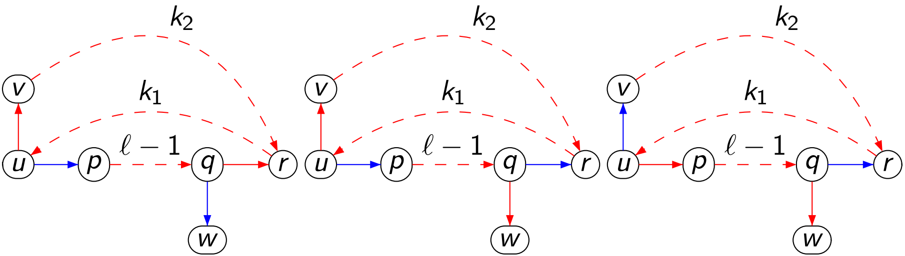


Figure: Case 5. There is an edge from q to $w \neq r$ (otherwise the pair $\{q, s\}$ would be stable)

If w has positive index (middle), then after the exchange all vertices of maximal index are ancestors of w.
Otherwise, if w is on a cycle distinct from C (right), after the second exchange, all states of maximal index are ancestors of u.
Finally, if w is on the cycle C, we proceed as in case 3 with w replacing r. Case 4 is then impossible.

Proof: second part

1. If there are stable pairs, consider the quotient of the graph by the equivalence identifying stable pairs of vertices. The quotient is still admissible and is road colorable by an induction argument. Lifting this coloring to the original graph gives a coloring with a homing sequence.
2. Otherwise, consider a coloring for which all vertices of maximal index (with respect to red) are ancestors of a common vertex. Let d be the minimal cardinality of a set of vertices all reachable by the same sequence (minimal rank). Such a set is called a minimal image.
   Let p be a vertex of maximal index ℓ. There is a minimal image I containing p. The index of all elements of I other than p is $< \ell$.

Let J be the set of vertices reached from I after $\ell - 1$ R. All elements of J other than the vertex q reached from p are of index 0.

Let k be a multiple of the lengths of the red cycles. Let K be the set of vertices reached from J after k R. Then J and K differ by at most one element each: the vertex q in J and the vertex r of K reached from q. But them r and q form a stable pair, a contradiction.

A still unsolved problem

Problem (Cerny, 1964)

Suppose that a graph with n vertices is road colorable. Is there a homing sequence of length at most n^2?

True in some particular cases:

- aperiodic colorings (Trahtman, 2002)
- graphs with an n-cycle (Pin, 1980)
- eulerian graphs (Kari, 2001)
- ...