Dimension groups and dynamical systems
(Substitutions, Bratteli diagrams and Cantor systems)

Dominique Perrin

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This book is the first self-contained exposition of the fascinating link between dynamical systems and dimension groups. The authors explore the rich interplay between topological properties of dynamical systems and the algebraic structures associated with them, with an emphasis on symbolic systems, particularly substitution systems. It is recommended for anybody with an interest in topological and symbolic dynamics, automata theory or combinatorics on words.

Intended to serve as an introduction for graduate students and other newcomers to the field as well as a reference for established researchers, the book includes a thorough account of the background notions as well as detailed exposition – with full proofs – of the major results of the subject. A wealth of examples and exercises, with solutions, serve to build intuition, while the many open problems collected at the end provide jumping-off points for future research.

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Dimension Groups and Dynamical Systems

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A quick tour in the land of dimension groups

Some definitions:

- Topological dynamical systems,
- ordered abelian groups and dimension groups
- Ordered cohomology groups,
- Bratteli diagrams and BV-systems,

some big theorems:

- The BV-representation theorem (every minimal Cantor system can be represented as a BV-system),
- the strong orbit equivalence theorem (dimension groups are a complete invariant for strong orbit equivalence),

and some perspectives.

Topological dynamical systems

A topological dynamical system is a pair (X, T) of a compact metric space X and a continuous map $T\colon X \to X$. It is invertible if T is a homeomorphism. It is a Cantor system if X is a Cantor space.
A continuous map $\phi\colon X \to X'$ is a morphism of dynamical systems if $\phi \circ T = T' \circ \phi$. An isomorphism of dynamical systems is called a conjugacy.
The orbit of a point $x \in X$ is the set $\{T^n x \mid n \in \mathbb{Z}\}$. Its forward orbit is $\{T^n x \mid n \in \mathbb{N}\}$.
A system (X, T) is irreducible if there is a point x with a dense forward orbit. It is minimal if it is nonempty and every point has a dense orbit.
A shift space (X, S) on a finite alphabet A is a closed and shift invariant subset X of $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$. The shift transformation $S\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $y = Sx$ if $y_n = x_{n+1}$.

Substitution shifts

Given a morphism $\sigma : A^* \to A^*$, let $\mathscr{L}(\sigma)$ be the set of factors of the words $\sigma^n(a)$ for $n \geq 0$ and $a \in A$. The shift generated by σ is the shift $X(\sigma)$ formed of all x such that all their factors are in $\mathscr{L}(\sigma)$. It is called a substitution shift.

The morphism σ is primitive if there is an $n \geq 1$ such that every letter $b \in A$ appears in every $\sigma^n(a)$ for $a \in A$. If σ is primitive and $\mathrm{Card}(A) \geq 2$, the shift $X(\sigma)$ is minimal.

Example

Let $\sigma : a \mapsto ab, b \mapsto a$. The shift $X(\sigma)$ is called the Fibonacci shift. It is minimal.

Odometers

Given a strictly increasing sequence $(p_n)_{n\geq 0}$ of natural integers with $p_0 = 1$ and $p_n | p_{n+1}$ for all $n \geq 0$, the set $X = \mathbb{Z}_{(p_n)}$ of expansions

$$x = a_0 + a_1 p_1 + a_2 p_2 + \dots$$

with $0 \leq a_n p_n < p_{n+1}$ is a topological ring in the same way as, for $p_n = p^n$ and p prime, we have the ring of p-adic integers. The map $T\colon x \mapsto x+1$ defines a topological dynamical system called the odometer in base (p_n). It is a minimal Cantor system.

Example

The system $(\mathbb{Z}_2, T)$ where $\mathbb{Z}_2$ is the ring of 2-adic integers and $T(x) = x + 1$ is called the 2-odometer.

Ordered abelian groups

An ordered abelian group G is given by a partial order on G such that $x \leq y$ implies $x+z \leq y+z$. The order is determined by the positive cone $G^+ = \{g \in G \mid g \geq 0\}$.
An order unit is an element $u \geq 0$ such that for every $g \geq 0$, there is an $n \geq 1$ with $g \leq nu$. A unital ordered group is a triple $(G, G^+, 1_G)$ where 1_G is an order unit.
An ordered group is simple if every nonzero element of G^+ is an ordered unit.
Let $(G, G^+, 1_G)$ and $(H, H^+, 1_H)$ be unital ordered groups. A group morphism $\phi : G \to H$ is a morphism of unital ordered groups if it is positive (that is $\phi(G^+) \subset H^+$) and such that $\phi(1_G) = 1_H$.

Direct limits of ordered groups

Let

$$G_0 \xrightarrow{\phi_0} G_1 \xrightarrow{\phi_1} G_2 \cdots$$

be a sequence of ordered abelian groups G_n connected by morphisms ϕ_n. The direct limit of this sequence is the quotient Δ/Δ^0 where

$$\begin{aligned} \Delta &= \{(g_n) \mid g_n \in G_n,\ \phi_n(g_n) = g_{n+1} \text{ for every } n \text{ large enough}\} \\ \Delta^0 &= \{(g_n) \mid g_n \in G_n,\ g_n = 0 \text{ for every } n \text{ large enough}\} \end{aligned}$$

It is an ordered group with positive cone Δ^+/Δ^0 where $\Delta^+ = \{(g_n) \in \Delta \mid g_n \in G_n^+, \text{ for } n \text{ large enough}\}$. If the G_n are unital, it is unital with unit the class of (1_{G_n}).

Examples

- Multiplication by 2.
The direct limit of the sequence

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \ldots$$

is isomorphic to the group $\mathbb{Z}[1/2]$ of dyadic rationals formed of the $p/2^k$ with positive cone $\mathbb{Z}_+[1/2]$ and unit 1.

- Action of a nonnegative matrix.
The direct limit of the sequence

$$\mathbb{Z}^2 \xrightarrow{M} \mathbb{Z}^2 \xrightarrow{M} \mathbb{Z}^2 \ldots$$

with $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is isomorphic to the group $\mathbb{Z} + \alpha\mathbb{Z}$ with $\alpha = (1 + \sqrt{5})/2$.

Ordered cohomology group

Let (X, T) be a topological dynamical system. Let ∂ be the operator on the group $C(X, \mathbb{Z})$ defined by

$$\partial f = f \circ T - f.$$

The map ∂f is called the coboundary of f.

Theorem

Let (X, T) be irreducible. The quotient $H(X, T, \mathbb{Z}) = C(X, \mathbb{Z})/\partial C(X, \mathbb{Z})$ is a unital ordered group with positive cone $C(X, \mathbb{N})/\partial C(X, \mathbb{Z})$ and order unit 1_X.

It is called the ordered cohomology group of (X, T), traditionally denoted by $K^0(X, T)$. It is invariant under conjugacy.

Dimension groups

A dimension group is a direct limit of a sequence

$$\mathbb{Z}^{k_1} \xrightarrow{\phi_1} \mathbb{Z}^{k_2} \xrightarrow{\phi_2} \cdots$$

of groups $\mathbb{Z}^{k_n}$ ordered in the usual way and with order unit $(1, 1, \cdots, 1)$.

Theorem (Herman, Putnam, Skau, 1992)

For every minimal Cantor system (X, T), the ordered group $K^0(X, T)$ is a simple dimension group.

Examples

Example

Let (X, T) be the periodic system $\{x_0, x_1, \ldots, x_{n-1}\}$ with $Tx_i = x_{i+1}$. Then $K^0(X, T) = \mathbb{Z}$ with order unit n.

Example

The dimension group of the Fibonacci shift is $\mathbb{Z} + \alpha\mathbb{Z}$ with $\alpha = (1 + \sqrt{5})/2$ (considered as an ordered subgroup of $(\mathbb{R}, \mathbb{R}_+, 1)$).

Example

The dimension group of the 2-odometer is the group $\mathbb{Z}[1/2]$ of dyadic rationals.

Invariant probability measures

A probability mesure μ on a system (X,T) is invariant if $\mu(T^{-1}U)=\mu(U)$ for every Borel set $U\subset X$. In this case, one has $\int f d\mu = 0$ for every $f\in\partial C(X,\mathbb{Z})$. Moreover, the map $f\mapsto \int f d\mu$ defines a group morphism $\alpha_\mu : H(X,T,\mathbb{Z})\to\mathbb{R}$.

Theorem

If (X,T) is irreducible, the map $\mu\mapsto\alpha_\mu$ is a bijection from the set of invariant probability measures on (X,T) onto the set of unital ordered group morphisms from $K^0(X,T)$ into $(\mathbb{R},\mathbb{R}_+,1)$.

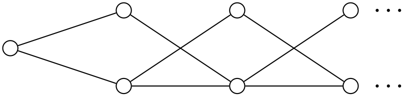
The unital ordered group morphisms from $(G,G^+,1)$ to $(\mathbb{R},\mathbb{R}_+,1)$ are called the states of the unital ordered group.

Example

The Fibonacci shift, as any primitive substitution shift, has a unique invariant probability measure. This corresponds to the fact that its dimension group, being $\mathbb{Z} + \alpha\mathbb{Z}$, is a subgroup of $(\mathbb{R}, \mathbb{R}_+, 1)$ and thus has a unique state.

Bratteli diagrams

A Bratteli diagram is a directed graph (V, E) with $V = V(0) \cup V(1) \cup \dots$ and $E = E(1) \cup E(2) \cup \dots$. We have $V(0) = \{v(0)\}$, every $V(n)$ is finite and the edges in $E(n)$ go from $V(n-1)$ to $V(n)$.



$v(0)$ $\quad$ $V(1)$ $\quad$ $V(2)$ $\quad$ $V(3)$

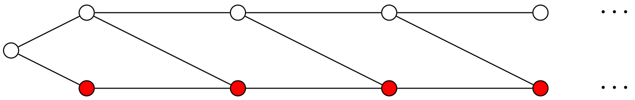
$E(1)$ $\quad$ $E(2)$ $\quad$ $E(3)$

Adjacency matrices

$$M(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; M(2) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Simple diagrams

A Bratteli diagram is simple if for every $m \geq 0$ there is an $n > m$ such that there is a path from every vertex in $V(m)$ to every vertex in $V(n)$, that is, if the matrix $M(n)M(n-1)\cdots M(m)$ is > 0.



The diagram above is not simple because the vertices of the lower level can never reach any of those at top level. We have

$$M(n) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Dimension groups of Bratteli diagrams

The dimension group of (V, E) is the direct limit of the sequence

$$G(0) \stackrel{M(1)}{\rightarrow} G(1) \stackrel{M(2)}{\rightarrow} G(2) \ldots$$

with $G(n) = \mathbb{Z}^{V(n)}$.

Proposition

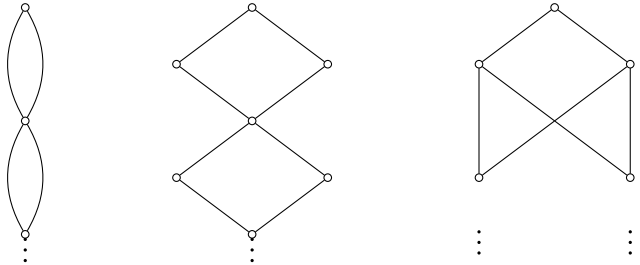
A Bratteli diagram is simple if and only if its dimension group is simple.

Telescoping equivalence

The telescoping of a Bratteli diagram (V, E) uses a sequence $m_0 = 0 < m_1 < m_2 < \dots$. It is the diagram (V', E') with $V'(n) = V(m_n)$ and $E(n) = E_{m_{n-1}+1, m_n}$.

Theorem (Eliott, 1976)

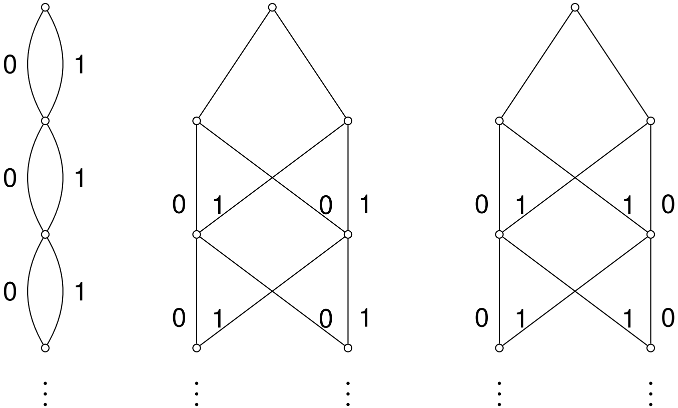
Two Bratteli diagrams are telescoping equivalent if and only if their dimension groups are isomorphic.



The dimension group is $\mathbb{Z}[1/2]$.

Ordered Bratteli diagrams

Assume that the set of edges with common range v is given for every $v \in V$ a total order. We extend this order to a lexicographic order on the set X_E of infinite paths starting at the root $v(0)$.



Th diagram is properly ordered if it is simple and if there is a unique minimal path $x^{\min}$ and a unique maximal path x^{max}.
The two first diagrams are properly ordered, the third one is not (there are two paths labeled $0, 0, 0, \dots$ and two paths labeled $1, 1, \dots$).
The morphism read on the second diagram is $0 \mapsto 01, 1 \mapsto 01$. The morphism read on the third is $0 \to 01, 1 \to 10$ (the Thue-Morse morphism).

The Vershik map

Let $(V, E, \leq)$ be a properly ordered Bratteli diagram.
The Vershik map on X_E is defined by

$$V_E(x) = \begin{cases} \text{successor of } x \text{ in lexicographic order} & \text{if } x \neq x^{\max} \\ x^{\min} & \text{otherwise} \end{cases}$$

The pair (X_E, V_E) is a minimal topological dynamical system called a BV-system.

The Model Theorem

A BV-representation of a system (X, T) is an isomorphism with a BV-system (X_E, V_E) for some properly ordered Bratteli diagram $(V, E, \leq)$.

Theorem (Herman, Putnam, Skau, 1992)

Every minimal Cantor system has a BV-representation.

There is no simple method to compute such a BV-representation. We will see how this can be done in the particular cases of odometers and substitution shifts.

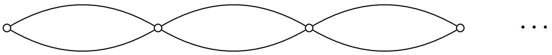
BV-representation of Odometers

Odometers are characterized by their BV-representations.

Theorem

A Cantor dynamical system is an odometer if and only if it has a BV-representation with one vertex at each level.

A BV-representation of the 2-odometer.



Strong orbit equivalence

Two topological dynamical systems (X, T) and (Y, S) are orbit equivalent if there is a homeomorphism $\phi : X \to Y$ which sends orbits to orbits. In this case, there are maps $\alpha, \beta : X \to \mathbb{Z}$ such that

$$\phi \circ Tx = S^{\alpha(x)} \circ \phi(x) \text{ and } \phi \circ T^{\beta(x)} x = S \circ \phi(x)$$

When α, β have at most one discontinuity point, the systems are strong orbit equivalent.

The strong orbit equivalence theorem

An intertwinning of two Bratteli diagrams (V, E) and (V', E') is a diagram such that telescoping at odd levels gives (V, E) and telescoping at even levels gives (V', E').

Theorem (Giordano, Putnam, Skau, 1995)

Let (X, T) and (X', T') be two invertible minimal Cantor dynamical systems. The following are equivalent.

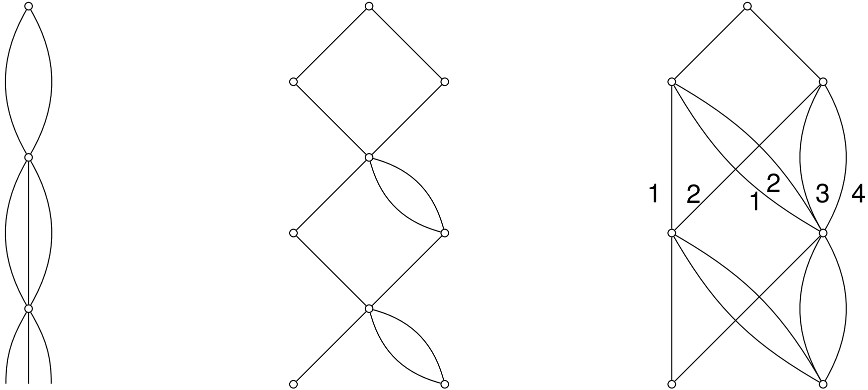
(i) *There exist two BV-representations, $(V, E, \leq)$ of (X, T) and $(V', E', \leq')$ of (X', T'), such that (V, E) and (V', E') have a common intertwining.*

(ii) *(X, T) and (X', T') are strong orbit equivalent.*

(iii) *The dimension groups $K^0(X, T)$ and $K^0(X', T')$ are isomorphic as unital ordered groups.*

Note that (iii)$\Rightarrow$(i) is Elliott Theorem.

Example



The diagram on the left is a BV-representation of an odometer. The diagram on the right is a BV-representation of the shift generated by the morphism $a \mapsto ab, b \mapsto a^2b^2$. They are strong orbit equivalent.

Stationary diagrams

A Bratteli diagram is stationary if all matrices $M(n)$ are equal for $n \geq 2$. An odomoter $\mathbb{Z}_{(p_n)}$ is stationary if the set of prime divisors of the p_n is finite.

Theorem (Durand, Host, Skau, 1999)

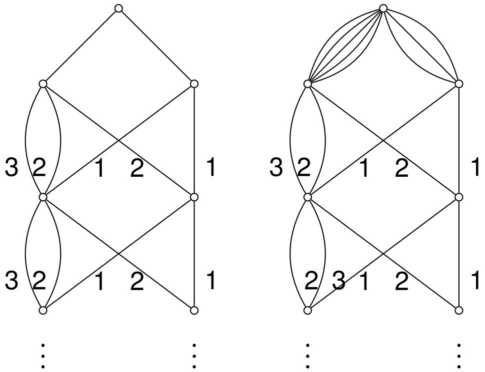
The class of infinite BV-systems associated with stationary Bratteli diagrams is the disjoint union of infinite substitution minimal shifts and stationary odometers.

The BV-representation of substitution shifts

A morphism $\sigma : A^* \to A^*$ is proper if all words $\sigma(a)$ for a$\in A$ begin with the same letter and end with the same letter. It is eventually proper if σ^n is proper for some $n \geq 1$.
If σ is eventually proper, the diagram $(V, E, \leq)$ with σ read on it is properly ordered and, provided $X(\sigma)$ is not periodic, it gives a BV-representation of $X(\sigma)$.
In the general case, use the following steps. Let $\sigma : A^* \to A^*$ be a morphism generating an infinite minimal shift space $X(\sigma)$.

- Compute an eventually proper morphism $\tau : B^* \to B^*$ and a morphism $\phi : B^* \to A^*$ such that $\phi \circ \tau = \sigma^k \circ \phi$.
- Build a BV-representation of $X(\tau)$ such that τ is read on (V, E).
- Split each edge $(v(0), b)$ of $E(1)$ in $\phi(b)$ edges.

Let $\sigma\colon a \mapsto ab, b \mapsto a$ be the Fibonacci morphism. Then $\sigma^2(a)$ begins and ends with a. We compute the set $\mathscr{R}(a \cdot a) = \{ababa, aba\}$ of words w without factor aa such that $awa \in \mathscr{L}(\sigma)$ ends and begins with aa. Let ϕ be the morphism defined by $\phi(x) = ababa$ and $\phi(y) = aba$. The morphism $\tau\colon x \mapsto yxx, y \mapsto yx$ is such that $\phi \circ \tau = \sigma^2 \circ \phi$. Since τ is proper, we are done.



We obtain in this way a computation of the dimension group of the Fibonacci shift as the direct limit of the sequence

$$\mathbb{Z}^2 \xrightarrow{M} \mathbb{Z}^2 \xrightarrow{M} \mathbb{Z}^2 \ldots$$

with

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2$$

and unit $\begin{bmatrix} 5 & 3 \end{bmatrix}^t = M^3 \begin{bmatrix} 1 & 0 \end{bmatrix}^t$. Thus we recover $K^0(X, S) = \mathbb{Z} + \alpha\mathbb{Z}$ with $\alpha = (1 + \sqrt{5})/2$.

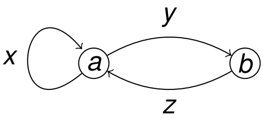
An alternative method

There is an alternative method to compute directly the dimension group of a substitution shift $X(\sigma)$. The steps are:

- compute the 2-block presentation σ_2 of σ such that $\pi_2 \circ \sigma_2 = \sigma \circ \pi_2$ where $\pi_2([ab]) = a$.
- Compute the Rauzy graph $\Gamma_2(X)$ with vertices ab from a to b whenever $ab \in \mathscr{L}_2(X)$.
- Compute the matrix N such that $PM(\sigma_2) = NP$ where P is a matrix with rows a basis of the cycles of the Rauzy graph $\Gamma_2(X)$.

The dimension group is the limit of $\mathbb{Z}^2 \xrightarrow{N} \mathbb{Z}^2 \xrightarrow{N} \mathbb{Z}^2 \ldots$ with order unit $P\mathbf{1}$.

We describe it on the example of the Fibonacci shift $\sigma\colon a \mapsto ab, b \mapsto a$. The 2-blocks are $x = aa$, $y = ab$, $z = ba$. The Rauzy graph $\Gamma_2(X)$ is



The 2-block presentation of σ is $\sigma_2\colon x \mapsto yz, y \mapsto yz, z \mapsto x$. Then $M(\sigma_2)$, the matrix P and the matrix N are

$$M_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; N = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

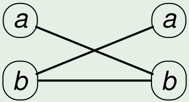
Thus we find again $\mathbb{Z} + \alpha\mathbb{Z}$ with $\alpha = (1 + \sqrt{5})/2$.

Dendric shifts

Let X be a shift space on the alphabet A and let $w \in \mathscr{L}(X)$. Set $L(w) = \{a \in A \mid aw \in \mathscr{L}(X)\}$ and $R(w) = \{a \in A \mid wa \in \mathscr{L}(X)\}$. The extension graph of w is the graph on the disjoint union of $L(w)$ and $R(w)$ with edges (a, b) if $awb \in \mathscr{L}(X)$. A shift space X is dendric if for every $w \in \mathscr{L}(X)$ the extension graph of w is a tree.

Example

The Fibonacci shift is dendric. The extension graph of a is shown below.



Dimension groups of dendric shifts

Theorem (Berthé, Cecchi, Durand, Leroy, P., Petite, 2021)

Every minimal dendric shift on A has a BV-representation $(V, E, \leq)$ such that the morphism read on $E(n)$ is for every $n \geq 2$ an automorphism of the free group on A.

Denote by $\mathscr{M}(X, S)$ the set of invariant probability measures on a shift space X.

Theorem (Berthé, Cecchi, Durand, Leroy, P., Petite, 2021)

The dimension group of a minimal dendric shift X on the alphabet A is $(G, G^+, 1_G)$ with $G = \mathbb{Z}^A$, $G^+ = \{x \in \mathbb{Z}^A \mid \langle x, \mu \rangle > 0, \mu \in \mathscr{M}(X, S)\} \cup \mathbf{0}$ and $1_G = \mathbf{1}$ where $\mathbf{1}$ is the vector with all components equal to 1 and μ is the vector $(\mu([a]))_{a \in A}$.

An intriguing question

To every minimal shift space X on A, one can associate its Schützenberger group $G(X)$, which is a group contained in the free profinite semigroup on A. It was shown by Almeida and Costa (2016) that $G(X)$ is the free profinite group on A for every minimal dendric shift X. This raises the following questions.

- Is it true for every minimal shift that $G(X)$ is free profinite if and only if $K^0(X,T)$ is free abelian?
- What is the relation between $G(X)$ and $K^0(X,T)$?