Densities of rational languages

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The space $A^{\mathbb{Z}}$

We consider the set $A^{\mathbb{Z}}$ of two-sided infinite sequences on a finite alphabet A as a compact metric space. The shift transformation is the continuous map $S\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $y = S(x)$ if

$$y_n = x_{n+1}$$

for every $n \in \mathbb{Z}$.
For $x \in A^{\mathbb{Z}}$, the language of x is the set $\mathcal{L}(x)$ of words which occur in x. For $X \subset A^{\mathbb{Z}}$, the language of X is the set $\mathcal{L}(X) = \cup_{x \in X} \mathcal{L}(x)$.

Shift spaces

A shift space is a subset of $A^{\mathbb{Z}}$ which is

- topologically closed,
- invariant under the shift

For $w \in A^*$, we denote

$$[w]_X = \{x \in X \mid x_{[0,|w|)} = w\}$$

the cylinder defined by the word w. For $L \subset A^*$, we denote $[L]_X = \cup_{w \in L}[w]_X$.
For $u, v \in A^*$, we use

$$[u \cdot v]_X = \{x \in X \mid x_{[-[u|,|v|)} = uv\}$$

and $[U \cdot V]_X = \bigcup_{u \in U, v \in V}[u \cdot v]_X$.

Topological dynamical systems

A shift space is a particular case of a topological dynamical system, which is pair (X, T) of a compact space X and a continuous transformation T on X.
A continuous map $\pi\colon X \to X'$ morphism of dynamical systems from (X, T) to (X', T') if φ intertwines T and T', that is $\pi \circ T = T' \circ \pi$.
It is a factor map if π is onto. It is a conjugacy if π is one-to-one.

Substitution shifts

A substitution is a monoid morphism $\sigma\colon A^* \to B^*$. A substitution $\sigma\colon A^* \to A^*$ is primitive if there is $n \geq 1$ such that every $a \in A$ appears in every $\sigma^n(b)$ for $b \in B$.
For $\sigma\colon A^* \to A^*$, the shift $\mathsf{X}(\sigma)$ is formed of the sequences x such that all blocks of x are factors of some $\sigma^n(a)$ for $a \in A$ and $n \geq 0$.

Example

The Fibonacci substitution $\sigma\colon a \mapsto ab, b \mapsto a$ is primitive. The shift $\mathsf{X}(\sigma)$ is the Fibonacci shift.

Example

The Thue-Morse substitution $\sigma\colon a \mapsto ab, b \mapsto ba$ is primitive. The shift $\mathsf{X}(\sigma)$ is the Thue-Morse shift. It contains the sequence

$$\sigma^\omega(a \cdot a) = \cdots abba \cdot abba \cdots$$

Minimal and irreducible shift spaces

A shift space X is minimal if there is no nonempty shift space properly contained in X. Equivalently, X is minimal if for every $n \geq 1$, there is $N \geq 1$ such that every word $u \in \mathcal{L}(X)$ of length n appears in every word of $\mathcal{L}(X)$ of length N.
As a weaker condition, a shift space is irreducible if, for every $u, v \in \mathcal{L}(X)$, there is $w \in \mathcal{L}(X)$ such that $uwv \in \mathcal{L}(X)$.
If σ is primitive distinct from the identity on one letter, the shift $\mathrm{X}(\sigma)$ is minimal.

Example

The Fibonacci shift and the Thue-Morse shift are minimal shifts.

Stochastic processes

Let $\mu \colon A^* \to [0,1]$ be such that $\mu(\varepsilon) = 1$ and

$$\mu(w) = \sum_{a \in A} \mu(wa)$$

for every $w \in A^*$. Thus, we can interpret $\pi(wa)/\pi(w)$ as the probability of seeing the letter a after the word w.
Such a map μ is called a stochastic process on A^*.
For $L \subset A^*$, we denote $\mu(L) = \sum_{w \in L} \mu(w)$, which is in $\mathbb{R} \cup \{\infty\}$.

Bernoulli processes

A simple example is a Bernoulli process, defined by a morphism $\mu\colon A^* \to [0,1]$ such that $\sum_{a\in A}\mu(a) = 1$. Equivalently, $\mu(wa)/\mu(w)$ does not depend on w.
If μ is a uniform Bernoulli process, that is if $\mu(a) = 1/\operatorname{card}(A)$, then

$$\mu(w) = \frac{1}{\operatorname{card}(A)^{|w|}}$$

Probability measures

The family of Borel sets of a topological space is the closure under countable unions and complement of the family of open sets.
A Borel probability measure on a topological space X is a map μ defined on the family of Borel sets of X such that $\mu(X) = 1$ and

$$\mu(\cup_{n\geq 0} U_i) = \sum_{n\geq 0} \mu(U_n)$$

for every family of paiwise disjoint Borel sets U_n.
Given a stochastic process μ, there is a unique Borel probability measure μ on $A^{\mathbb{Z}}$ such that $\mu([w]) = \mu(w)$ for every $w \in A^*$.
One has $\mu([L]) = \mu(L)$ provided the cylinders $[w]$ for $w \in L$ are disjoint, in particular when L is a prefix code, that is, no element of L is a proper prefix of another one.

Support of a mesure

Given a Borel probability measure μ on $A^{\mathbb{Z}}$, the support of μ is the set

$$X = \{x \in A^{\mathbb{Z}} \mid \mu(w) > 0 \text{ for every } w \in \mathcal{L}(x)\}.$$

It is a closed subset and $\mu(X) = 1$. Thus μ is a Borel probability measure on X.

Invariant measures

A measure μ on $A^{\mathbb{Z}}$ is invariant if $\mu(S^{-1}U) = \mu(U)$ for every Borel set U, where S denotes the shift transformation.
The measure μ is invariant if the associated stochatic process satisfies

$$\mu(w) = \sum_{a \in A} \mu(aw)$$

for every $w \in A^*$.
The support of an invariant measure is closed and invariant. Thus, it is a shift space. Conversely, for every shift space X, there exists an invariant measure supported by X.
A Bernoulli measure is invariant.

Ergodic measures

An invariant probability measure μ on $A^{\mathbb{Z}}$ is ergodic if every invariant Borel set has measure 0 or 1. As an equivalent condition, μ is ergodic if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(U \cap S^{-i}V) = \mu(U)\mu(V)$$

for every pair U, V of Borel sets.

Every shift space has ergodic measures. If there is a unique invariant measure, it is ergodic. The shift is said to be uniquely ergodic.

The support of an ergodic measure is an irreducible shift space.

A Bernoulli measure is ergodic.

Theorem (Michel)

Every primitive substitution shift is uniquely ergodic.

Mixing

An invariant probability measure on $A^{\mathbb{Z}}$ is mixing if

$$\lim_{n\to\infty} \mu(U \cap S^{-n} V) = \mu(U)\mu(V)$$

Thus mixing implies ergodic. The contrary is false (think of a periodic system with $p > 1$ points).

Invariant measures on substitution shifts

The matrix of a substitution $\sigma\colon A^* \to B^*$ is the $B \times A$-matrix $M(\sigma)$ defined by

$$M(\sigma)_{b,a} = |\sigma(a)|_b.$$

If σ is primitive, the unique invariant measure μ on $\mathsf{X}(\sigma)$ is such that $(\mu(a))_{a\in A}$ is a Perron eigenvector of $M(\sigma)$.
As a consequence, the Perron eigenvalue of $M(\sigma)$ is the average length $\sum_{a\in A} |\sigma(a)|\mu(a)$ of σ.

Example

The matrix of the Fibonacci shift $\sigma\colon a \mapsto ab, b \mapsto a$ is

$$M(\sigma) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The invariant measure μ on the Fibonacci shift is such that $\mu(a) = \lambda^{-1}$ and $\mu(b) = \lambda^{-2}$, where $\lambda = (1+\sqrt{5})/2$ is the Perron eigenvalue of M.

A useful formula (Kac's formula)

Let $\sigma \colon A^* \to A^*$ be a primitive substitution such that $X = \mathsf{X}(\sigma)$ is not periodic. Let μ be the unique invariant probability measure on X. Then

$$\lambda\mu(\sigma(X)) = 1$$

where λ is the average length of σ.
If, for example, σ is the Thue-Morse substitution and $X = \mathsf{X}(\sigma)$, the invariant probability measure on X is such that

$$\mu(\sigma(X)) = \frac{1}{2}.$$

The space of probability measures

The set $\mathcal{M}(X)$ of probability measures on a shift space X is a topological space for the weak-$*$ topology making continuous the maps $\mu \mapsto \int f d\mu$, for $f \in C(X, \mathbb{R})$. By the Banach-Alaoglu theorem, and since X is compact, the space $\mathcal{M}(X)$ is compact for this topology. Let μ be an invariant measure with support X. By the ergodic decomposition theorem, there is a measure τ on the compact space $E(X)$ of ergodic measures λ on X such that

$$\mu = \int_{E(X)} \lambda d\tau.$$

The relation $\lambda \prec \mu$ if $\mu(U) = 0$ implies $\nu(U) = 0$ for every Borel set U, defines a preorder on $\mathcal{M}(X)$. The ergodic measures are the minimal elements of this preorder.

Rational languages, automata and monoids

A finite automaton $\mathcal{A} = (Q, E, I, T)$ on the alphabet A is given by a finite set Q of states, a finite set $E \subset Q \times A \times Q$ of edges, a set I of initial states and a set T of terminal states.
A path in the automaton is a sequence $(p_i, a_i, p_{i+1})_{0 \leq i \leq n-1}$ of consecutive edges. Its label is the word $a_0 a_1 \cdots a_{n-1}$.
The language recognized by $\mathcal{A}$ is the set of labels of paths from I to T.
A language is rational if it can be recognized by a finite automaton.
As an equivalent definition, a language $L \subset A^*$ is rational if and only if it can be recognized by a finite monoid, that is, if the exists a morphism $\varphi \colon A^* \to M$ onto a finite monoid M such that $L = \varphi^{-1}(P)$ for some $P \subset M$.

Density of a language

The density of a language $L \subset A^*$ with respect to a probability measure μ on $A^{\mathbb{Z}}$ is

$$\delta_\mu(L) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(L \cap A^i)$$

whenever the limit exists. It is in the strong sense if the limit $\lim_{n\to\infty} \mu(L \cap A^n)$ exists.

Our aim is to show that the density of a rational language exists for every invariant measure μ and to give a way to compute it. The result is known when μ is a Bernoulli measure (Berstel, 1972).

Densities with respect to Bernoulli measures

Let μ be a Bernoulli measure. The following result proves, since the density of left or right ideals is easy to compute, the existence of densities for rational languages with respect to μ.

Theorem (Schützenberger, 1965)

Let $\varphi \colon A^* \to M$ be a morphism onto a finite monoid. Let J be the minimal ideal of M. For every $m \in M$, one has

$$\nu(m) = \begin{cases} 0 & \text{if } m \notin J \\ \frac{\nu(mM)\nu(Mm)}{\mathrm{Card}(mM \cap mM)} & \text{otherwise.} \end{cases}$$

where $\nu(m) = \delta_\mu(\varphi^{-1}(m))$.

It also exhibits a property of equidistribution since $\delta_\mu(\varphi^{-1}(m))$ is constant on each $\mathcal{H}$-class of J.
For example, if $M = G$ is a group, then $\nu(g) = 1/\,\mathrm{Card}(G)$.

Elementary properties of densities

If the density of L exists, then

$$0 \leq \delta_\mu(L) \leq 1.$$

The density is finitely additive, that is if L, L' have densities and $L \cap L' = \emptyset$, then $L \cup L'$ has a density and

$$\delta_\mu(L \cup L') = \delta_\mu(L) + \delta_\mu(L').$$

Moreover

$$\delta_\mu(A^* \setminus L) = 1 - \delta_\mu(L).$$

Reduction to ergodic measures

Proposition

If a language L has a density with respect to every ergodic measure, it has a density with respect to every invariant measure.

Let μ be an invariant measure with support X. Assume that L has a density with respect to every ergodic measure λ on X. Then, by the ergodic decomposition theorem, there is a measure τ on the space $E(X)$ of ergodic measures on X such that $\mu = \int_{E(X)} \lambda d\tau$ and thus

$$\begin{aligned}\delta_\mu(L) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu(L\cap A^i) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \int_{E(X)} \lambda(L\cap A^i) d\tau \\ &= \int_{E(X)} \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \lambda(L\cap A^i) d\tau = \int_{E(X)} \delta_\lambda(L) d\tau.\end{aligned}$$

Density of right ideals

Proposition

Let μ be an probability measure on $A^{\mathbb{Z}}$ and let $w \in A^*$. Then

$$\delta_\mu(wA^*) = \mu(w)$$

in the strong sense.

Indeed, we have $wA^* \cap A^n = wA^{n-|w|}$ whenever $n \geq |w|$ and thus

$$\lim_{n\to\infty} \mu(wA^* \cap A^n) = \lim_{n\to\infty} \mu(wA^{n-|w|}) = \mu(w).$$

More generally, for any right ideal $L \subset A^*$, we have

$$\delta_\mu(L) = \mu(D)$$

where D is the prefix code such that $L = DA^*$ and $\mu(D) = \sum_{d \in D} \mu(d)$.

Density of left ideals

Proposition

If μ is invariant, then

$$\delta_\mu(A^*w) = \mu(w)$$

in the strong sense.

Indeed, we have $A^*w \cap A^n = A^{n-|w|}w$ whenever $n \geq |w|$ and thus

$$\lim_{n\to\infty} \mu(A^*w \cap A^n) = \lim_{n\to\infty} \mu(A^{n-|w|}w) = \mu(w)$$

since μ is invariant.
More generally, for any left ideal L, we have

$$\delta_\mu(L) = \mu(G)$$

where G is the suffix code such that $L = A^*G$ and $\mu(G) = \sum_{g\in G} \mu(g)$.

Quasi-ideals

Proposition

If μ is ergodic then

$$\delta_\mu(uA^* \cap A^*v) = \mu(u)\mu(v).$$

The density exists in the strong sense if μ is mixing.

Indeed, we have for $i \geq |v|$,

$$[uA^* \cap A^*v \cap A^i] = [u] \cap S^{|v|-i}[v]$$

and thus

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu(uA^* \cap A^*v \cap A^i) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu([u] \cap S^{-i}[v]) \\ &= \mu(u)\mu(v)\end{aligned}$$

The formula extends to arbitrary quasi-ideals. Let $L = DA^* \cap A^*G$ with D a prefix code and G a suffix code. Then $\delta_\mu(L) = \mu(D)\mu(G)$.

Example

Let X be the Fibonacci shift and μ its unique invariant measure. Then

$$\delta_\mu(aA^* \cap A^*a) = \mu(a)^2 = \lambda^{-2}.$$

Two-sided ideals

For every ergodic measure, the density of $L = A^*wA^*$ exists in the strong sense, and is

$$\delta_\mu(L) = \begin{cases} 1 & \text{if } \mu(w) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have a 0-1 law for two-sided ideals.
Indeed, set $D = L \setminus LA^+$ and $G = L \setminus A^+L$. We have $L = DA^* = A^*G = DA^* \cap A^*G$ and thus

$$\delta_\mu(L) = \delta_\mu(DA^*)\delta_\mu(A^*G) = \delta_\mu(L)^2$$

whence the result since $\delta_\mu(L) > 0$ if and only if $\mu(w) > 0$.
The formula extends to an arbitrary two-sided ideal L. One has $\delta_\mu(L) = 1$ if $\mu(w) > 0$ for some $w \in L$ and 0 otherwise.

Part 2. Density of Group languages

A group language is of the form $L = \varphi^{-1}(H)$, where $\varphi\colon A^* \to G$ is a morphism onto a finite group and $H \subset G$.

Theorem (Berthé, Goulet-Ouellet, Nyberg-Brodda, P., Petersen)

Let L be a group language and μ be an invariant measure. Then $\delta_\mu(L)$ exists.

The proof uses four steps:

1. Use the ergodic decomposition to reduce to the case of an ergodic measure.
2. Define the skew product $G \rtimes_\varphi X$, where X is the support of μ.
3. Lift the ergodic measure μ to an ergodic measure on $G \rtimes_\varphi X$.
4. Give a formula for $\delta_\mu(\varphi^{-1}(g))$ for $g \in G$.

Skew product with a group

The skew product $G \rtimes_\varphi X$ of the group G and the shift X relative to a morphism $\varphi\colon A^* \to G$, is the topological dynamical system $(G \times X, T)$ with

$$T(g, x) = (g\varphi(x_0), Sx).$$

The map $\pi\colon (g, x) \to x$ is a factor map.

Lifting of ergodic measures

Proposition

For each ergodic measure μ on X, there is an ergodic measure $\bar{\mu}$ on $G \rtimes_\varphi X$ which projects on μ.

Let ζ be the product of the counting measure on G with μ. It is an invariant measure on $G \rtimes_\varphi X$ which projects on μ. Finally, any ergodic measure $\bar{\mu} \prec \zeta$ also projects on μ.

A formula for the density

Let X be a shift space on a finite alphabet A with an ergodic measure μ and let $\varphi\colon A^* \to G$ be a morphism onto a finite group G. Let $\bar{\mu}$ be an ergodic measure on $G \rtimes_\varphi X$ that projects to μ. For every group language $L = \varphi^{-1}(g)$, where $g \in G$, the density $\delta_\mu(L)$ exists and is given by the following formula,

$$\delta_\mu(L) = \sum_{g \in G} \bar{\mu}(U_g)\,\bar{\mu}(U_{hg}). \tag{1}$$

where for $h \in G$, $U_h = \{h\} \times X$

We find

$$\{h\} \times [L \cap A^i]_X = (\{h\} \times X) \cap T^{-i}(\{hg\} \times X) = U_h \cap T^{-i} U_{hg}.$$

Next,

$$\begin{aligned}\mu(L \cap A^i) = \bar{\mu}\left(G \times [L \cap A^i]_X\right) &= \sum_{h \in G} \bar{\mu}(\{h\} \times [L \cap A^i]_X) \\ &= \sum_{h \in G} \bar{\mu}(U_h \cap T^{-i} U_{hg}).\end{aligned}$$

Since $\bar{\mu}$ is ergodic,

$$\begin{aligned}\delta_\mu(L) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(L \cap A^i) &= \sum_{h \in G} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \bar{\mu}(U_h \cap T^{-i} U_{hg}) \\ &= \sum_{h \in G} \bar{\mu}(U_h)\, \bar{\mu}(U_{hg}).\end{aligned}$$

Equidistibuted densities

Let $\varphi \colon A^* \to G$ be a morphism onto a finite group G. We say then that δ_μ is equidistributed on G if

$$\delta_\mu(\varphi^{-1}(g)) = \frac{1}{\mathrm{Card}(G)}$$

for every $g \in G$.

Theorem

When the product measure $\nu \times \mu$, with ν the counting measure on G, is ergodic, then δ_μ is equidistributed on G.

This follows from the above formula since $\delta_{\nu\times\mu}(U_h) = 1/\mathrm{Card}(G)$ and thus

$$\delta_\mu(\varphi^{-1}(g)) = \sum_{h\in G} (\nu\times\mu)(Uh)(\nu\times\mu)(U_{hg}) = \sum_{h\in G} \frac{1}{\mathrm{Card}(G)^2} = \frac{1}{\mathrm{Card}(G)}.$$

Three points example

The following example shows that the density is not always well distributed when $\nu \times \mu$ is not ergodic.
Let X be the orbit of $x = (abc)^\omega$. Thus $X = \{x, y, z\}$ with $y = Sx$, $z = Sy$. Let $\varphi \colon A^* \to \mathbb{Z}/2\mathbb{Z}$ be defined by $\varphi(a) = 0$, $\varphi(b) = \varphi(c) = 1$. Let $L = \varphi^{-1}(0)$. We have

$$L \cap \mathcal{L}(X) = (abc)^*\{\varepsilon, a\} \cup (bca)^*\{\varepsilon, bc\} \cup (cab)^*\{\varepsilon\}.$$

Thus

$$\mu(L \cap A^i) = \begin{cases} 1 & \text{if } i \equiv 0 \bmod 3 \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

This shows that

$$\delta_\mu(L) = \frac{1}{3}(1 + \frac{1}{3} + \frac{1}{3}) = \frac{5}{9}$$

(and not $1/2$). The measure $\nu \times \mu$ is not ergodic.

The skew product $G \rtimes_\varphi X$ is formed of 6 elements with the transformation T represented below. It has two orbits.

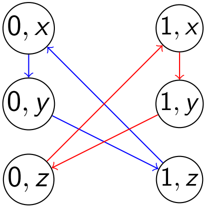


Figure: The skew product $G \times X$.

We have actually $\nu \times \mu = \frac{1}{2}(\lambda_1 + \lambda_2)$ where λ_1, λ_2 are the invariant mesures on the two orbits of T.

Part 2. Density of rational languages

Theorem (Berthé, Goulet-Ouellet, P., ICALP 2025)

Let μ be an invariant measure on $A^{\mathbb{Z}}$. Then every rational language on A has a density with respect to μ.

The proof is in five steps. We use a morphism $\varphi\colon A^* \to M$ onto a finite monoid M.

1. Use the ergodic decomposition theorem to restrict to the case where μ is ergodic.
2. Define the X-minimal $\mathcal{J}$-class $J_X(M)$, where X is the support of μ.
3. Define a skew product $(R \cup \{0\}) \rtimes_\varphi X$, where R is an $\mathcal{R}$-class of $J_X(M)$.
4. Lift μ to an ergodic measure on $(R \cup \{0\}) \rtimes_\varphi X$.
5. Give a formula for $\delta_\mu(\varphi^{-1}(m))$, where $m \in M$.

The X-minimal $\mathcal{J}$-class $J_X(M)$

Recall the Green relations in a monoid M.

- $m\mathcal{R}n \Leftrightarrow mM = nM \Leftrightarrow m, n$ generate the same right ideal
- $m\mathcal{L}n \Leftrightarrow Mm = Mn \Leftrightarrow m, n$ generate the same left ideal.
- $m\mathcal{J}n \Leftrightarrow MmM = MnM \Leftrightarrow m, n$ generate the same ideal.
- $m\mathcal{H}n \Leftrightarrow m\mathcal{R}n$ and $m\mathcal{L}n$.

A $\mathcal{J}$-class J is regular if it contains an idempotent. We have

- All $\mathcal{H}$-classes contained in J have the same number of elements.
- Each $\mathcal{H}$-class containing an idempotent is a group and there is one in each $\mathcal{R}$-class and each $\mathcal{L}$-class.
- All groups in J are isomorphic to the Schützenberger group of J.

When M is a group, it is a single $\mathcal{H}$-class.

The X-minimal $\mathcal{J}$-class

Let $\varphi\colon A^* \to M$ be a morphism onto a finite monoid M. Let X be an irreducible shift space. Let $K_X(M)$ be the intersection of all ideals in M which meet $\varphi(\mathcal{L}(X))$. The X-minimal $\mathcal{J}$-class of M is the set

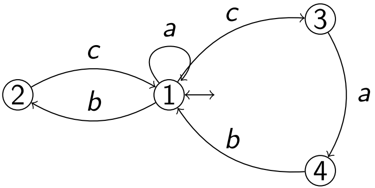
$$J_X(M) = \{m \in K_X(M) \mid MmM \cap \varphi(\mathcal{L}(X)) \neq \emptyset\}.$$

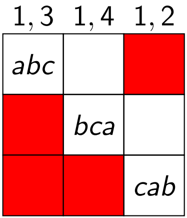
It is the unique 0-minimal ideal of the quotient M/I of M by the largest ideal I having empty intersection with $\varphi(\mathcal{L}(X))$. As such,

1. it is a regular $\mathcal{J}$-class
2. its $\mathcal{R}$-classes are the 0-minimal right ideals.
3. its $\mathcal{L}$-classes are the 0-minimal left ideals.

Example

Consider the automaton below on the left. Let φ be the morphism onto its transition monoid M.

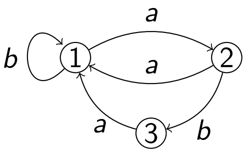




Let X be the three-point set $\{x, Sx, S^2x\}$ with $x = (abc)^\infty$. The $\mathcal{J}$-class $J_X(M)$ is represented on the right. Its group is trivial.

Example

Let $\mathcal{A}$ be the automaton represented below on the left. Let X be the Fibonacci shift.



$1, 2$	$1, 3$
a, a^2	ab, a^2b
ba, ba^2	b, bab

The $\mathcal{J}$-class $J_X(M)$ is represented on the right. Its group is $\mathbb{Z}/2\mathbb{Z}$.

Density of aperiodic languages

A rational language L on the alphabet A is aperiodic if it can be recognized by an aperiodic monoid, that is, having only trivial subgroups.

Theorem

The density of an aperiodic language with respect to an invariant measure exists in the strong sense.

Indeed, let $\varphi\colon A^* \to M$ be a morphism onto a finite aperiodic monoid. Let μ be an ergodic measure with support μ. Let $L = \varphi^{-1}(m)$ for $m \in M$. One has $\delta_\mu(L) = 0$ if $m \notin J_X(M)$. Next, if $m \in J_X(M)$, we have

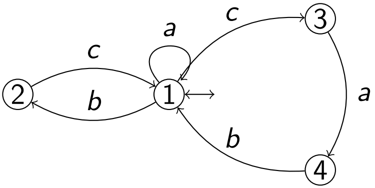
$$L \cap \mathcal{L}(X) = LA^* \cap A^*L \cap \mathcal{L}(X).$$

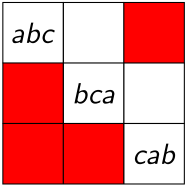
therefore

$$\delta_\mu(L) = \delta_\mu(LA^*)\delta_\mu(A^*L).$$

Example

Let $\mathcal{A}$ be the aperiodic automaton below and let X be the 3-point shift as above. Let L be stabilizer of 1. It coincides with $\psi^{-1}(0) \cap \mathcal{L}(X)$ with $\psi\colon A^* \to \mathbb{Z}/2\mathbb{Z}$ the morphism $\psi(a) = 0$ and $\psi(b) = \psi(c) = 1$ (merging $2, 3, 4$ gives the group automaton for the parity of b, c).

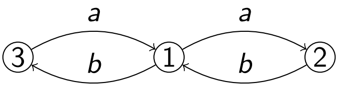




The density $\delta_\mu(L)$ is the sum of the densities corresponding to the white $\mathcal{H}$-classes, that is $\delta_\mu(L) = 5/9$.

Example

The transition monoid of the automaton below is aperiodic. Let $X = \mathsf{X}(\sigma)$ be the Thue-Morse shift. The X-minimal $\mathcal{J}$-class $J = J_X(M)$ is represented on the right.



	2	1	3	
3	a^2	a^2b	a^2b^2	$1/4$
1	ba^2	ba^2b	ba^2b^2	$1/2$
2	b^2a^2	b^2a^2b	b^2	$1/4$
	$1/4$	$1/2$	$1/4$	

The density of the language $L = \{ab, ba\}^*$ is $\delta_\mu(L) = 1/4$. Indeed, one has $\delta_\mu(LA^* \cap \varphi^{-1}(J)) = \delta_\mu(A^*L \cap \varphi^{-1}(J)) = \mu(\sigma(X)) = 1/2$ by Kac's formula.

Step 2:The skew product $(R \cup \{0\}) \rtimes_{\varphi} X$

Let $\varphi \colon A^* \to M$ be a morphism onto a finite monoid M. Let μ be an ergodic measure with support X. Let $J = J_X(M)$ and let R be an $\mathcal{R}$-class of J.
The skew product $(R \cup \{0\}) \rtimes_{\varphi} X$ is the topological dynamical system $((R \cup \{0\}) \times X, T)$ with the continuous transformation T defined by

$$T(r, x) = (r \cdot \varphi(x_0), Sx)$$

where $r \cdot m = rm$ if $rm \in R$ and 0 otherwise.
When M is a group G, we have $R = G$ and $(R \cup \{0\}) \rtimes X = G \rtimes_{\varphi} X$.

Step 3: Lifting of ergodic measures

The following generalizes the case where M is a group.

Proposition

Let $\varphi\colon A^* \to M$ be a morphism onto a finite monoid M. Let μ be an ergodic measure with support $X \subset A^Z$ and let $J = J_X(M)$. Let R be an $\mathcal{R}$-class of J. There is an ergodic measure ν on $R \rtimes_\varphi X$ which projects on μ and satisfies $\nu(\{0\} \times X) = 0$.

A formula for the density

Let $\varphi \colon A^* \to M$ be a morphism onto a finite monoid M. Let μ be an ergodic measure with support X. Let R be an $\mathcal{R}$-class of $J_X(M)$. Let ν be an ergodic measure on $(R \cup \{0\}) \rtimes_\varphi X$ that projects on μ and such that $\nu(\{0\} \times X) = 0$. Let $m \in M$ and $L = \varphi^{-1}(m)$. We have

$$\delta_\mu(L) = \begin{cases} 0 & \text{if } m \notin J_X(M), \\ \sum_{r, rm \in R} \nu(U_{r,[L]})\nu(U_{r,X}) & \text{otherwise} \end{cases}$$

where $U_{r,V} = \{r\} \times V$.

We may assume that $m \in J_X(M)$. Let C be the prefix code such that $LA^* = CA^*$. For $i \geq 0$, let

$$C_{\leq i} = \{u \in C \mid |u| \leq i\}, \quad C_{>i} = \{u \in C \mid |u| > i\}.$$

We claim that for every $r \in R$ such that $rm \in R$, one has

$$U_{r,[L \cap A^i]_X} = U_{r,[C_{\leq i}]_X} \cap T^{-i}(U_{rm,X}).$$

As a result, we have

$$\mu(L \cap A^i) = \bar{\mu}(R \times [L \cap A^i]_X) = \sum_{r,rm \in R} \bar{\mu}\Big(U_{r,[C_{\leq i}]_X} \cap T^{-i}(U_{rm,X})\Big).$$

Next we claim that for $\varepsilon > 0$, there is $i_0 \geq 0$ such that $\bar{\mu}(U_{r,[C_{>i_0}]_X}) < \varepsilon$ for every $r \in R$.

Using the above claim together with the ergodicity of $\bar{\mu}$, this gives

$$\delta_\mu(L) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=i_0}^{n-1} \mu(L \cap A^i) = \sum_{r,rm\in R} \lim_{n\to\infty} \frac{1}{n} \sum_{i=i_0}^{n-1} \bar{\mu}\big(U_{r,[C_{\leq i}]_X} \cap T^{-i}(U_{rm,X}$$

$$\geq \sum_{r,rm\in R} \lim_{n\to\infty} \frac{1}{n} \sum_{i=i_0}^{n-1} \bar{\mu}\big(U_{r,[C]_X} \cap T^{-i}(U_{rm,X})\big) - \varepsilon$$

$$\geq \sum_{r,rm\in R} \bar{\mu}(U_{r,[L]_X})\bar{\mu}(U_{rm,X}) - \varepsilon.$$

On the other hand, we have

$$\delta_\mu(L) = \sum_{r,rm\in R} \lim_{n\to\infty} \frac{1}{n} \sum_{i=i_0}^{n-1} \bar{\mu}\big(U_{r,[C_{\leq i}]_X} \cap T^{-i}(U_{rm,X})\big)$$

$$\leq \sum_{r,rm\in R} \lim_{n\to\infty} \frac{1}{n} \sum_{i=i_0}^{n-1} \bar{\mu}\big(U_{r,[C]_X} \cap T^{-i}(U_{rm,X})\big) \leq \sum_{r,rm\in R} \bar{\mu}(U_{r,[L]_X})\bar{\mu}(U_r$$

concluding the proof.

The weighted counting measure

Let $\varphi\colon A^* \to M$ be a morphism onto a finite monoid M and let μ be an ergodic measure with support X. Let R be an $\mathcal{R}$-class of the $\mathcal{J}$-class $J = J_X(M)$. Let d be the cardinality of the $\mathcal{H}$-classes in J. The weighted counting measure is the measure ν on $(R \cup \{0\}) \times X$ defined by $\nu(\{0\} \times X) = 0$ and for $r \in R$ and $w \in \mathcal{L}(X)$ by

$$\nu(\{r\}, [w]) = \frac{1}{d}\mu(G_r w)$$

where G_r is the suffix code such that $\varphi^{-1}(Mr) = A^* G_r$.

Proposition

The weighted counting measure is an invariant probability measure on $(R \cup \{0\}) \rtimes_\varphi X$.

When M is a group G, it is the product of the counting measure on G with μ.

Equidistributed densities

Let $\varphi\colon A^* \to M$ be a morphism onto a finite monoid M and let μ be an ergodic measure with support X. We say that δ_μ is equidistributed on M if for every $m \in M$, the density of $L = \varphi^{-1}(m)$ is

$$\delta_\mu(L) = \begin{cases} \frac{1}{d}\delta_\mu(LA^*)\delta_\mu(A^*L) & \text{if } m \in J_X(M), \\ 0 & \text{otherwise} \end{cases}$$

where d is the cadinality of $\mathcal{H}$-classes of $J_X(M)$. Thus, the density is the same within each $\mathcal{H}$-class of J.

Theorem (Berthé, Goulet-Ouellet, P.)

If the weighted counting measure is ergodic, then δ_μ is equidistributed on M.

Let ν be the weigthted counting measure on $(R \cup \{0\}) \rtimes X$. Let D_m be the prefix code such that $LA^* = D_m A^*$. The formula above reduces to

$$\begin{aligned}
\delta_\mu(L) &= \sum_{r, rm \in R} \nu(U_{r,[L]})\nu(U_{rm,X}) \\
&= \frac{1}{d^2} \sum_{r, rm \in R} \mu([G_r \cdot D_m])\mu(G_{rm}) = \frac{1}{d^2}\mu(G_m) \sum_{r, rm \in R} \mu([G_r \cdot D_m]) \\
&= \frac{1}{d}\mu(G_m) \sum_{H \subset R} \mu([G_H \cdot D_m])
\end{aligned}$$

where H runs over the $\mathcal{H}$-classes of R and G_H is the common value of G_r for the d elements $r \in H$. We have

$$\sum_{H \subset R} \mu([G_H \cdot D_m]) = \mu(D_m).$$

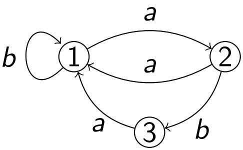
Therefore,

$$\delta_\mu(L) = \frac{1}{d}\mu(G_m)\mu(D_m) = \frac{1}{d}\delta_\mu(A^*L)\delta_\mu(LA^*),$$

using the formulas for the density of left and right ideals.

Example

Let $\mathcal{A}$ be the automaton represented below on the left. Let X be the Fibonacci shift.



$1, 2$	$1, 3$
a, a^2	$ab, a^2 b$
ba, ba^2	b, bab

Let $L = \{aa, aba, bb\}^*$. Then $\varphi(L)$ has one element in each of the four $\mathcal{H}$-classes and $\delta_\mu(L) = 1/2$. This could be anticipated since L has the same intersection with $\mathcal{L}(X)$ as the group language :constituted of words with an even number of a (merging 2 and 3 gives the group automaton for the parity of a).

Values of the densities

Let $\varphi\colon A^* \to M$ be a morphism onto a finite monoid M and μ be an ergodic measure. If

(i) the weighted counting measure is ergodic,

(ii) the values of $\mu(L)$ belong, whenever finite, to an extension K of $\mathbb{Q}$ for every rational language L.

Then the density of every rational language belongs to K.
This generalizes to property known for Bernoulli measures (Berstel, 1972).

Morphic shifts

A letter coding is a substitution $\phi\colon A^* \to B^*$ such that $\phi(a) \in B$ for every $a \in A$.
Let $\sigma\colon A^* \to A^*$ be a substitution and let $\phi\colon A^* \to B^*$ be a letter coding. The set $\mathsf{X}(\sigma, \phi) = \phi(\mathsf{X}(\sigma))$ is a shift space, called a morphic shift

Example

Let $\sigma\colon a \mapsto ab, b \mapsto ac, c \mapsto db, d \mapsto dc$ and $\phi\colon a \mapsto 0, b \mapsto 0, c \mapsto 1, d \mapsto 1$. The morphic shift $\mathsf{X}(\sigma, \phi)$ is the Rudin-Shapiro shift.

Minimal morphic shifts

Let X be a mimimal shift space on the alphabet A. A return word to $u \in \mathcal{L}(X)$ is a word w such that wu is in $\mathcal{L}(X)$ and has exactly two occurrences of u, one as a prefix and the other one as a suffix. Let $\mathcal{R}_X(u)$ denote the set of return words to u.

Let $\phi_u \colon A_u^* \to A^*$ be a substitution defining a bijection from A_u onto $\mathcal{R}_X(u)$. The set of $y \in A_u^{\mathbb{Z}}$ such that $\phi_u(w) \in \mathcal{L}(X)$ for every block w of y is a shift space, called the derivative of X with respect to u.

Theorem (Durand, 1998)

A minimal shift X is morphic if and only if it has a finite number of derivatives with respect to words in $\mathcal{L}(X)$.

Morphic skew products

Theorem (Berthé, Carton, Goulet-Ouellet, P.)

Let X be a minimal morphic shift and let $\varphi\colon A^* \to M$ be a morphism onto a finite monoid. Let R be an $\mathcal{R}$-class of $J_X(M)$. Every minimal component of $(R \cup \{0\} \rtimes_\varphi X)$ is morphic.

Since minimal morphic shifts are uniquely ergodic, this implies the following result.

Corollary

Let X be a minimal morphic shift and let μ be its invariant probability measure. Let $\varphi\colon A^* \to M$ be a morphism onto a finite monoid. Then δ_μ is equidistributed on M.

Open problems

1. Average length of prefix codes
2. Sofic measures
3. Idempotent measures

Average length of prefix codes

If μ is a Bernoulli measure and C is a rational prefix code such that $\mu(C) = 1$, then

$$\delta_\mu(C^*) = \frac{1}{\lambda(C)} \tag{2}$$

This is a particular case of a result due to Erdös, Feller and Pollard (1949). It follows from the fact that $A^* = C^*P$ where P is the set of proper prefixes of the words of C. Indeed, since $\mu(C) = 1$, we have $\lambda(C) = \mu(P)$ and therefore $1 = \delta_\mu(C^*)\mu(P) = \delta_\mu(C^*)\lambda(C)$.
Question: under what hypotheses does Formula (2) hold for an arbitrary invariant measure?

Sofic measure and k-step Markov measures

A sofic measure is a measure μ on $A^{\mathbb{Z}}$ such that for every $w \in A^*$

$$\mu(w) = i\varphi(w)t$$

for some morphism $\varphi\colon A^* \to M_n(\mathbb{R}_+)$, a row vector $i \in \mathbb{R}_+^n$ and a column vector $t \in \mathbb{R}_+^n$. Thus sofic measures are such that $\sum_{w \in A^*} \mu(w)w$ is an $\mathbb{R}_+$-rational series.
A measure μ is a k-step Markov measure if one has

$$\mu(uv) = \mu(u'v)$$

for every words u, u' of the same length and v of length $k+1$.
A k-step Markov measure is sofic.
It has been shown that if a sofic measure on $A^{\mathbb{Z}}$ given by a linear representation of dimension n is a k-step Markov measure for some k, then we can bound k in terms of n and Card(A) (Boyle, Petersen, 2010).
Question: Is there a reasonable bound on k?

Idempotent measures

If ν, ν' are two probability measures on a finite monoid M, their convolution product is the probability measure

$$\nu * \nu'(m) = \sum_{m=uv} \nu(u)\nu'(v).$$

A probability measure ν is idempotent if $\nu * \nu = \nu$.
When μ is a Bernouli measure on $A^{\mathbb{Z}}$ and $\varphi \colon A^* \to M$ is a morphism onto a finite monoid, then $\nu = \delta_\mu \circ \varphi^{-1}$ is an idempotent measure.
Question: Is there a definition of the convolution product such that the above property is true for a general invariant measure μ?