Generating sequences of bifix Codes

Definition: set of words both a prefix and a suffix code
=antichains both for the prefix order and for the suffix order.

Old references:
Gilbert and Moore (1961)
Schützenberger (1956)
Berstel, Perrin (1984)
...
New ones:
R. Ahlswede and B. Balkenhol and L. Khachatrian, 1997
Chunxuan Ye and Raymond W. Yeung, 2000

New references linked to the use of bifix codes for image compression.

Examples

1. words of fixed length : $aaa + aab + \ldots$
2. $a + ba^*b$
3. group codes : $X^* = \varphi^{-1}(e)$ for $\varphi A^* \to G$ morphism into a group.
4. first returns to zero in coin tossing.
5. others : $aaa + ab + bba + bbb$.

generating sequences

Given a set X of words, we denote

$$u_X(z) = \sum_{n \geq 1} u_n z^n$$

the generating sequence of the set X with

$$u_n = \text{number of words of length } n \text{ in } X$$

The indicator

$$\begin{aligned} L_X(s) &= 1 + \text{ length of } s \\ &\quad -\text{number of factors of } s \text{ in } X \\ &= \text{ \# prefixes of } s \text{ without a suffix in } X \\ &= \text{ \# suffixes of } s \text{ without a prefix in } X \end{aligned}$$

Actually

$$\begin{aligned} L_X &= A^*(1-X)A^* \\ &= A^*U \\ &= VA^* \end{aligned}$$

where $U = A^* - XA^*$ and $V = A^* - A^*X$.

How does L_X increase?

$$L_X(ua) - L_X(u) = \begin{cases} 0 & \text{if } ua \in A^*X \\ 1 & \text{otherwise} \end{cases}$$

Maximality

Maximal biprefix codes: for set inclusion.

For a biprefix code which is thin (i. e. there is some word w which is not a factor of X), the following are equivalent:

- X is a maximal biprefix code
- X is complete (i.e. each word is a factor of X^*).
- L_X is bounded.

The maximal value of L_X is called the *degree* of X.

Example

The following example already appears in the paper of Schützenberger of 1956 ('Une théorie algébrique du codage').

$$
\begin{aligned}
X &= aaa + aaba + aabb + ab \\
&\quad + baa + baba + babb + bba + bbb
\end{aligned}
$$

	prefix order		suffix order
1	aaa	1	aaa
2	$aaba$	5	baa
3	$aabb$	2	$aaba$
4	ab	6	$baba$
5	baa	8	bba
6	$baba$	4	ab
7	$babb$	3	$aabb$
8	bba	7	$babb$
9	bbb	9	bbb

Binary trees

Completion

Theorem (Zhang and Shen, 1994) *Any rational biprefix code is included in a maximal one.*

Solution of a conjecture by Jean Berstel and D. P. (1984).

Previous results

1. Analogous statement for general codes: Ehrenfeucht and Rosenberg, 1983
2. statement for finite biprefix codes : D. P. 1984.
3. Statement for codes with bounded deciphering delay: V. Bruyère and L. Zhang, 1990.
4. Statement for codes with bounded synchronization delay : V. Bruyère, 1997.

Example

Any biprefix code containing $a + bb$ is infinite: one may not add ba nor baa nor $baaa$ nor ...

Rational completion:

$$a + ba^*b$$

A method for finite biprefix codes

Idea: add to X a set Y such that no word of Y is a factor of another.

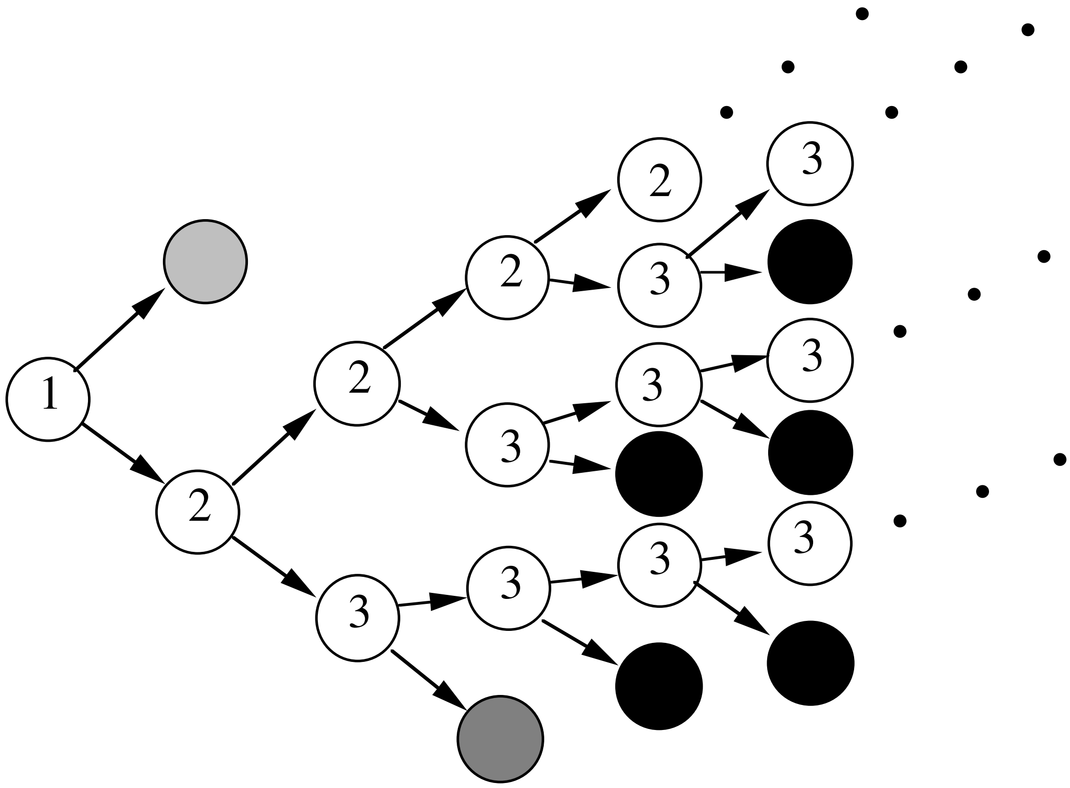
Theorem : *Let X be a finite biprefix code and let $n > \max\{L_X(x) | x \in X\}$.*

Let Y be the set of words y minimal for the factor ordering such that $L_X(y) = n$.

Then $Z = X+Y$ is a maximal rational biprefix code.

Proof:

- Z is biprefix: any $x \in X$ and $y \in Y$ are incomparable for the prefix and the suffix orders.
- Z is maximal: for any $u \in A^*$ there is a $v \in A^*$ such that $L_X(uv) > n$. Then uv has a prefix in Z.
- Z is rational: obvious.



Example

Let

$$X = a + bbb$$

Then

$$Z = a + ba^*ba^*b$$

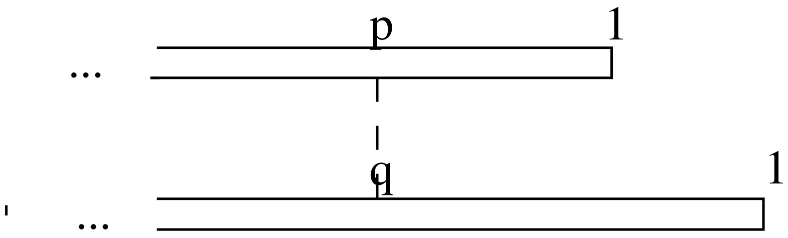
The chinese way

When X is infinite, L_X might be unbounded on X. The idea is to progressively complete X, beginning with the set M_X={words of $P(I_X)$ minimal for the factor order} where

I_X is the set of words incomparable to X for the prefix order and

$P(S) = A^* - SA^*$ is the set of *first words* of S.

Miracle : L_X is bounded M_X (by the number of states of the automaton).



Chains

Let X be a rational biprefix code recognized by an n-state automaton. Let

p= the maximal length of chains of prefixes of X for the suffix order.

s= the maximal length of chains of suffixes of X for the prefix order.

m= the maximal value of L_X on M_X

Then

$$m \leq p \leq n, \quad s \leq n$$

The algorithm

Completion in two nested loops:

while $I_X \neq \emptyset$ **do**
$\{$
  $F := M_X;$
  $n := \max\{L_X(u) \mid u \in F\};$
  while $F \neq \emptyset$ **do**
  $\{$
    $Z := P\{u \in FA^* \mid L_X(u) = n\};$
    $F = Z \cap A^- X$
    $Y := Z - A^- X;$
    $X := X + Y;$
    $n := n + 1;$
  $\}$
$\}$

where
$A^- X$={suffixes of X}.

Analysis

Correction: The result is clearly a maximal biprefix code. If the algorithm terminates, it is also rational.
Termination:

- Inside loop: at most s iterations.
- External loop: at most p iterations.

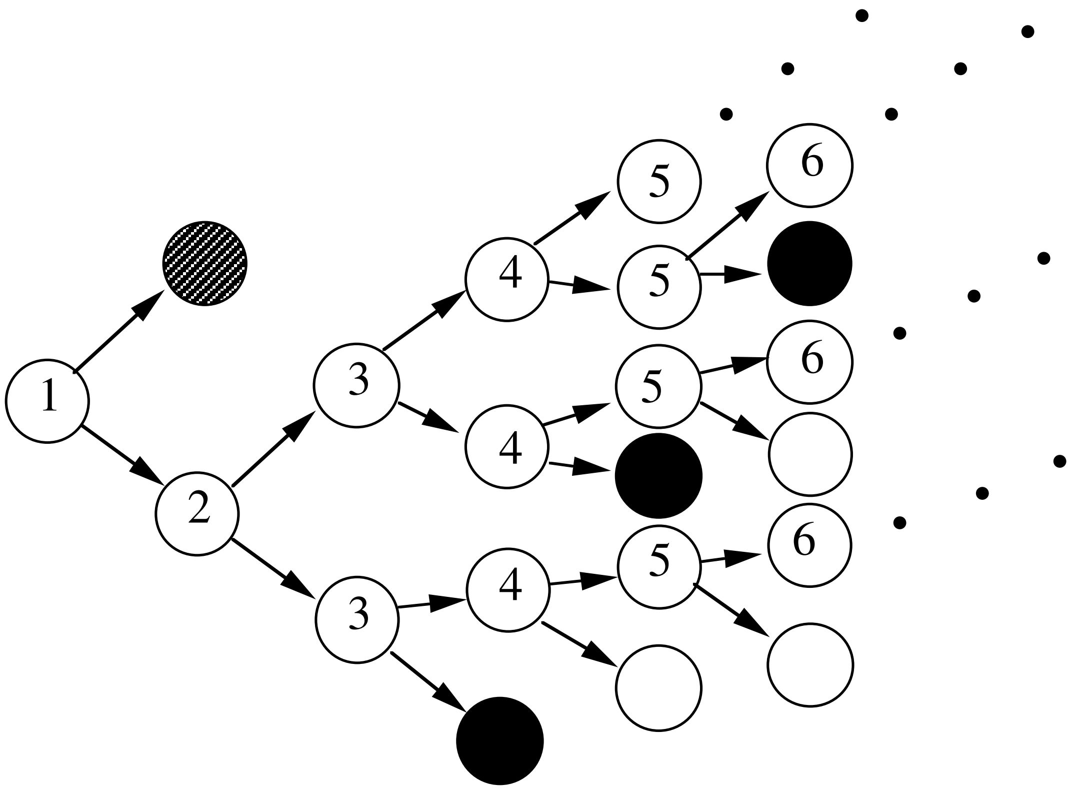
Example

Let

$$X = ba^*bb$$

We have $M_X = a$ and completing M_X gives

$$X = a + ba^*bb$$


1
2
3
3
4
4
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6
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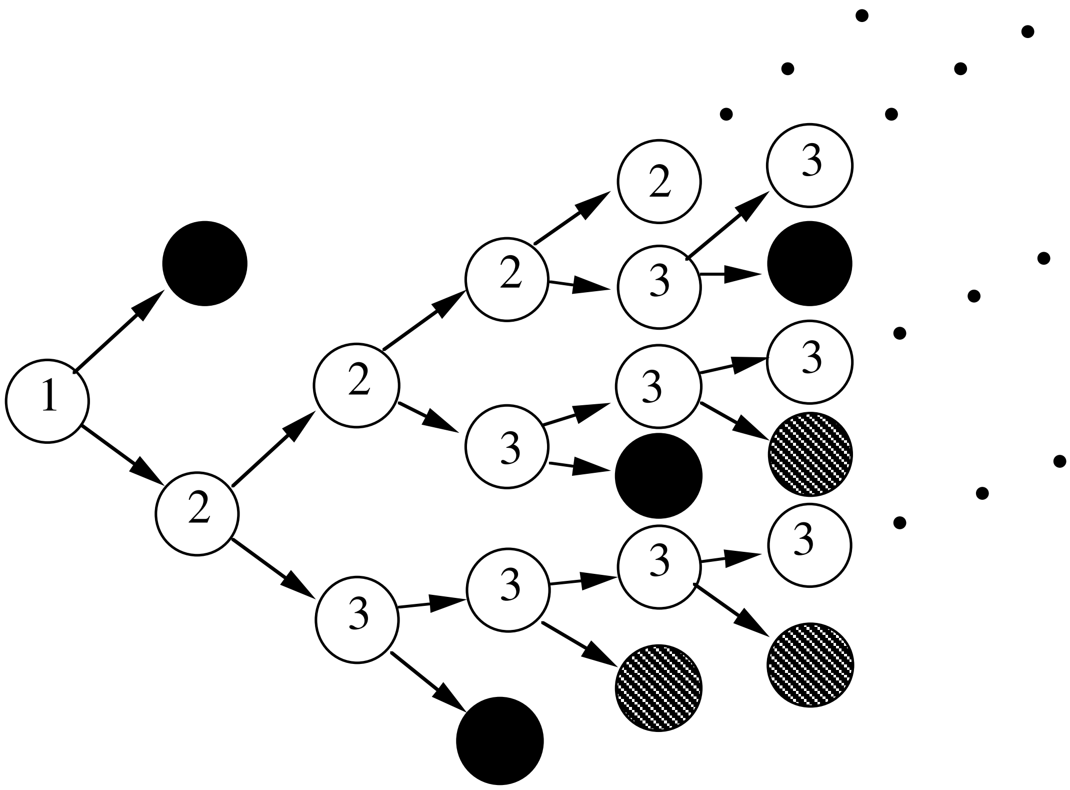
6

A second iteration allows to complete X as

$$X = a + ba^*ba^*b$$


1
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My way

Let X be a biprefix code and let Y be the set of words

- incomparable with X for the prefix and the suffix order
- minimal for this property in the factor order

Then

$$X' = X + Y$$

is again biprefix. Continue...

Can this really loop for ever when X is rational?

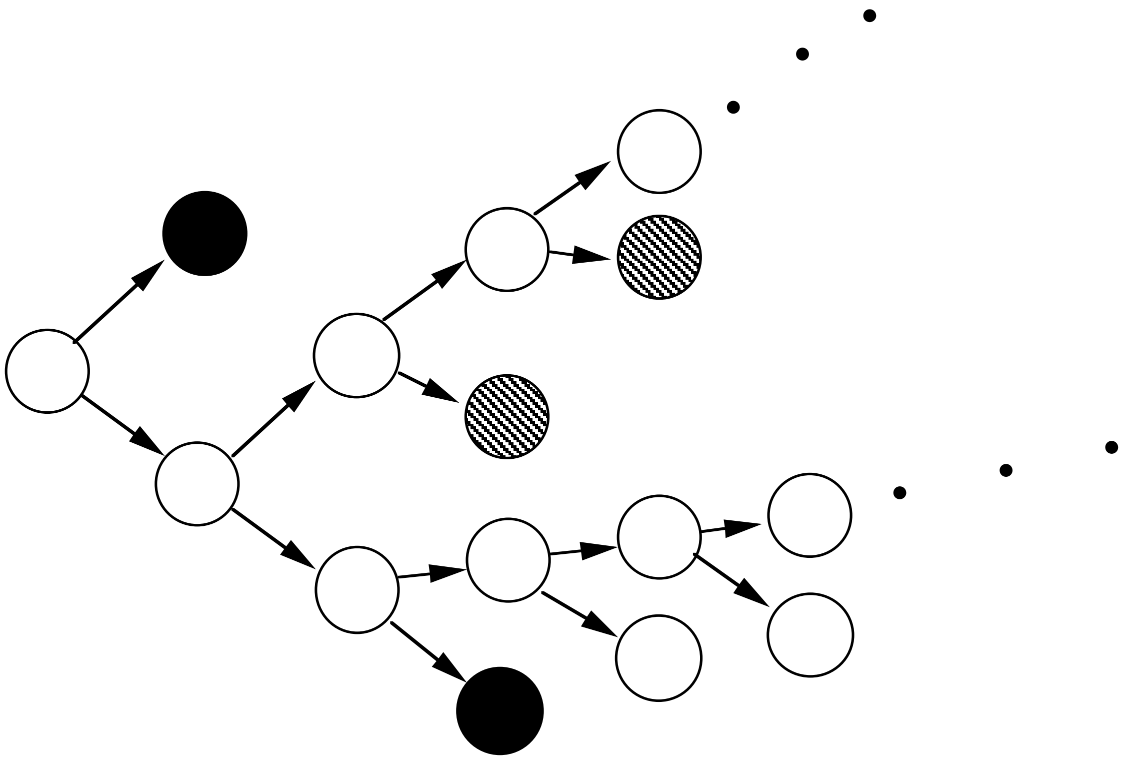
Example

Let

$$X = a + bbb$$

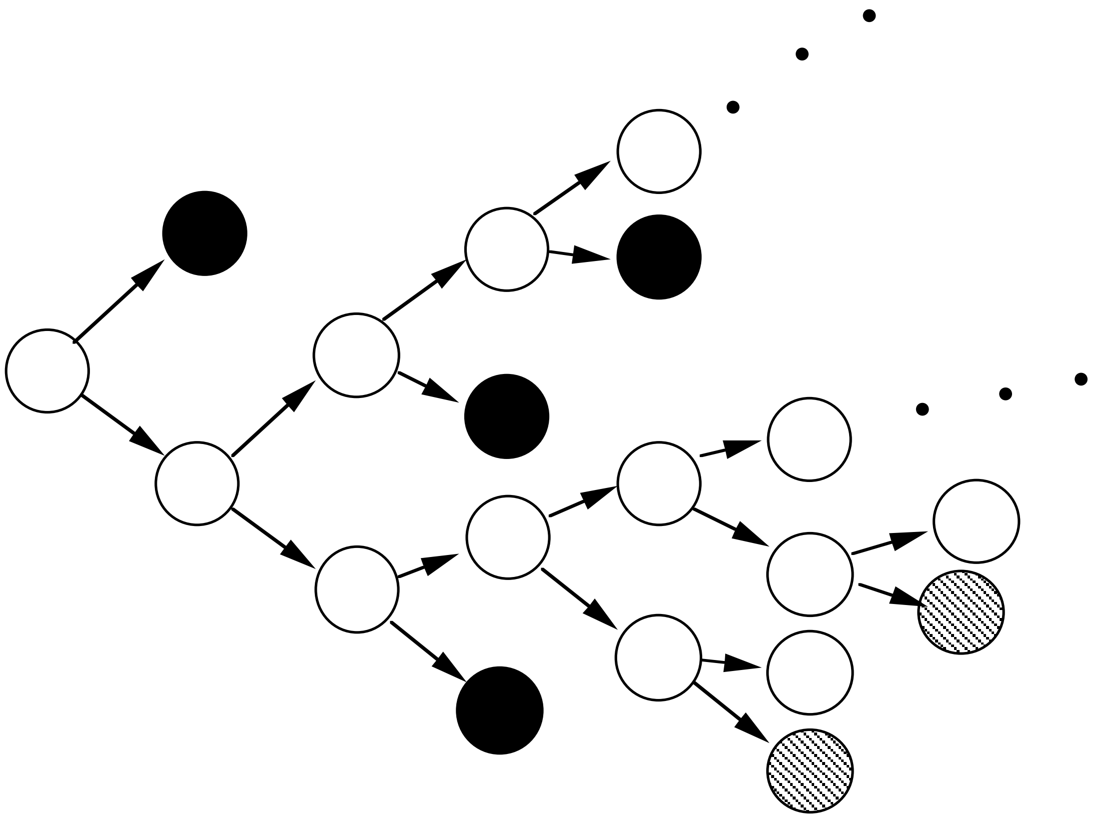
Then

$$Y = ba^+b$$



A second iteration gives

$$Y' = bb(a^+b)^*b$$



A new result

Theorem (Véronique Bruyère & D.P.)
A bifix code X is a subset of a thin maximal bifix code iff

(i) the chains of prefixes for the suffix order are bounded.

(ii) the chains of suffixes for the prefix order are bounded.

Examples:

$$\begin{aligned} X &= \{a^n b^n \mid n \geq 1\} \\ X &= \{ba^n b^n a \mid n \geq 1\} \end{aligned}$$

the second one is included in

$$X' = aa + ab + ba^+b^+a + bb$$