The mysteries of free profinite semigroups

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- Let **V** be a *pseudovariety* of semigroups (class of finite semigroups closed under taking homomorphic images, subsemigroups, and finite direct products).
- Finite semigroups are viewed as topological semigroups under the discrete topology.
- *Pro-**V** semigroup*: compact semigroup that is *residually* **V** as a topological semigroup.
- *Profinite semigroup*: compact semigroup that is residually finite as a topological semigroup.
  Equivalently: compact zero-dimensional (or totally disconnected) semigroup.

- $\overline{\Omega}_A \mathbf{V}$: pro-$\mathbf{V}$ semigroup freely generated by the totally disconnected space A. Usually, we assume that $|A| \geq 2$.
- It is characterized by the following *universal property*:

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\varphi"'] & \overline{\Omega}_A \mathbf{V} \arrow[d, dashed, "\hat{\varphi}"] \\ & S \end{tikzcd}$$

  where $\varphi : A \to S$ denotes an arbitrary continuous mapping into a pro-$\mathbf{V}$ semigroup.
- It may be constructed as the *projective limit* of all *A-generated* members of $\mathbf{V}$.
- Elements of $\overline{\Omega}_A \mathbf{V}$ will be called *pseudowords (over $\mathbf{V}$)*.
- For A finite, they may be viewed as *(implicit) operations* on pro-$\mathbf{V}$ semigroups: for $w \in \overline{\Omega}_A \mathbf{V}$ and S pro-$\mathbf{V}$,

$$\begin{aligned} w_S : S^A &\longrightarrow S \\ \varphi &\longmapsto \hat{\varphi}(w) \end{aligned}$$

- Denote by $\iota_{\mathbf{V}}$ the natural homomorphism $A^+ \to \overline{\Omega}_A\mathbf{V}$.

Theorem (JA'1989)

A language $L \subseteq A^+$ is $\mathbf{V}$-recognizable if and only if the following conditions hold:

1. *the closure $\overline{\iota_{\mathbf{V}}(L)}$ of $\iota_{\mathbf{V}}(L)$ in $\overline{\Omega}_A\mathbf{V}$ is open;*
2. $L = \iota_{\mathbf{V}}^{-1}\left(\overline{\iota_{\mathbf{V}}(L)}\right)$.

- Note: the second condition is superfluous if $\iota_{\mathbf{V}}$ is injective and the induced topology on A^+ (i.e., the *pro-*$\mathbf{V}$ *topology*) is discrete.
- This is the case for instance for any pseudovariety containing all finite nilpotent semigroups.

- In other words: the topological space $\overline{\Omega}_A\mathbf{V}$ is the *Stone dual* of the Boolean algebra of $\mathbf{V}$-recognizable subsets of A^+.
- Gehrke-Grigorieff-Pin'2008: the multiplication operation on $\overline{\Omega}_A\mathbf{V}$ is the dual of the residuation operations.
- Thus, to understand the algebraic/topological structure of $\overline{\Omega}_A\mathbf{V}$ is equivalent to understanding the algebra/combinatorics of $\mathbf{V}$-recognizable languages over the alphabet A.

- From hereon, we assume that A is a finite set with $|A| \geq 2$.

THEOREM (BINZ-NEUKIRCH-WENZEL'1971)

Every open subgroup of a free profinite group is also a free profinite group.

THEOREM (RIBES'1970)

A profinite group is projective if and only if it is isomorphic to a closed subgroup of a free profinite group.

THEOREM (NOVIKOV-SEGAL'2003,2007)

Every subgroup of finite index of a finitely generated profinite group is open.

- In other words, every homomorphism from a finitely generated profinite group into a finite group is continuous.
  - It follows that every homomorphism from a finitely generated profinite semigroup into a finite group is continuous.
  - This is not the case for all finite semigroups: for instance, $\chi_{A^+} : \overline{\Omega}_A\mathbf{S} \to \{0, 1\}$ is a discontinuous homomorphism into the two-element semilattice.
  - Problem. *For which finite semigroups is it true that every homomorphism from a finitely generated profinite semigroup into it is continuous?*
- In particular, the topology of a finitely generated profinite group is completely determined by its algebraic structure.

THEOREM (RHODES-STEINBERG'2008)

The closed subgroups of $\overline{\Omega}_A\mathbf{S}$ are precisely the projective profinite groups. In particular, every subgroup of $\overline{\Omega}_A\mathbf{S}$ is torsion-free.

- Combining with Ribes' Theorem, we deduce that $\overline{\Omega}_A\mathbf{S}$ has the same closed subgroups as $\overline{\Omega}_A\mathbf{G}$.
- Zalesskiĭ asked which profinite groups may appear as *maximal subgroups* of $\overline{\Omega}_A\mathbf{S}$.
- In particular, can a free pro-p group appear as a maximal subgroup of $\overline{\Omega}_A\mathbf{S}$?
- More generally, the theorem holds for every pseudovariety $\mathbf{V}$ such that $(\mathbf{V} \cap \mathbf{Ab}) * \mathbf{V} = \mathbf{V}$.

THEOREM (RHODES-STEINBERG'2008)

Every finite subsemigroup of $\overline{\Omega}_A\mathbf{S}$ *is a band.*

- More generally, this holds for every pseudovariety $\mathbf{V}$ such that $\mathbf{A} ⓜ \mathbf{V} = \mathbf{V}$ (that is for which the corresponding variety of languages is closed under concatenation) provided we replace "*a band*" by "*completely regular*".

Theorem (JA-Steinberg'2008)

A clopen subsemigroup of $\overline{\Omega}_A\mathbf{S}$ is a free profinite semigroup if and only if it is the closure of a rational free subsemigroup of A^+.

- Margolis-Sapir-Weil'1998: reverse direction for a finitely generated free subsemigroup of A^+.
- Holds more generally for pseudovarieties of the form $\overline{\mathbf{H}}$ (all finite semigroups whose subgroups lie in $\mathbf{H}$) for an arbitrary pseudovariety $\mathbf{H}$ of groups which is *extension-closed*.

THEOREM (STEINBERG'2008)

The maximal subgroups of the minimum ideal of $\overline{\Omega}_A\mathbf{S}$ are free profinite groups of countable rank.

- This holds more generally for pseudovarieties of the form $\overline{\mathbf{H}}$ for a pseudovariety $\mathbf{H}$ of groups which is extension-closed and *contains cyclic groups of infinitely many prime orders*.
- Steinberg asked whether the latter hypothesis may be dropped for nontrivial pseudovarieties of groups.

Theorem (Steinberg'2008)

Let G be a maximal subgroup of the minimum ideal of $\overline{\Omega}_A\mathbf{S}$ and let $\varphi : G \to \overline{\Omega}_A\mathbf{G}$ be the restriction to G of the natural continuous homomorphism $\overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{G}$. Then $\ker\varphi$ is a free profinite group of countable rank.

- This holds more under the same more general hypotheses as in the preceding theorem if we replace the pair of pseudovarieties $(\mathbf{S}, \mathbf{G})$ by $(\overline{\mathbf{H}}, \mathbf{H})$.
- Steinberg also conjectured that the maximal subgroup of the subsemigroup of $\overline{\Omega}_A\mathbf{S}$ generated by the idempotents in the minimum ideal is free profinite.

index

- For a topological semigroup S, denote by $End(S)$ its endomorphism monoid.

THEOREM (HUNTER'1983 (JA'2003, STEINBERG'2011))

Let S be a finitely generated profinite semigroup. Then $End(S)$ is a profinite semigroup under the pointwise convergence topology (i.e., with the subspace topology induced from the direct power S^S). Moreover, this topology coincides with the compact-open topology, which entails the continuity of the evaluation mapping

$$\begin{aligned} End(S) \times S &\longrightarrow S \\ (\varphi, s) &\longmapsto \varphi(s). \end{aligned}$$

- In a compact semigroup S, given $s \in S$, the minimum ideal of $\overline{\langle s\rangle}$ is a group, and therefore has a unique idempotent, denoted s^ω. The inverse of $s^{\omega+1} = ss^\omega$ is denoted $s^{\omega-1}$.
- In case S is profinite, $s^{\omega(-1)} = \lim_{n\to\infty} s^{n!(-1)}$.

- Call the elements of $\mathrm{End}(\overline{\Omega}_A\mathbf{S})$ *substitutions*.
- Given a substitution φ, we may therefore consider its "infinite iterate" φ^ω, which is an idempotent substitution.
- Examples for $A = \{a, b\}$:
  - *Fibonacci substitution*: $\varphi(a) = ab$, $\varphi(b) = a$;

    $$\varphi^7(a) = abaababaabaababaababaabaababaabaab$$

  - *Prouhet-Thue-Morse substitution*: $\tau(a) = ab$, $\tau(b) = ba$;

    $$\tau^5(a) = abbabaabbaababbabaababbaabbabaab.$$

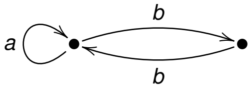
- From hereon, we let $\mathbf{V}$ be a pseudovariety containing **LSI**, so that, in particular, $\iota_{\mathbf{V}} : A^+ \to \overline{\Omega}_A\mathbf{V}$ is injective.
- We identify each $u \in A^+$ with its image under $\iota_{\mathbf{V}}$ and call the elements of A^+ the *finite elements* of $\overline{\Omega}_A\mathbf{V}$, while the remaining elements are said to be *infinite*.
- We say that $w \in \overline{\Omega}_A\mathbf{V}$ is *uniformly recurrent* if, for every finite factor u of w, there is some positive integer N such that every finite factor v of w of length at least N admits u as a factor.

THEOREM (JA'2005)

An infinite element of $\overline{\Omega}_A\mathbf{V}$ is uniformly recurrent if and only if it is $\mathcal{J}$-maximal.

- Note: since every infinite pseudoword has some idempotent factor, $\mathcal{J}$-maximal infinite pseudowords are regular.

- If $\mathcal{X} \subseteq A^{\mathbb{Z}}$ is a subshift, then one may consider the language $L(\mathcal{X})$ of its finite factors and two associated subsets of $\overline{\Omega}_A\mathbf{V}$:
  - $\overline{L(\mathcal{X})}$, the closure of $L(\mathcal{X})$ in $\overline{\Omega}_A\mathbf{V}$;
  - $\mathcal{M}(\mathcal{X}) = \{w \in \overline{\Omega}_A\mathbf{V} : F(w) \subseteq L(\mathcal{X})\}$, where $F(w)$ is set the of all finite factors of w.
- Note that $\overline{L(\mathcal{X})} \subseteq \mathcal{M}(\mathcal{X})$.
- Costa'2006: the preceding inclusion may be strict, which is the case for example for the even subshift



- Recall that a subshift is *irreducible* if, for all $u, v \in L(\mathcal{X})$, there exists w such that $uwv \in L(\mathcal{X})$.

Proposition (JA, Costa)

If $\mathcal{X}$ is an irreducible subshift, then each of the sets $\overline{L(\mathcal{X})}$ and $\mathcal{M}(\mathcal{X})$ contains $\mathcal{J}$-minimal elements, and these elements are regular and $\mathcal{J}$-equivalent. In case $\mathcal{X}$ is a minimal subshift, then $\overline{L(\mathcal{X})} = \mathcal{M}(\mathcal{X})$.

- $J(\mathcal{X})$ and $J\mathcal{M}(\mathcal{X})$ denote the $\mathcal{J}$-classes of those $\mathcal{J}$-minimal elements.

Theorem (JA'2005)

A regular $\mathcal{J}$-class of $\overline{\Omega}_A\mathbf{V}$ is $\mathcal{J}$-maximal if and only if it is of the form $J(\mathcal{X})$ for some minimal subshift $\mathcal{X} \subseteq A^{\mathbb{Z}}$

- As regular $\mathcal{J}$-classes, $\mathcal{J}(\mathcal{X})$ and $\mathcal{J}\mathcal{M}(\mathcal{X})$ contain maximal subgroups.
- As Green's relations are closed, such groups are profinite groups.
- Up to isomorphism of topological groups, they depend only on the $\mathcal{J}$-class.
- We call the group $G(\mathcal{X})$ thus associated to an irreducible subshift $\mathcal{X}$ via the regular $\mathcal{J}$-class $\mathcal{J}(\mathcal{X})$ the *Schützenberger group* of $\mathcal{X}$.
- If the irreducible subshift $\mathcal{X}$ is given by some finite description, such as a sofic subshift or a minimal subshift determined by a primitive substitution, a natural problem is to find a finite description of the group $G(\mathcal{X})$.
- In case $\mathcal{X}$ is the minimal subshift determined by a prinitive substitution φ, we also denote $G(\mathcal{X})$ by $G(\varphi)$.

- If G is a profinite group and N is a closed normal subgroup, then the quotient G/N has a natural structure of profinite group: for a subgroup H of G containing N, H/N is open in G/N if and only if H is open in G.
- A *group profinite presentation* is a pair $\langle A; R\rangle$, where A is a set and R a binary relation on the free profinite group $\overline{\Omega}_A\mathbf{G}$.
- In case A and R are finite sets, the presentation $\langle A; R\rangle$ is said to be *finite*.
- The profinite group *presented* by the profinite presentation $\langle A; R\rangle$ is the quotient of $\overline{\Omega}_A\mathbf{G}$ by the closed normal subgroup generated by the set $\{uv^{-1} : (u, v) \in R\}$.

THEOREM (JA'2005)

If $\mathcal{X} \subseteq A^{\mathbb{Z}}$ is an Arnoux-Rauzy subshift, then $G(\mathcal{X})$ is a free profinite group of rank $|A|$.

- JA'2005: let $\mathcal{X}$ be the subshift determined by the substitution $a \mapsto ab$, $b \mapsto a^3b$; then $G(\mathcal{X})$ is a profinite group of rank 2 that is not free.

THEOREM (COSTA-STEINBERG'2011)

Let $\mathcal{X}$ be an irreducible non-periodic sofic subshift. Then the profinite group $G(\mathcal{X})$ is free of countable rank.

- We say that a substitution φ over a finite alphabet A is *proper* if there are letters $a, b \in A$ such that, for every $d \in A$, the word $\varphi(d)$ starts with a and ends with b.

Theorem (JA-Costa'2013)

Let φ be a non-periodic proper primitive substitution over a finite alphabet A. Then $G(\varphi)$ admits the presentation

$$\langle A \mid \varphi_{\mathbf{G}}^{\omega}(a) = a \ (a \in A)\rangle.$$

- Let φ be a primitive substitution over a finite alphabet A, determining a subshift $\mathcal{X}_\varphi \subseteq A^{\mathbb{Z}}$.
- We say that ba ($a, b \in A$) is a *connection* for φ if $ba \in L(\mathcal{X}_\varphi)$, $\varphi^\omega(a)$ starts with a and $\varphi^\omega(b)$ ends with b.
- For a word $u \in L(\mathcal{X}_\varphi)$, $R(u)$ denotes the set of all return words of u in $\mathcal{X}_\varphi$, i.e., the set of all words v such that $vu \in L(\mathcal{X}_\varphi)$ and u appears as a factor of vu only at the prefix and suffix positions.

- For a connection ba, of the primitive substitution φ, we let $X_\varphi(a,b) = b^{-1}(R(ba)b)$.

THEOREM (JA-COSTA'2013)

Let φ be a non-periodic primitive substitution over the alphabet A. Let ba be a connection of φ and let $X = X_\varphi(a,b)$. Then, φ induces a continuous endomorphism $\tilde{\varphi}_{\mathbf{G}}$ of $\overline{\Omega}_X\mathbf{G}$ and the profinite group $G(\varphi)$ admits the presentation

$$\langle X \mid \tilde{\varphi}^{\omega}_{X,\mathbf{G}}(x) = x \ (x \in X)\rangle_{\mathbf{G}}.$$

THEOREM (JA-COSTA'2013)

Let φ be a primitive substitution over a finite alphabet. Then the profinite group $G(\varphi)$ is decidable in the sense that it is decidable for a finite group G whether or not there is an onto (continuous) homomorphism $G(\varphi) \to G$.

- Let A be the two-letter alphabet $\{a, b\}$. The *Prouhet-Thue-Morse substitution* is the non-periodic primitive substitution τ over A given by $\tau(a) = ab$ and $\tau(b) = ba$.

Theorem (JA-Costa'2013)

The group $G(\tau)$ admits the following presentation:

$$\langle x, y, z \mid \Psi^{\omega}(x) = x,\ \Psi^{\omega}(y) = y,\ \Psi^{\omega}(z) = z\rangle_{\mathbf{G}}$$

where Ψ is the unique continuous endomorphism of $\overline{\Omega}_{\{x,y,z\}}\mathbf{G}$ such that $\Psi(x) = zxy$, $\Psi(y) = zyx^{-1}zxy$, and $\Psi(z) = zxyx^{-1}zy$. The profinite group $G(\tau)$ has rank 3 and it is not free.

- Let $\mathcal{X} \subseteq A^{\mathbb{Z}}$ be a subshift.
- Denote by $L_n(\mathcal{X})$ the set $L(\mathcal{X}) \cap A^n$.
- The *Rauzy graph of order n* of $\mathcal{X}$, denoted $\Sigma_n(\mathcal{X})$ is defined by:
  - set of vertices: $L_n(\mathcal{X})$;
  - set of edges: $L_{n+1}(\mathcal{X})$;
  - incidence:

$$a_1 a_2 \cdots a_n \xrightarrow{a_1 a_2 \cdots a_n a_{n+1}} a_2 \cdots a_n a_{n+1}.$$

- We label edges of the Rauzy graph Σ_{2n} *centrally*:

$$\mu_n(a_1 a_2 \cdots a_{2n+1}) = a_{n+1}.$$

- For $x \in \mathcal{X}$, we denote $\hat{\Pi}_{2n}(\mathcal{X}, x)$ the profinite completion of the (free) fundamental group of the Rauzy graph $\Sigma_{2n}(\mathcal{X})$ based at the vertex $x_{-n}x_{-n+1}\cdots x_{n-1}$.
- The mapping

$$\begin{aligned} \Sigma_{2n+2}(\mathcal{X}) &\to \Sigma_{2n}(\mathcal{X}) \\ aub \in L_{2n+2}(\mathcal{X}) &\mapsto u \\ aub \in L_{2n+3}(\mathcal{X}) &\mapsto u \end{aligned}$$

  (where $a, b \in A$) is a graph homomorphism respecting labelings.
- It induces a homomorphism

$$\hat{\Pi}_{2n+2}(\mathcal{X}, x) \to \hat{\Pi}_{2n}(\mathcal{X}, x).$$

Theorem

If $\mathcal{X}$ is a minimal subshift then $G(\mathcal{X})$ is isomorphic with $\varprojlim \hat{\Pi}_{2n}(\mathcal{X}, x)$ as a profinite group, for every $x \in \mathcal{X}$.

- The inverse limit $\varprojlim \Sigma_{2n}(\mathcal{X})$ is isomorphic with the *graph of the subshift* $\mathcal{X}$, with
  - set of vertices: $\mathcal{X}$;
  - edges: $x \to \sigma(x)$.
- It all becomes more interesting if we take the free profinite semigroupoid $\hat{\Sigma}_{2n}(\mathcal{X})$ generated by the Rauzy graph $\Sigma_{2n}(\mathcal{X})$ and the inverse limit

$$\hat{\Sigma}(\mathcal{X}) = \varprojlim \hat{\Sigma}_{2n}(\mathcal{X}).$$

THEOREM (JA-COSTA'2009)

If $\mathcal{X}$ is a minimal subshift, then the free semigroupoid generated by $\Sigma(\mathcal{X})$ is dense in $\hat{\Sigma}(\mathcal{X})$.

- In general, the labeling mapping μ_n induces

$$\begin{tikzcd}
E(\Sigma_{2n}(\mathcal{X})) \arrow[r, "\mu_n"] \arrow[d] & A \arrow[d] \\
\Sigma_{2n}(\mathcal{X})^+ \arrow[r, "\mu_n"] \arrow[d] & A^+ \arrow[d] \\
\hat{\Sigma}_{2n}(\mathcal{X}) \arrow[r, "\hat{\mu}_n"] & \overline{\Omega}_A \mathbf{V}
\end{tikzcd}$$

- The sequence of mappings $(\hat{\mu}_n)_n$ passes to the inverse limit as a continuous semigroupoid homomorphism

$$\hat{\mu} : \hat{\Sigma}(\mathcal{X}) \to \overline{\Omega}_A \mathbf{V}.$$

Theorem (JA-Costa'2009)

The mapping $\hat{\mu}$ is faithful and its image is $\mathcal{M}(\mathcal{X})$.

- Consider the "infinite part" of $\hat{\Sigma}(\mathcal{X})$, which is defined by:

$$\hat{\Sigma}(\mathcal{X})_\infty = \hat{\Sigma}(\mathcal{X}) \setminus E(\Sigma(\mathcal{X})^+).$$

THEOREM

If $\mathcal{X}$ is minimal, then $\hat{\Sigma}(\mathcal{X})_\infty$ is a profinite connected groupoid whose local group at the vertex $x \in \mathcal{X}$ is mapped isomorphically by $\hat{\mu}$ to the $\mathcal{H}$-class G_x of $\mathcal{J}(\mathcal{X})$ consisting of the elements that start with $\overrightarrow{x} = x_0 x_1 \cdots$ and end with $\overleftarrow{x} = \cdots x_{-2} x_{-1}$.

- For finite words u, v and a subshift $\mathcal{X}$, let

$$R(u,v) = u^{-1}(R(uv)u).$$

- For $x \in \mathcal{X}$, let

$$R_n(x) = R(x_{[-n,-1]}, x_{[0,n-1]}).$$

Lemma

Let $\mathcal{X}$ be a minimal subshift and let $x \in \mathcal{X}$. Then we have

$$\bigcap_{n\geq 1} \left(\overline{\langle R_n(x) \rangle} \setminus A^+ \right) = G_x.$$

THEOREM

Let $\mathcal{X}$ be a minimal non-periodic subshift and $x \in \mathcal{X}$. Let A be the set of letters occurring in $\mathcal{X}$. Suppose there is a subgroup K of $FG(A)$ and an infinite set P of positive integers such that, for every $n \in P$, the set $R_n(x)$ is a free basis of K. Let $\overline{K}$ be the topological closure of K in $\overline{\Omega}_A\mathbf{G}$. Then the restriction to G_x of the canonical projection $p_{\mathbf{G}}: \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{G}$ is a continuous isomorphism from G_x onto $\overline{K}$.

- The following is an application extending the JA'2005 result on Arnoux-Rauzy subshifts, which depends on the *Return Theorem* (Berthé et.al.'2015):

THEOREM

If $\mathcal{X}$ is a minimal subshift satisfying the tree condition, then $G(\mathcal{X})$ is a free profinite group with rank the number of letters occurring in $\mathcal{X}$.

- Denote by $\Pi_{2n}(\mathcal{X})$ the fundamental groupoid of the graph $\Sigma_{2n}(\mathcal{X})$.
- Let $\hat{\Pi}_{2n}(\mathcal{X})$ be its profinite completion.
- Paths in $\Sigma_{2n}(\mathcal{X})$ determine homotopy classes in $\Pi_{2n}(\mathcal{X})$, which yields a continuous semigroupoid homomorphism

$$\hat{h}_n : \hat{\Sigma}_{2n}(\mathcal{X}) \to \hat{\Pi}_{2n}(\mathcal{X})$$

inducing a continuous semigroupoid homomorphism

$$\hat{h} : \hat{\Sigma}(\mathcal{X}) \to \varprojlim \hat{\Pi}_{2n}(\mathcal{X}).$$

THEOREM

Let $\mathcal{X}$ be a minimal subshift. Then, the restriction of the mapping $\hat{h}$ to $\hat{\Sigma}(\mathcal{X})_\infty$ is an isomorphism of topological groupoids onto $\varprojlim \hat{\Pi}_{2n}(\mathcal{X})$.

- Let $w \in \overline{\Omega}_A \mathbf{V}$.
- For each positive integer n, let $q_w(n) = |F(w) \cap A^n|$.
- Note that $q_w(n)$ is defined by counting certain finite factors of w, which do not depend on $\mathbf{V}$ (for $\mathbf{V} \supseteq \mathbf{LSI}$).
- It is easy to see that the *complexity sequence* $q_w(n)$ satisfies the subadditive inequality

$$q_w(r+s) \leq q_w(r) q_w(s)$$

- It follows that the following limit exists

$$h(w) = \lim_{n \to \infty} \frac{1}{n} \log_{|A|} q_w(n)$$

- It is called the *entropy* of w.

PROPOSITION

The minimum ideal of $\overline{\Omega}_A\mathbf{V}$ *consists of the pseudowords of entropy 1.*

THEOREM

Let $w \in \overline{\Omega}_A\mathbf{V}$, $v_1, \ldots, v_r \in \overline{\Omega}_B\mathbf{S}$, *where* $r = |A|$, *and let* $u = w_{\overline{\Omega}_B\mathbf{V}}(v_1, \ldots, v_r)$ *and* $m = |B|$. *Then*

$$h(u) \leq \max\{h(w)\log_m r, h(v_1), \ldots, h(v_r)\}.$$

THEOREM

Let $\varphi \in End(\overline{\Omega}_A\mathbf{V})$ *be a substitution. Then,*

$$\max_{a\in A} h\big(\varphi^\omega(a)\big) \leq \max_{a\in A} h\big(\varphi(a)\big).$$

COROLLARY

The complement of the minimum ideal of $\overline{\Omega}_A\mathbf{V}$ is a subsemigroup that is closed under composition and iteration.

- A weaker form of this result was previously obtained by the same authors in 2003 by a completely different approach:

THEOREM

Let $\mathbf{H}$ be a pseudovariety of finite groups such that $\mathbf{H} \supseteq \mathbf{Ab}$. The smallest subset of $\overline{\Omega}_A\overline{\mathbf{H}}$ that contains A and is closed under multiplication, composition of operations and arbitrary powers has empty intersection with the minimum ideal.

- This was then obtained as a corollary of the following result, whose proof in turn depends on the theory of Burnside semigroups as developed by McCammond'1991, de Luca and Varricchio'1992, Guba'1993, and do Lago'1996:

Theorem

Let $\mathbf{H}$ be a pseudovariety of finite groups. Then the pseudovariety $\overline{\mathbf{H}}$ can be defined by a system of pseudoidentities using only multiplication and arbitrary powers if and only if membership of a finite group in $\mathbf{H}$ depends only on its cyclic subgroups.

- This also implies the following (\$100) conjecture of Rhodes'1986, which he had previously proved for $n \geq 665$, based on Adian's solution of the Burnside problem for groups:

Corollary

For all $n \geq 3$, the pseudovariety $\overline{\mathbf{Ab}} \cap [\![x^{2n} = x^n]\!]$ is not equational.

- We know that the finite words are at the *top* of $\overline{\Omega}_A\mathbf{S}$ in the sense that the complement is an ideal.
- We have also seen that we cannot go very deep in $\overline{\Omega}_A\mathbf{S}$ by adding the ability to take arbitrary powers.

PROBLEM

Does the subalgebra of $\overline{\Omega}_A\mathbf{S}$ with respect to the signature consisting of multiplication and arbitrary powers generated by A also lie at the top?

- This problem becomes much more tractable if we reduce significantly the range of powers to be considered by identifying all infinite powers, that is in the *aperiodic* case.

- In the aperiodic case, we can take advantage of McCammond's normal form for the elements in our subalgebra of $\overline{\Omega}_A\mathbf{A}$, denoted $\Omega_A^\omega\mathbf{A}$.
- McCammond'2001 proved that it solves the word problem for $\Omega_A^\omega\mathbf{A}$ by applying his earlier solution of the word problem for free aperiodic Burnside semigroups.
- We have obtained a direct proof which also leads to new applications, among which the following

THEOREM

$\Omega_A^\omega\mathbf{A}$ *sits at the top of* $\overline{\Omega}_A\mathbf{A}$.

Theorem

An element w of $\overline{\Omega}_A\mathbf{A}$ belongs to $\Omega_A^\omega\mathbf{A}$ if and only if it satisfies the following two finiteness conditions:

1. *there are no infinite anti-chains of factors of w;*
2. *the language of McCammond normal forms of elements of $\Omega_A^\omega\mathbf{A}$ that are factors of w is rational.*

- We do not know whether the first condition is superfluous, that is whether it follows from the second one.

index