Outline

(from a joint work with Valérie Berthé, Vincent Delecroix, Francesco Dolce, Christophe Reutenauer and Giuseppina Rindone)

For the birthday of Arto Salomaa

- Words on curves
- Words on surfaces
- Linear involutions
- Natural codings
- Suspension surfaces
Words on curves: Gauss codes


Gauss code: abba (or any conjugate). Used to describe knots.
Rotations and Sturmian words

Rotation of angle $\alpha$. $R(z) = z + \alpha \mod 1$. Natural coding: let $s(\alpha) = (s_n)$ be the sequence

$$s_n = \begin{cases} 
  a & \text{if } \lfloor (n+1)\alpha \rfloor = \lfloor n\alpha \rfloor, \\
  b & \text{otherwise}
\end{cases}$$

For $\alpha = (3 - \sqrt{5})/2$ this gives the Fibonacci sequence which is the fixpoint of $a \mapsto ab$, $b \mapsto a$. 

![Graph showing the Fibonacci sequence in a grid]
Words on surfaces

- Coding of geodesics according to a partition (Hadamard, Morse and Hedlund)
- Interval exchange transformations (Rauzy)
- Linear involutions (Nogueira)
- Train tracks and foliations (Thurston)
Interval exchange transformations

A generalization of rotations: piecewise isometries on $[0, 1]$. 

\[ 0 \quad a \quad 1 - \alpha \quad b \quad 1 \]

\[ 0 \quad \alpha \quad b \quad a \quad 1 \]
What is a linear involution?

The transformation is a product of two involutions

- The first one is an isometry (translation or symmetry),
- The second one exchanges top and bottom

\[
T^2(z) = a^{-1}b^{-1}d^{-1}c^{-1} \quad a, \; d \text{ are symmetries and } b, \; d \text{ are translations.}
\]
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\[ T(z) \]

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\[
T^2(z) \quad \text{T}(z)
\]

\[
a, \ d \text{ are symmetries and } b, \ d \text{ are translations.}
\]
A particular case: interval exchange transformations

- Only translations are used.
- The level is preserved
A connection of a linear involution $T$ is a triple $(x, y, n)$ where $x$ is a singularity of $T^{-1}$, $y$ is a singularity of $T$ and $T^n x = y$.

**Theorem (Keane, 1975)**

A linear involution without connection is minimal.
A second example

- $a$ is a translation
- $b$, $c$ are symmetries

→ nonorientable foliations of orientable surfaces (see further).
The natural coding

Let $I_a$ be the interval labeled $a$ for $a \in A \cup A^{-1}$.

$$\Sigma_T(z) = a_0 a_1 \cdots$$

with

$$a_n = a \text{ if } T^n(z) \in I_a.$$ 

When $T$ is minimal, the set $S$ of factors of the natural coding does not depend on the point $z$.

The set $S$ can actually be defined directly as the set of factors of the fixpoint of the substitution

$$f : a \mapsto cb^{-1}, \quad b \mapsto c, \quad c \mapsto ab^{-1}.$$
The suspension foliation

\[
\begin{align*}
& a \quad b \quad b^{-1} \\
& c \quad c^{-1} \quad a^{-1}
\end{align*}
\]
A uniformly recurrent infinite word factorized as an infinite product \( \cdots r stu \cdots \) of first return words to \( w \) (stop after \( w \) and before \( w^{-1} \)).

**Theorem**

Let \( S \) be the natural coding of a linear involution without connection on the alphabet \( A \). For any \( w \in S \), the set of first return words to \( w \) is a monoidal basis of the free group on \( A \).
The first returns on $b$ form the symmetric set

$$\mathcal{R}_S(b) = \{\bar{ac}b, \bar{ac}, \bar{ca}, \bar{b}cb, \bar{b}\bar{ca}, \bar{b}\bar{c}b\}$$

(stop after $b$ and before $b^{-1}$).

- Cancelling by $\bar{ac}$ and its inverse gives $\{b, \bar{ac}, \bar{ca}, \bar{b}cb, \bar{b}, \bar{b}\bar{c}b\}$
- Cancelling by $b$ and $b^{-1}$ gives $\{b, \bar{ac}, \bar{ca}, c, \bar{b}, \bar{c}\}$
- Cancelling by $c$ and $c^{-1}$ gives $A \cup A^{-1}$.

Thus $\mathcal{R}_S(b)$ is a monoidal basis of the free group on $A$. 
The Subgroup Return Theorem

Let $G$ be a subgroup of the free group. The set of nonempty words of $G \cap S$ without a proper nonempty prefix in $G \cap S$ is the set of first returns to $G$.

**Theorem**

Let $T$ be a linear involution on $A$ without connection and let $S = L(T)$. For any subgroup $G$ of finite index of the free group $F_A$, the set of first return words to $G$ in $S$ is a monoidal basis of $G$. 
Example: The subgroup of even words

Let $G$ be the group of even words (having an even number of odd letters such as $b, c$). It is a subgroup of index 2. The set of first returns to $G$ is $X \cup X^{-1}$ with

$$X = \{a, ba^{-1}c, bc^{-1}, b^{-1}c^{-1}, b^{-1}c\}.$$ 

It corresponds to the first returns at the same level.
Linear involutions and measured foliations

Measured foliation: \((X, \Sigma, F, \mu)\) with \(X\) a compact surface, \(\Sigma\) a finite set of singularities, \(F\) a foliation defined on \(X \setminus \Sigma\) and \(\mu\) a measure on arcs transverse to \(F\).

In the neighborhood of a singularity:

- Linefield given by \(z^p(dz)^2 = I\) (here \(p = 1\)).
  - Linear involution \(T\) on an interval \(I \to\) measured foliation with \(I\) admissible (suspension on \(T\))
  - measured foliation + admissible interval \(I \to\) linear involution \(T\) on \(I\)
Rotations, Sturmian words and tori

A rotation

A torus
Fundamental groups

We consider punctured surfaces \((X, \Sigma)\). Fixing a base point \(x_0\), recall that the fundamental group \(\pi_1(X \setminus \Sigma, x_0)\) is the set of equivalence classes of loops in \(X \setminus \Sigma\) based at \(x_0\) up to homotopy.

- The fundamental group of the suspension on a linear involution \(T\) on \(A\) is the free group on \(A\).
- The interval \(I_w\) formed by the points with a natural coding beginning with \(w\) is admissible and the transformation induced by \(T\) on \(I_w\) has a suspension with a fundamental group which is free with monoidal basis \(R_S(w)\).
Subgroups and coverings

A covering of degree $d$ of a compact connected surface $X$ is a compact connected surface $Y$ with a continuous map $f : Y \to X$ such that for each $x \in X \setminus \Sigma$ there exists a connected neighborhood $U$ of $x$ such that $f^{-1}(U)$ is a disjoint union of $d$ open sets $f^{-1}(U) = U_1 \cup U_2 \cup \ldots \cup U_d$ such that for each $i \in \{1, \ldots, d\}$, $f : U_i \to U$ is a homeomorphism.

If $\gamma$ is a loop in $Y$ then $f(\gamma)$ is a loop in $X$. Hence, for any $y_0 \in Y$ we get a map $f_* : \pi_1(Y \setminus f^{-1}(\Sigma), y_0) \to \pi_1(X \setminus \Sigma, f(y_0))$. The map $f_*$ is injective and its image is of finite index in $\pi_1(X \setminus \Sigma, f(y_0))$.

**Theorem**

Let $X$ be a compact connected surface and let $\Sigma \subset X$ be a finite set. Then, the map $(f : Y \to X) \mapsto f_*(\pi_1(Y \setminus f^{-1}(\Sigma)))$ induces a bijection between equivalence classes of coverings of degree $d$ ramified over $\Sigma$ and conjugacy classes of subgroups of $\pi_1(X \setminus \Sigma)$ of index $d$. 
Perspectives

- Investigate words in other geometrical settings (hyperbolic spaces)
- Is automata theory in such environments less complex (linear factor complexity)?
- What is the role played by semigroups vs. groups?