Symbolic dynamics and automata

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Shift spaces

A shift space on the alphabet A is a shift-invariant subset of $A^{\mathbb{Z}}$ which is closed in the topology. The set $A^{\mathbb{Z}}$ itself is a shift space called the full shift.

For a set $W \subset A^*$ of words (whose elements are called the forbidden factors), we denote by $X^{(W)}$ the set of $x \in A^{\mathbb{Z}}$ such that no $w \in W$ is a factor of x.

Proposition

The shift spaces on the alphabet A are the sets $X^{(W)}$, for $W \subset A^*$.

A shift space X is of finite type if there is a finite set $W \subset A^*$ such that $X = X^{(W)}$.

Example

Let $A = \{a, b\}$, and let $W = \{bb\}$. The shift $X^{(W)}$ is composed of the sequences without two consecutive b's. It is a shift of finite type, called the golden mean shift.

A shift space X is said to be sofic if there is a recognizable set W such that $X = X^{(W)}$. Since a finite set is recognizable, any shift of finite type is sofic.

Example

Let $A = \{a, b\}$, and let $W = a(bb)^*ba$. The shift $X^{(W)}$ is composed of the sequences where two consecutive occurrences of the symbol a are separated by an even number of b's. It is a sofic shift called the even shift. It is not a shift of finite type.

Edge shifts

The edge shift on the graph G is the set of biinfinite paths in G. It is denoted by X_G and is a shift of finite type on the alphabet of edges. Indeed, it can be defined by taking the set of non-consecutive edges for the set of forbidden factors. The converse does not hold, since the golden mean shift is not an edge shift.

However, every shift of finite type is conjugate to an edge shift.

A graph is essential if every state has at least one incoming and one outgoing edge. This implies that every edge is on a biinfinite path. The essential part of a graph G is the subgraph obtained by restricting to the set of vertices and edges which are on a biinfinite path.

Morphisms

Let X be a shift space on an alphabet A, and let Y be a shift space on an alphabet B.

A morphism φ from X into Y is a continuous map from X into Y which commutes with the shift. This means that $\varphi \circ \sigma_A = \sigma_B \circ \varphi$.

Let k be a positive integer. We denote by $\mathcal{B}_k(X)$ the set of k-blocks of X. A function $f : \mathcal{B}_k(X) \to B$ is called a k-block substitution Let now m, n be fixed nonnegative integers with $k = m + 1 + n$. Then the function f defines a map φ called sliding block map with memory m and anticipation n as follows. The image of $x \in X$ is the element $y = \varphi(x) \in B^{\mathbb{Z}}$ given by

$$y_i = f(x_{i-m} \cdots x_i \cdots x_{i+n}) .$$

We denote $\varphi = f_\infty^{[m,n]}$.

Theorem (Curtis–Lyndon–Hedlund)

A map from a shift space X into a shift space Y is a morphism if and only if it is a sliding block map.

Conjugacies of shifts

A morphism from a shift X onto a shift Y is called a conjugacy if it is one-to-one from X onto Y. The inverse mapping is also a morphism, and thus a conjugacy.
The n-th higher block shift $X^{[n]}$ of a shift X has alphabet the set $B = \mathcal{B}_n(X)$ of blocks of length n of X.

Proposition

The shifts X and $X^{[n]}$ for $n \geq 1$ are conjugate.

For $G = (Q, \mathcal{E})$ and an integer $n \geq 1$, $G^{[n]}$ denotes the n-th higher edge graph of G. The set of states of $G^{[n]}$ is the set of paths of length $n-1$ in G. The edges of $G^{[n]}$ are the paths of length n of G.

The following result shows that the higher block shifts of an edge shift are again edge shifts.

Proposition

Let G be a graph. For $n \geq 1$, one has $X_G^{[n]} = X_{G^{[n]}}$.

A shift of finite type need not be an edge shift. For example the golden mean shift is not an edge shift. However, any shift of finite type comes from an edge shift in the following sense.

Proposition

Every shift of finite type is conjugate to an edge shift.

Proposition

A shift space that is conjugate to a shift of finite type is itself of finite type.

Conjugacy invariants

Several quantities are known to be invariant under conjugacy. The entropy of a shift space X is defined by

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Card}(\mathcal{B}_n(X)) .$$

Theorem

If X, Y are conjugate shift spaces, then $h(X) = h(Y)$.

Example

Let X be the golden mean shift. Then a block of length $n+1$ is either a block of length $n-1$ followed by ab or a block of length n followed by a. Thus $s_{n+1} = s_n + s_{n-1}$. As a classical result, $h(X) = \log\lambda$ where $\lambda = (1+\sqrt{5})/2$ is the golden mean.

An element x of a shift space X over the alphabet A has period n if $\sigma_A^n(x) = x$. If $\varphi : X \to Y$ is a conjugacy, then an element x of X has period n if and only if $\varphi(x)$ has period n.

The zeta function of a shift space X is the power series

$$\zeta_X(z) = \exp \sum_{n \geq 0} \frac{p_n}{n} z^n ,$$

where p_n is the number of elements x of X of period n.

It follows from the definition that the sequence $(p_n)_{n \in \mathbb{N}}$ is invariant under conjugacy, and thus the zeta function of a shift space is invariant under conjugacy.

Example

Let $X = A^{\mathbb{Z}}$. Then $\zeta_X(z) = \frac{1}{1-kz}$, where $k = \mathrm{Card}(A)$. Indeed, one has $p_n = k^n$, since an element x of $A^{\mathbb{Z}}$ has period n if and only if it is a biinfinite repetition of a word of length n over A.

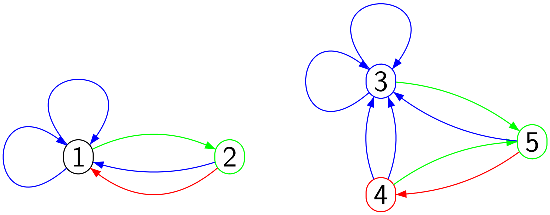
State splitting

Let $G = (Q, \mathcal{E})$ and $H = (R, \mathcal{F})$ be graphs. A pair (h, k) of surjective maps $k : R \to Q$ and $h : \mathcal{F} \to \mathcal{E}$ is called a graph morphism from H onto G if the two diagrams below are commutative.

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{h} & \mathcal{E} \\
\downarrow i & & \downarrow i \\
R & \xrightarrow{k} & Q
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{h} & \mathcal{E} \\
\downarrow t & & \downarrow t \\
R & \xrightarrow{k} & Q
\end{array}
$$

A graph morphism (h, k) from H onto G is an in-merge from H onto G if for each $p, q \in Q$ there is a partition $(\mathcal{E}_p^q(t))_{t \in k^{-1}(q)}$ of the set $\mathcal{E}_p^q$ such that for each $r \in k^{-1}(p)$ and $t \in k^{-1}(q)$, the map h is a bijection from $\mathcal{F}_r^t$ onto $\mathcal{E}_p^q(t)$. If this holds, then G is called an in-merge of H, and H is an in-split of G.

Thus an in-split H is obtained from a graph G as follows : each state $q \in Q$ is split into copies which are the states of H in the set $k^{-1}(q)$. Each of these states t receives a copy of $\mathcal{E}_p^q(t)$ starting in r and ending in t for each r in $k^{-1}(p)$.
Each r in $k^{-1}(p)$ has the same number of edges going out of r and coming in s, for any $s \in R$.
Moreover, for any $p, q \in Q$ and $e \in \mathcal{E}_p^q$, all edges in $h^{-1}(e)$ have the same terminal vertex, namely the state t such that $e \in \mathcal{E}_p^q(t)$.



The following result is well-known. It shows that if H is an in-split of a graph G, then X_G and X_H are conjugate.

Proposition

If (h, k) is an in-merge of a graph H onto a graph G, then h_∞ is a 1-block conjugacy from X_H onto X_G and its inverse is 2-block.

The map h_∞ from X_H to X_G is called an edge in-merging map and its inverse an edge in-splitting map.

Division matrices

A column division matrix over two sets R, Q is an $R \times Q$-matrix D with elements in $\{0,1\}$ such that each column has at least one 1 and each row has exactly one 1. Thus, the columns of such a matrix represent a partition of R into $\mathrm{Card}(Q)$ sets.

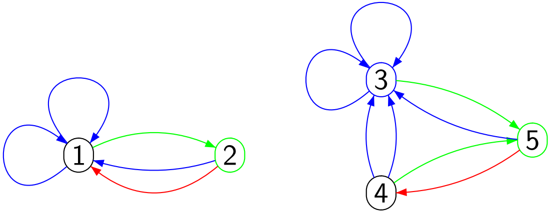
Proposition

Let G and H be essential graphs. The graph H is an in-split of the graph G if and only if there is an $R \times Q$-column division matrix D and a $Q \times R$-matrix E with nonnegative integer entries such that

$$M(G) = ED, \quad M(H) = DE. \tag{1}$$

In the previous example :

$$E = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$



The Decomposition Theorem

Theorem

Every conjugacy from an edge shift onto another is the composition of a sequence of edge splitting maps followed by a sequence of edge merging maps.

The proof relies on the following statement.

Lemma

Let G, H be graphs and let $\varphi : X_G \to X_H$ be a 1-block conjugacy whose inverse has memory $m \geq 1$ and anticipation $n \geq 0$. There are in-splittings $\overline{G}, \overline{H}$ of the graphs G, H and a 1-block conjugacy with memory $m-1$ and anticipation n $\overline{\varphi} : X_{\overline{G}} \to X_{\overline{H}}$ such that the following diagram commutes.

$$\begin{array}{ccc} X_G & \longrightarrow & X_{\overline{G}} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \overline{\varphi}} \\ X_H & \longrightarrow & X_{\overline{H}} \end{array}$$

The horizontal edges in the above diagram represent the edge in-splitting maps from X_G to $X_{\overline{G}}$ and from X_H to $X_{\overline{H}}$ respectively.

The Classification Theorem

Two nonnegative integral square matrices M, N are elementary equivalent if there exists a pair R, S of nonnegative integral matrices such that

$$M = RS\,, \quad N = SR\,.$$

The matrices M and N are strong shift equivalent if there is a sequence $(M_0, M_1, \ldots, M_n)$ of nonnegative integral matrices such that M_i and M_{i+1} are elementary equivalent for $0 \leq i < n$ with $M_0 = M$ and $M_n = N$.

Theorem (Williams, 1973)

Let G and H be two graphs. The edge shifts X_G and X_H are conjugate if and only if the matrices $M(G)$ and $M(H)$ are strong shift equivalent.

Flow equivalence

Set $B = A \cup \omega$. The symbol expansion of a set $W \subset A^+$ relative to $a \in A$ is the image of W by the semigroup morphism $\varphi : A^+ \to B^+$ such that $\varphi(a) = a\omega$ and $\varphi(b) = b$ for all $b \in A \setminus a$.
Let X be a shift space on the alphabet A. The symbol expansion of X relative to a is the least shift space X' on the alphabet $B = A \cup \omega$ which contains the symbol expansion of $\mathcal{B}(X)$.
Two shift spaces X, Y are said to be flow equivalent if there is a sequence $X_0, \ldots, X_n$ of shift spaces such that $X_0 = X$, $Y_n = Y$ and for $0 \leq i \leq n-1$, either X_{i+1} is the image of X_i by a conjugacy, a symbol expansion or a symbol contraction.

Example

Let $A = \{a, b\}$. The symbol expansion of the full shift $A^{\mathbb{Z}}$ relative to b is conjugate to the golden mean shift.

For edge shifts, symbol expansion can be replaced by another operation. Let G be a graph and let p be a vertex of G. The graph expansion of G relative to p is the graph G' obtained by replacing p by an edge from a new vertex p' to p to and replacing all edges coming in p by edges coming in p'. The inverse of a graph expansion is called a graph contraction.

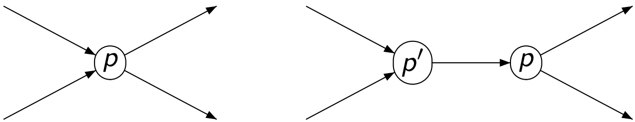


FIG.: Graph expansion

The Bowen-Franks group

The Bowen-Franks group of a square $n \times n$-matrix M with integer elements is the Abelian group

$$BF(M) = \mathbb{Z}^n / \mathbb{Z}^n(I - M).$$

In other terms, $\mathbb{Z}^n(I - M)$ is the Abelian group generated by the rows of the matrix $I - M$. This notion is due to Bowen and Franks, who have shown that it is an invariant for flow equivalence. We say that a graph is trivial if it is reduced to one cycle.

Theorem (Franks, 1984)

Let G, G' be two strongly connected nontrivial graphs and let M, M' be their adjacency matrices. The edge shifts $X_G, X_{G'}$ are flow equivalent if and only if $\det(I - M) = \det(I - M')$ *and the groups $BF(M)$, $BF(M')$ are isomorphic.*

Example

Let

$$M = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}, \quad M' = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}.$$

One has $\det(I - M) = \det(I - M') = -4$. Moreover $BF(M) \sim \mathbb{Z}/4\mathbb{Z}$. Indeed, the rows of the matrix $I - M$ are $\begin{bmatrix} -3 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \end{bmatrix}$. They generate the same group as $\begin{bmatrix} 4 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \end{bmatrix}$. Thus $BF(M) \sim \mathbb{Z}/4\mathbb{Z}$. In the same way, $BF(M') \sim \mathbb{Z}/4\mathbb{Z}$. Thus, the edge shifts X_G and $X_{G'}$ are flow equivalent.
Actually X_G and $X_{G'}$ are both flow equivalent to the full shift on 5 symbols.

Automata and sofic shifts

An automaton is denoted by $\mathcal{A} = (Q, E)$ where Q is the finite set of states and $E \subset Q \times A \times Q$ is the set of edges. The edge (p, a, q) has initial state p, label a and terminal state q. The underlying graph of $\mathcal{A}$ is the same as $\mathcal{A}$ except that the labels of the edges are not used.

An automaton is essential if its underlying graph is essential. The essential part of an automaton is its restriction to the essential part of its underlying graph.

We denote by $X_{\mathcal{A}}$ the set of biinfinite paths in $\mathcal{A}$. It is the edge shift of the underlying graph of $\mathcal{A}$. We denote by $L_{\mathcal{A}}$ the set of labels of biinfinite paths in $\mathcal{A}$. We denote by $\lambda_{\mathcal{A}}$ the 1-block map from $X_{\mathcal{A}}$ into the full shift $A^{\mathbb{Z}}$ which assigns to a path its label. Thus $L_{\mathcal{A}} = \lambda_{\mathcal{A}}(X_{\mathcal{A}})$. If this holds, we say that $L_{\mathcal{A}}$ is the shift space recognized by $\mathcal{A}$.

Proposition

Let $W \subset A^*$ be a recognizable set and let $\mathcal{A} = (Q, I, T)$ be a trim finite automaton recognizing the set $A^* \setminus A^*WA^*$. Then $L_{\mathcal{A}} = X^{(W)}$.

The following proposition states in some sense the converse.

Proposition

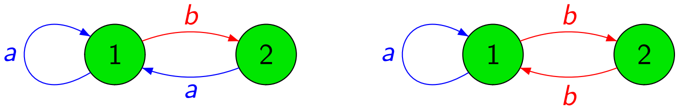
Let X be a sofic shift over A, and let $\mathcal{A} = (Q, I, T)$ be a trim finite automaton recognizing the set $\mathcal{B}(X)$ of blocks of X. Then $L_{\mathcal{A}} = X$.

Proposition

A shift X over A is sofic if and only if there is a finite automaton $\mathcal{A}$ such that $X = L_{\mathcal{A}}$.

The golden mean shift and the even shift

The golden mean shift of is the shift of finite type recognized by the automaton on the left. The even shift is the sofic shift recognized by the automaton on the right.



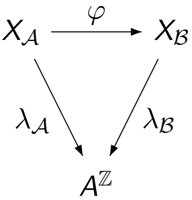
The adjacency matrix of the automaton $\mathcal{A} = (Q, E)$ is the $Q \times Q$-matrix $M(\mathcal{A})$ with elements in $\mathbb{N}\langle A \rangle$ defined by

$$(M(\mathcal{A})_{pq}, a) = \begin{cases} 1 & \text{if } (p, a, q) \in E\,, \\ 0 & \text{otherwise.} \end{cases}$$

We write M for $M(\mathcal{A})$ when the automaton is understood.
A matrix M is called alphabetic over the alphabet A if its elements are homogeneous polynomials of degree 1 over A with nonnegative coefficients. Adjacency matrices are special cases of alphabetic matrices. Indeed, its elements are homogeneous polynomials of degree 1 with coefficients 0 or 1.

Labeled conjugacy

Let $\mathcal{A}$ and $\mathcal{B}$ be two automata on the alphabet A. A labeled conjugacy from $X_\mathcal{A}$ onto $X_\mathcal{B}$ is a conjugacy φ such that $\lambda_\mathcal{A} = \lambda_\mathcal{B}\varphi$, that is such that the following diagram is commutative.



We say that $\mathcal{A}$ and $\mathcal{B}$ are conjugate if there exists a labeled conjugacy from $X_\mathcal{A}$ to $X_\mathcal{B}$.

Labeled split and merge

Let $\mathcal{A} = (Q, E)$ and $\mathcal{B} = (R, F)$ be two automata. Let G, H be the underlying graphs of $\mathcal{A}$ and $\mathcal{B}$ respectively.
A labeled in-merge from $\mathcal{B}$ onto $\mathcal{A}$ is an in-merge (h, k) from H onto G such that for each $f \in F$ the labels of f and $h(f)$ are equal. We say that $\mathcal{B}$ is a labeled in-split of $\mathcal{A}$, or that $\mathcal{A}$ is a labeled in-merge of $\mathcal{B}$.

Proposition

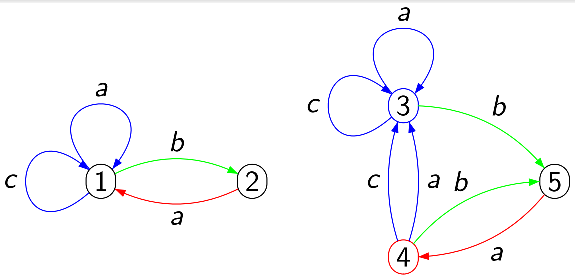
If (h, k) is a labeled in-merge from the automaton $\mathcal{B}$ onto the automaton $\mathcal{A}$, then the map h_∞ is a labeled conjugacy from $X_{\mathcal{B}}$ onto $X_{\mathcal{A}}$.

Proposition

An automaton $\mathcal{B} = (R, F)$ is a labeled in-split of the automaton $\mathcal{A} = (Q, E)$ if and only if there is an $R \times Q$-column division matrix D and an alphabetic $Q \times R$-matrix N such that

$$M(\mathcal{A}) = ND, \quad M(\mathcal{B}) = DN.$$

An in-split



Let $\mathcal{A}$ and $\mathcal{B}$ be the automata represented above. One has $M(\mathcal{A}) = ND$ and $M(\mathcal{B}) = DN$ with

$$N = \begin{bmatrix} a+c & 0 & b \\ 0 & a & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $\mathcal{A} = (Q, E)$ be an automaton. For a pair of integers $m, n \geq 0$, denote by $\mathcal{A}^{[m,n]}$ the following automaton called the (m, n)-th extension of $\mathcal{A}$. The underlying graph of $\mathcal{A}^{[m,n]}$ is the higher edge graph $G^{[k]}$ for $k = m + n + 1$. The label of an edge

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \cdots \xrightarrow{a_m} p_m \xrightarrow{a_{m+1}} p_{m+1} \xrightarrow{a_{m+2}} \cdots \xrightarrow{a_{m+n}} p_{m+n} \xrightarrow{a_{m+n+1}} p_{m+n+1}$$

is the letter a_{m+1}. Observe that $\mathcal{A}^{[0,0]} = \mathcal{A}$. By this construction, each graph $G^{[k]}$ produces k extensions according to the choice of the labeling.

Proposition

For $m \geq 1, n \geq 0$, the automaton $\mathcal{A}^{[m-1,n]}$ is a labeled in-merge of the automaton $\mathcal{A}^{[m,n]}$ and for $m \geq 0, n \geq 1$, the automaton $\mathcal{A}^{[m,n-1]}$ is a labeled out-merge of the automaton $\mathcal{A}^{[m,n]}$.

Decomposition Theorem

The following result is the analogue, for automata, of the Decomposition Theorem.

Theorem

Every conjugacy of automata is a composition of labeled splits and merges.

Classification Theorem for automata

Let M and M' be two alphabetic square matrices over the same alphabet A. We say that M and M' are elementary equivalent if there exists a nonnegative integral matrix D and an alphabetic matrix N such that

$$M = DN\,, \quad M' = ND \quad \text{or vice-versa.}$$

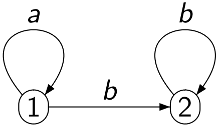
We say that M, M' are strong shift equivalent if there is a sequence $(M_0, M_1, \ldots, M_n)$ such that M_i and M_{i+1} are elementary equivalent for $0 \leq i < n$ with $M_0 = M$ and $M_n = M'$.

Theorem

Two automata are conjugate if and only if their adjacency matrices are strong shift equivalent.

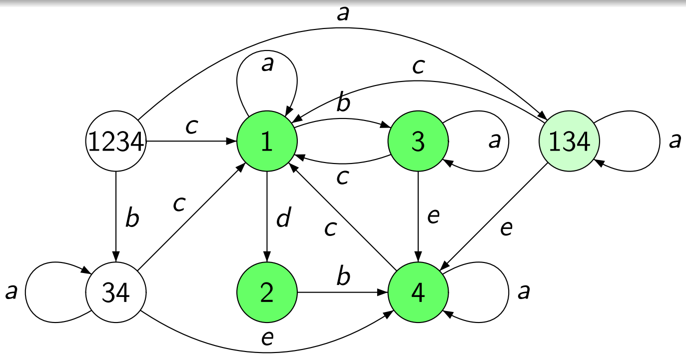
Krieger automata

For $y \in A^{-\mathbb{N}}$, the set of right contexts of y is the set $C_X(y) = \{z \in A^{\mathbb{N}} \mid y \cdot z \in X\}$.
The Krieger automaton of a shift space X is the deterministic automaton whose states are the nonempty sets of the form $C_X(y)$ for $y \in A^{-\mathbb{N}}$, and whose edges are the triples (p, a, q) where $p = C_X(y)$ for some left infinite word, $a \in A$ and $q = C_X(ya)$.



Proposition (Krieger, 1984)

The Krieger automaton of a shift space X is reduced and recognizes X. It is finite if and only if X is sofic.



This automaton is obtained using the subset construction starting from the set $\{1, 2, 3, 4\}$.
The subautomaton with dark shaded states $1, 2, 3, 4$ is strongly connected and recognizes an irreducible sofic shift X. The whole automaton is the minimal automaton of the blocks of X. The Krieger automaton of X is on the five shaded states.

Fischer automata

A shift space $X \subset A^{\mathbb{Z}}$ is called irreducible if for any $u, v \in \mathcal{B}(X)$ there exists a $w \in \mathcal{B}(X)$ such that $uwv \in \mathcal{B}(X)$.

An automaton is said to be strongly connected if its underlying graph is strongly connected. Clearly a shift recognized by a strongly connected automaton is irreducible.

A strongly connected component of an automaton $\mathcal{A}$ is minimal if all successors of vertices of the component are themselves in the component. One may verify that a minimal strongly connected component is the same as a strongly connected subautomaton.

Proposition (Fischer,1975)

The Krieger automaton of an irreducible sofic shift X is synchronized and has a unique minimal strongly connected component.

Let $\mathcal{A} = (Q, E)$ and $B = (R, F)$ be two deterministic automata. A reduction from $\mathcal{A}$ onto $\mathcal{B}$ is a map h from Q onto R such that for any letter $a \in A$, one has $(p, a, q) \in E$ if and only if $(h(p), a, h(q)) \in F$. Thus any labeled in or out-merge is a reduction. However the converse is not true since a reduction is not, in general, a conjugacy.

Proposition

Let X be an irreducible shift space. For any strongly connected deterministic automaton $\mathcal{A}$ recognizing X there is a reduction from $\mathcal{A}$ onto the Fischer automaton of X.

This statement shows that the Fischer automaton of an irreducible shift X is minimal in the sense that it has the minimal number of states among all deterministic strongly connected automata recognizing X.

Ordered semigroups

Recall that a preorder on a set is a relation which is reflexive and transitive. The equivalence associated to a preorder is the equivalence relation defined by $u \equiv v$ if and only if $u \leq v$ and $v \leq u$.

Let S be a semigroup. A preorder on S is said to be stable if $s \leq s'$ implies $us \leq us'$ and $su \leq s'u$ for all $s, s', u \in S$. An ordered semigroup S is a semigroup equipped with a stable preorder. Any semigroup can be considered as an ordered semigroup equipped with the equality order.

A congruence in an ordered semigroup S is the equivalence associated to a stable preorder which is coarser than the preorder of S. The quotient of an ordered semigroup by a congruence is the ordered semigroup formed by the classes of the congruence.

Syntactic semigroup

Set $\Gamma_W(u) = \{(\ell, r) \in A^* \times A^* \mid \ell u r \in W\}$. The preorder on A^+ defined by $u \leq_W v$ if $\Gamma_W(u) \subset \Gamma_W(v)$ is stable and thus defines a congruence of the semigroup A^+ equipped with the equality order called the syntactic congruence. The syntactic semigroup of a set $W \subset A^*$ is the quotient of the semigroup A^+ by the syntactic congruence.

For a deterministic automaton $\mathcal{A} = (Q, E)$, the preorder defined on A^+ by $u \leq_{\mathcal{A}} v$ if $p \cdot u \subset p \cdot v$ for all $p \in Q$ is stable. The quotient of A^+ by the congruence associated to this preorder is the transition semigroup of $\mathcal{A}$.

The syntactic semigroup of a shift space X is by definition the syntactic semigroup of $\mathcal{B}(X)$.

Proposition

Let X be a sofic shift and let S be its syntactic semigroup. The transition semigroup of the Krieger automaton of X is isomorphic to S. Moreover, if X is irreducible, then it is isomorphic to the transition semigroup of its Fischer automaton.

Symbolic conjugacy

We introduce now a new notion of conjugacy between automata called symbolic conjugacy. It extends the notion of labeled conjugacy and captures the fact that the automata may be over different alphabets. The table below summarizes the various notions.

object type	isomorphism	elementary transformat
shift spaces	conjugacy	split/merge
edge shifts	conjugacy	edge split/merge
integer matrices	strong shift equiv.	elementary equivalence
automata (same alph.)	labeled conjugacy	labeled split/merge
automata	symbolic conjugacy	split/merge
alphabetic matrices	symbolic strong shift	elementary symbolic

An automaton is now a pair $\mathcal{A} = (G, \lambda)$ of a graph $G = (Q, \mathcal{E})$ and a map assigning to each edge $e \in \mathcal{E}$ of a label $\lambda(e) \in A$. The adjacency matrix of $\mathcal{A}$ is the $Q \times Q$-matrix $M(\mathcal{A})$ with elements in $\mathbb{N}\langle A\rangle$ defined by

$$(M(\mathcal{A})_{pq}, a) = \mathrm{Card}\{e \in \mathcal{E} \mid \lambda(e) = a\}. \tag{2}$$

We denote by $X_{\mathcal{A}}$ the edge shift on G and by $L_{\mathcal{A}}$ the set of labels of infinite paths in G.

Let $\mathcal{A}, \mathcal{B}$ be two automata. A symbolic conjugacy from $\mathcal{A}$ onto $\mathcal{B}$ is a pair (φ, ψ) of conjugacies $\varphi : X_{\mathcal{A}} \to X_{\mathcal{B}}$ and $\psi : L_{\mathcal{A}} \to L_{\mathcal{B}}$ such that the following diagram is commutative.

$$\begin{array}{ccc} X_{\mathcal{A}} & \xrightarrow{\varphi} & X_{\mathcal{B}} \\ \downarrow{\lambda_{\mathcal{A}}} & & \downarrow{\lambda_{\mathcal{B}}} \\ L_{\mathcal{A}} & \xrightarrow{\psi} & L_{\mathcal{B}} \end{array}$$

Splitting and merging maps

Let A, B be two alphabets and let $f : A \to B$ be a map. We consider the set of words $A' = \{f(a_1)a_2 \mid a_1 a_2 \in \mathcal{B}_2(X)\}$ as a new alphabet. For a shift space X, let $g : \mathcal{B}_2(X) \to A'$ be the 2-block substitution defined by $g(a_1 a_2) = f(a_1)a_2$.
The in-splitting map defined on X and relative to f or to g is the sliding block map $g_\infty^{1,0}$ corresponding to g. It is a conjugacy from X onto its image by $X' = g_\infty^{1,0}(X)$ since its inverse is 1-block. The shift space X', is called the textcolorredin-splitting of X, relative to f or g. The inverse of an in-splitting map is called an in-merging map.

Example

Let $A = B$ and let f be the identity on A. The out-splitting of a shift X relative to f is the second higher block shift of X.

Symmetrically an out-splitting map is defined by the substitution $g(ab) = af(b)$. Its inverse is an out-merging map.

We use the term splitting to mean either a in-splitting or out-splitting. The same convention holds for a merging.

The following result, is a generalization of the Decomposition Theorem to arbitrary shift spaces.

Theorem (Nasu, 1986)

Any conjugacy between shift spaces is a composition of splitting and merging maps.

The proof is similar to the proof of classical decomposition Theorem. It relies on the following lemma.

Lemma

Let $\varphi : X \to Y$ be a 1-block conjugacy whose inverse has memory $m \geq 1$ and anticipation $n \geq 0$. There are in-splitting maps from X, Y to $\tilde{X}, \tilde{Y}$ respectively such that the 1-block conjugacy $\tilde{\varphi}$ making the diagram below commutative has an inverse with memory $m - 1$ and anticipation n.

$$\begin{array}{ccc} X & \longrightarrow & \tilde{X} \\ \downarrow \varphi & & \downarrow \tilde{\varphi} \\ Y & \longrightarrow & \tilde{Y} \end{array}$$

Symbolic strong shift equivalence

Two alphabetic $Q \times Q$-matrices M, M' over the alphabets A and B, are similar if they are equal up to a bijection of A onto B. We write $M \leftrightarrow M'$.

Two alphabetic square matrices M and M' over the alphabets A and B respectively are symbolic elementary equivalent if there exist two alphabetic matrices R, S over the alphabets C and D respectively such that

$$M \leftrightarrow RS, \quad M' \leftrightarrow SR\,.$$

Two matrices M, M' are symbolic strong shift equivalent if there is a sequence $(M_0, M_1, \dots, M_n)$ of alphabetic matrices such that M_i and M_{i+1} are symbolic elementary equivalent for $0 \leq i < n$ with $M_0 = M$ and $M_n = M'$.

Bipartite automata

An automaton $\mathcal{A}$ on the alphabet A is said to be bipartite if its adjacency matrix has the form

$$M(\mathcal{A}) = \begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix}$$

The automata $\mathcal{A}_1$ and $\mathcal{A}_2$ which have $M_1 M_2$ and $M_2 M_1$ respectively as adjacency matrix are called the components of $\mathcal{A}$.

Proposition

Let $\mathcal{A} = (Q, E)$ be a bipartite deterministic essential automaton. Its components $\mathcal{A}_1, \mathcal{A}_2$ are deterministic essential automata which are symbolic conjugate. If moreover $\mathcal{A}$ is strongly connected (resp. reduced, resp. synchronized), then $\mathcal{A}_1, \mathcal{A}_2$ are strongly connected (resp.reduced, resp. synchronized).

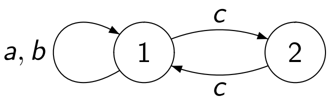
Proposition

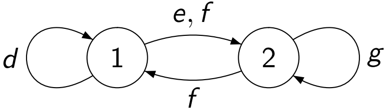
Let $\mathcal{A}, \mathcal{B}$ be two automata such that $M(\mathcal{A})$ and $M(\mathcal{B})$ are symbolic elementary equivalent. Then there is a bipartite automaton $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ such that $M(\mathcal{C}_1), M(\mathcal{C}_2)$ are similar to $M(\mathcal{A}), M(\mathcal{B})$ respectively.

Proposition

If the adjacency matrices of two automata are symbolic strong shift equivalent, the automata are symbolic conjugate.

Example





We have $M(\mathcal{A}) \leftrightarrow RS$ and $M(\mathcal{B}) \leftrightarrow SR$ for

$$R = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}, \quad S = \begin{bmatrix} z & t \\ t & 0 \end{bmatrix}.$$

$$RS = \begin{bmatrix} xz + yt & xt \\ xt & 0 \end{bmatrix}, \quad SR = \begin{bmatrix} zx & zy + tx \\ tx & ty \end{bmatrix}.$$

Bijections between the alphabets.

a	b	c
xz	yt	xt

,

d	e	f	g
zx	zy	tx	ty

.

The following result shows in particular that the Krieger (resp. Fischer) automaton is invariant under conjugacy. The equivalence between conditions (i) and (ii) is a version, for sofic shifts, of the Classification Theorem. The equivalence between conditions (i) and (iii) is due to Krieger (1984).

Theorem (Nasu,1986)

Let X, X' be two sofic shifts (resp. irreducible sofic shifts) and let $\mathcal{A}, \mathcal{A}'$ be their Krieger (resp. Fischer) automata. The following conditions are equivalent.

(i) *X, X' are conjugate.*

(ii) *The adjacency matrices of $\mathcal{A}, \mathcal{A}'$ are symbolic strong shift equivalent.*

(iii) *$\mathcal{A}, \mathcal{A}'$ are symbolic conjugate.*

A classification theorem for automata

The following result, due to is a version for automata of the Classification Theorem. It shows that, in the previous theorem, the equivalence between conditions (ii) and (iii) holds for automata which are not reduced.

Theorem (Hamachi, Nasu, 1988)

Two essential automata are symbolic conjugate if and only if their adjacency matrices are symbolic strong shift equivalent.

The first element of the proof is a version of the Decomposition Theorem for automata.

A decomposition theorem for automata

Let $\mathcal{A}, \mathcal{A}'$ be two automata. An in-split from $\mathcal{A}$ onto $\mathcal{A}'$ is a symbolic conjugacy (φ, ψ) such that $\varphi : X_{\mathcal{A}} \to X_{\mathcal{A}'}$ and $\psi : L_{\mathcal{A}} \to L_{\mathcal{A}'}$ are in-splitting maps. A similar definition holds for out-splits.

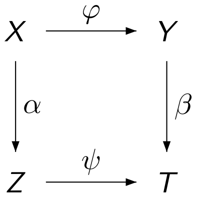
Theorem

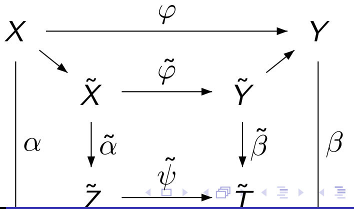
Any symbolic conjugacy between automata is a composition of splits and merges.

The proof relies on the following lemma.

Lemma

Let α, β be 1-block maps and φ, ψ be 1-block conjugacies be as below. If the inverses of φ, ψ have memory $m \geq 1$ and anticipation $n \geq 0$, there exist in-splits $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{T}$ of X, Y, Z, T respectively and 1-block maps $\tilde{\alpha} : \tilde{X} \to \tilde{Z}$, $\tilde{\beta} : \tilde{Y} \to \tilde{T}$ such that the 1-block conjugacies $\tilde{\varphi}, \tilde{\psi}$ below have inverses with memory $m - 1$ and anticipation n.





The second step for the proof of the classification theorem is the following statement.

Proposition

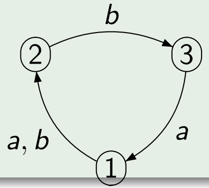
Let $\mathcal{A}, \mathcal{A}'$ be two essential automata. If $\mathcal{A}'$ is an in-split of $\mathcal{A}$, the matrices $M(\mathcal{A})$ and $M(\mathcal{A}')$ are symbolic elementary equivalent.

Local automata

Let $m, n \geq 0$. An automaton $\mathcal{A} = (Q, E)$ is said to be (m, n)-local if whenever $p \xrightarrow{u} q \xrightarrow{v} r$ and $p' \xrightarrow{u} q' \xrightarrow{v} r'$ are two paths with $|u| = m$ and $|v| = n$, then $q = q'$. It is local if it is (m, n)-local for some m, n.

Example

The automaton represented below is $(3, 0)$-local.



We say that an automaton $\mathcal{A} = (Q, E)$ is contained in an automaton $\mathcal{A}' = (Q', E')$ if $Q \subset Q'$ and $E \subset E'$. We note that if $\mathcal{A}$ is contained in $\mathcal{A}'$ and if $\mathcal{A}'$ is local, then $\mathcal{A}$ is local.

Proposition

An essential automaton $\mathcal{A}$ is local if and only if the map $\lambda_{\mathcal{A}} : X_{\mathcal{A}} \to L_{\mathcal{A}}$ is a conjugacy from $X_{\mathcal{A}}$ onto $L_{\mathcal{A}}$.

Proposition

The following conditions are equivalent for a strongly connected finite automaton $\mathcal{A}$.

(i) *$\mathcal{A}$ is local ;*

(ii) *distinct cycles have distinct labels.*

Two cycles in this statement are considered to be distinct if, viewed as paths, they are distinct.

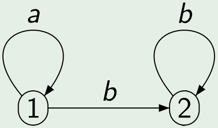
The following result shows the strong connection between shifts of finite type and local automata.

Proposition

A shift space (resp. an irreducible shift space) is of finite type if and only if its Krieger automaton (resp. its Fischer automaton) is local.

Example

Let X be the shift of finite type on the alphabet $A = \{a, b\}$ defined by the forbidden factor ba. The Krieger automaton of X is represented below. It is $(1, 0)$-local.

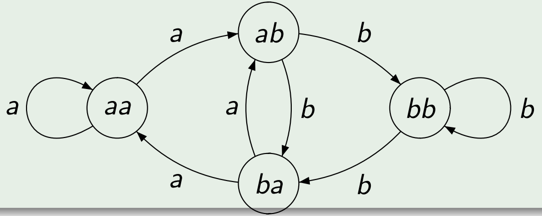


For $m, n \geq 0$, the standard (m, n)-local automaton is the automaton with states the set of words of length $m + n$ and edges the triples $(uv, a, u'v')$ for $u, u' \in A^m$, $a \in A$ and $v, v' \in A^n$ such that for some letters $b, c \in A$, one has $uvc = bu'v'$ and a is the first letter of vc.

The standard $(m, 0)$-local automaton = De Bruijn automaton of order m.

Example

The standard $(1, 1)$-local automaton on the alphabet $\{a, b\}$:



Complete automata

An automaton $\mathcal{A}$ on the alphabet A is called complete if any word on A is the label of some path in $\mathcal{A}$. As an example, the standard (m, n)-local automaton is complete.

Theorem (Béal, Lombardy, P. ,2008)

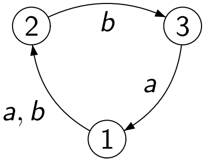
Any local automaton is contained in a complete local automaton.

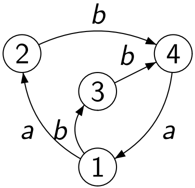
The proof relies on the following version of the masking lemma.

Proposition (Masking lemma)

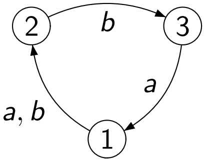
Let $\mathcal{A}$ and $\mathcal{B}$ be two automata and assume that $M(\mathcal{A})$ and $M(\mathcal{B})$ are elementary equivalent. If $\mathcal{B}$ is contained in an automaton $\mathcal{B}'$, then $\mathcal{A}$ is contained in some automaton $\mathcal{A}'$ which is conjugate to $\mathcal{B}'$.

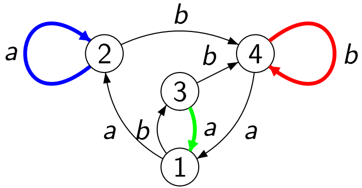
Example



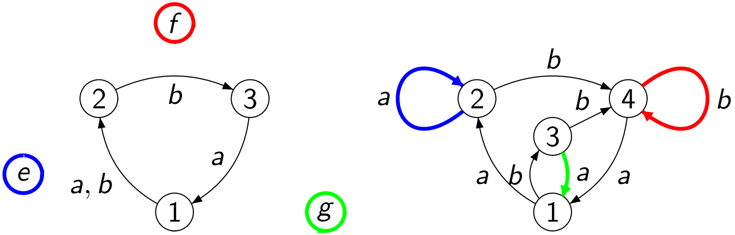


Example

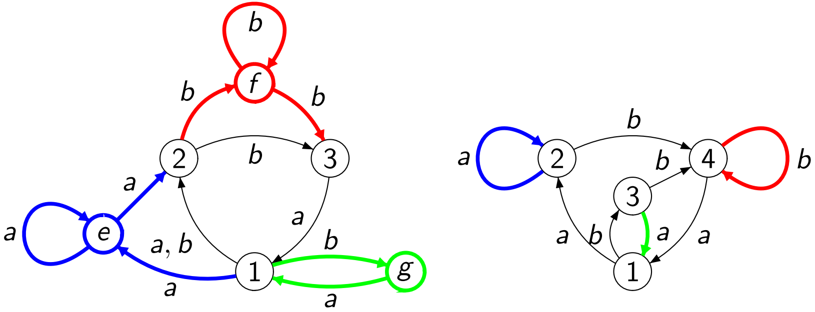




Example



Example



In terms of adjacency matrices, we have $M(\mathcal{A}') = N'D'$, $M(\mathcal{B}') = D'N'$ with

$$N' = \begin{bmatrix} 0 & a & b & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \end{bmatrix}, \qquad D' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Automata with finite delay

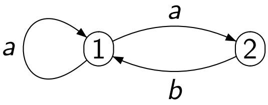
An automaton is said to have right delay $d \geq 0$ if for any pair of paths

$$p \xrightarrow{a} q \xrightarrow{z} r, \quad p \xrightarrow{a} q' \xrightarrow{z} r'$$

with $a \in A$, if $|z| = d$, then $q = q'$. Thus a deterministic automaton has right delay 0. An automaton has finite right delay if it has right delay d for some (finite) integer d. Otherwise, it is said to have infinite right delay.

Example

The automaton represented below has right delay 1.



Proposition

An automaton has finite right delay if and only if it is conjugate to a deterministic automaton.

In the same way the automaton is said to have left delay $d \geq 0$ if for any pair of paths $p \xrightarrow{z} q \xrightarrow{a} r$ and $p' \xrightarrow{z} q' \xrightarrow{a} r$ with $a \in A$, if $|z| = d$, then $q = q'$.

Corollary

If two automata are conjugate, and if one has finite right (left) delay, then the other also has.

Proposition

An essential (m, n)-local automaton has right delay n and left delay m.

Shifts of almost finite type

A shift space is said to have almost finite type if it can be recognized by a strongly connected automaton with both finite left and finite right delay.
An irreducible shift of finite type is also of almost finite type since a local automaton has finite right and left delay.

Example

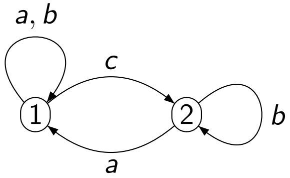
The even shift has almost finite type. Indeed, its Fischer automaton has right and left delay 0.

Proposition (Nasu, 1985)

An irreducible shift space is of almost finite type if and only if its Fischer automaton has finite left delay.

example

The deterministic automaton represented below has infinite left delay. Indeed, there are paths $\cdots 1 \xrightarrow{b} 1 \xrightarrow{a} 1$ and $\cdots 2 \xrightarrow{b} 2 \xrightarrow{a} 1$. Since this automaton cannot be reduced, $X = L_{\mathcal{A}}$ is not of almost finite type.

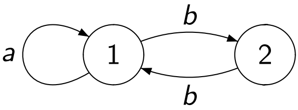


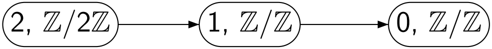
Syntactic graph

We associate with $\mathcal{A}$ a labeled graph $G(\mathcal{A})$ called its syntactic graph. The vertices of $G(\mathcal{A})$ are the regular $\mathcal{D}$-classes of the transition semigroup of $\mathcal{A}$. Each vertex is labeled by the rank of the $\mathcal{D}$-class and its structure group. There is an edge from the vertex associated with a $\mathcal{D}$-class D to the vertex associated to a $\mathcal{D}$-class D' if and only if $D \geq_{\mathcal{J}} D'$.

Example

The automaton $\mathcal{A}$ on the left is the Fischer automaton of the even shift . The semigroup of transitions of $\mathcal{A}$ has 3 regular $\mathcal{D}$-classes of ranks 2 (containing $\varphi_{\mathcal{A}}(b)$), 1 (containing $\varphi_{\mathcal{A}}(a)$), and 0 (containing $\varphi_{\mathcal{A}}(aba)$). Its syntactic graph is represented on the right.





Theorem (Béal, Fiorenzi, P.,2006)

Two symbolic conjugate automata have isomorphic syntactic graphs.

The proof uses the following result.

Proposition

Let $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ be a bipartite automaton. The syntactic graphs of $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic.

Flow equivalent automata

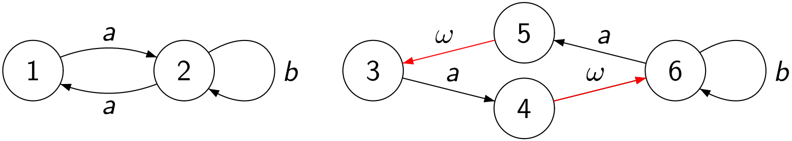
Let $\mathcal{A}$ be an automaton on the alphabet A and let G be its underlying graph. An expansion of $\mathcal{A}$ is a pair (φ, ψ) of a graph expansion of G and a symbol expansion of $L_{\mathcal{A}}$ such that the diagram below is commutative.

$$\begin{array}{ccc} X_{\mathcal{A}} & \xrightarrow{\varphi} & X_{\mathcal{B}} \\ \downarrow \lambda_{\mathcal{A}} & & \downarrow \lambda_{\mathcal{B}} \\ L_{\mathcal{A}} & \xrightarrow{\psi} & L_{\mathcal{B}} \end{array}$$

The inverse of an automaton expansion is called a contraction.

example

Let $\mathcal{A}$ and $\mathcal{B}$ be the automata represented below. The second automaton is an expansion of the first one.



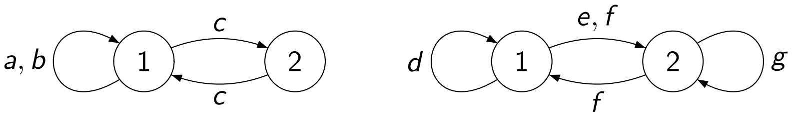
The flow equivalence of automata is the equivalence generated by symbolic conjugacies, expansions and contractions.
The invariance of the syntactic graph under symbolic conjugacy has been generalized to flow equivalence.

Theorem (Costa and Steinberg, 2010)

Two flow equivalent automata have isomorphic syntactic graphs.

example

The syntactic graphs of the automata $\mathcal{A}$, $\mathcal{B}$ below are isomorphic to the syntactic graph of the Fischer automaton $\mathcal{C}$ of the even shift.



Note that the automata $\mathcal{A}, \mathcal{B}$ are not flow equivalent to $\mathcal{C}$. Indeed, the edge shifts $X_{\mathcal{A}}$, $X_{\mathcal{B}}$ on the underlying graphs of the automata $\mathcal{A}$, $\mathcal{B}$ are flow equivalent to the full shift on 3 symbols while the edge shift $X_{\mathcal{C}}$ is flow equivalent to the full shift on 2 symbols. Thus the converse of the theorem is false.

Pseudovarieties

A morphism of ordered semigroups φ from S into T is an order compatible semigroup morphism, that is such that $s \leq s'$ implies $\varphi(s) \leq \varphi(s')$. An ordered subsemigroup of S is a subsemigroup equipped with the restriction of the preorder.

A pseudovariety of finite ordered semigroups is a class of ordered semigroups closed under taking ordered subsemigroups, finite direct products and image under morphisms of ordered semigroups.

Let V be a pseudovariety of ordered semigroups. We say that a semigroup S is locally in V if all the submonoids of S are in V. The class of these semigroups is a pseudovariety of ordered semigroups.

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Theorem (Costa, 2007)

Let V be a pseudovariety of finite ordered semigroups containing the class of commutative ordered monoids such that every element is idempotent and greater than the identity. The class of shifts whose syntactic semigroup is locally in V is invariant under conjugacy.

The following statements give examples of pseudovarieties satisfying the above condition.

Proposition

An irreducible shift space is of finite type if and only if its syntactic semigroup is locally commutative.

An inverse semigroup is a semigroup which can be represented as a semigroup of partial one-to-one maps from a finite set Q into itself. According to Ash's theorem (1987), the variety generated by inverse semigroups is characterized by the property that the idempotents commute.

Theorem (Costa, 2007)

An irreducible shift space is of almost finite type if and only if its syntactic semigroup is locally in the pseudovariety generated by inverse semigroups.