Algorithms for Combinatorial Systems:

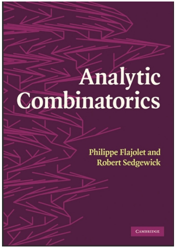
Between Analytic Combinatorics and Species Theory.

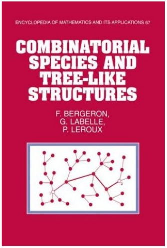
Carine Pivoteau - LIGM - december 2011

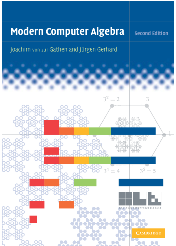
joint work with Bruno Salvy and Michèle Soria

I Introduction

Context







Aim: Algorithms for analytic combinatorics.
Well-defined input provided by species theory.
Efficiency by computer algebra.

Bonus: Unified framework for constructible combinatorial classes.

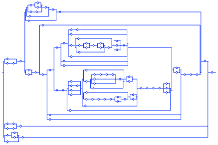
Framework: Constructible Species (Flajolet, Sedgewick)

A small set of species

- $\mathcal{E}$ (or 1), $\mathcal{Z}, \times, +, \textsc{Seq}, \textsc{Set}, \textsc{Cyc}$,
- cardinality constraints that are finite unions of intervals,
- can be used recursively (implicit systems).

Examples:

- Regular languages
- Unambiguous context-free languages
- Trees ($\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{B}^2$, $\mathcal{T} = \mathcal{Z} \times \textsc{Set}(\mathcal{T})$)
- Mappings, ...



Two related problems:

1. Enumeration: number of objects of size n for $n = 0, 1, 2, \dots$.
2. Random sampling: all objects of size n with the same proba.

Two contexts: labelled/unlabelled.

What do we need for Random Sampling?

We have:

- A combinatorial specification
- A recursive algorithm to compute structures
- Implicit equations on generating functions

$$\begin{cases} \mathcal{S} & = \textsc{Seq}(\mathcal{Z} + \mathcal{P}), \\ \mathcal{P} & = \textsc{Set}_{>0}(\mathcal{Z} + \mathcal{S}). \end{cases}$$

$$\begin{cases} S(z) & = 1/(1-(z+P(z)), \\ P(z) & = \exp(z+S(z)) - 1. \end{cases}$$

or

$$P(z) = \exp(\textstyle\sum_{k\geq 0}(z^k + S(z^k))/k) - 1.$$

We need:

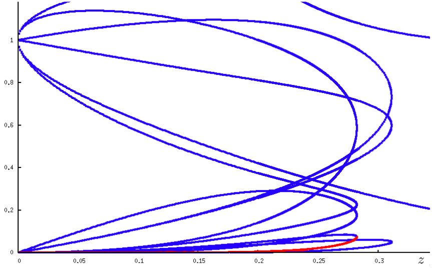
- Recursive Method (Flajolet, Zimmerman, Van Cutsem 94):
  ▷ Enumeration sequences for each class of the specification.
- Boltzmann Method (Duchon *et al.* 04):
  ▷ A Numerical Oracle that gives values of the gfs at a given (well chosen) point.
  ▷ Ultimately, a way to evaluate the singularities of the gfs.

Oracle ≡ solving large systems?

The generating series are given by implicit systems of equations.

We need:

- **only one solution;**
- **the right one;**
- **only numerically.**



In the worst case, these requirements would make no difference.
But these systems inherit structure from combinatorics.

▷ Even more complicated in the **unlabeled** case.

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In the worst case, these requirements would make no difference.
But these systems inherit structure from combinatorics.

$$sys := \left[C0 = x\, C1\, C2\, C3\, (C1 + C3),\ Z = x,\ C1 = x + \frac{x}{1 - C1^2 C3^2},\ C2 = 2x + \frac{x}{(1 - x\, C1^2 C2^2)(1 - C2)},\ C3 = x + \frac{x(3x + x^2 + x^2 C1\, C3)}{1 - C1^2} \right]$$

```
> 
> [seq(subs(t,C0),t=solve(subs(x=0.1,sys)))];
```

[0.0003125169973, 0.0007429960174, 0.01391132169, −0.01391089776, 0.06534819752, 0.1516695772, 0.5931967039, −0.5909308297, −0.002843524044, −0.006587551424, −0.02496904471, 0.02486320262, 1.016379119, 0.2631789750 + 0.1384080116 I, −0.3391146531, 0.2631789750 − 0.1384080116 I, −0.002894993353, −0.006718005666, −0.02619777844, 0.02609673139, −0.07632515320, −0.1768253273, −0.6704728314, 0.6676342030, 1.015911152, 0.2617092228 + 0.1379131433 I, −0.3359391708, 0.2617092228 − 0.1379131433 I]

▷ Even more complicated in the **unlabeled** case.

Idea

By iteration

Numerical Oracle:
numerical iteration that converges
to the unique **relevant** solution

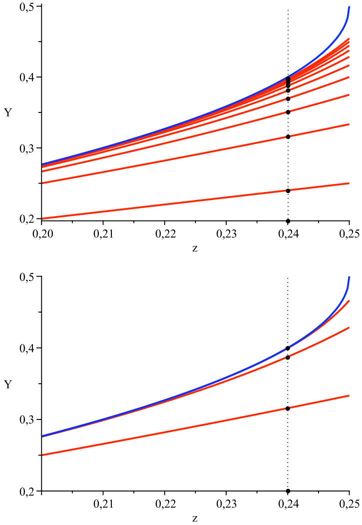
$\Uparrow$

convergence of the iteration
on counting series

$\Uparrow$

convergence of the iteration on
combinatorial functional systems

Fast oracle: Newton iteration



Binary trees: $B(z) = z + B(z)^2$

Results (1/2): Fast Enumeration

Theorem (Enumeration in Quasi-Optimal Complexity)

First N coefficients of gfs of **constructible species** *in*

1. *arithmetic complexity:*
   - $O(N \log N)$ *(both ogf and egf);*
2. *binary complexity:*
   - $O(N^2 \log^2 N \log \log N)$ *(unlabeled, ogf);*
   - $O(N^2 \log^3 N \log \log N)$ *(labeled, egf).*

▷ Quasi-optimal: linear with respect to the size of the output, up to possibly logarithmic factors.

▷ Very simple method, easy to implement.

Bonus : Differential Systems

$$\mathcal{Y}(\mathcal{Z}) = \mathcal{H}(\mathcal{Z}, \mathcal{Y}(\mathcal{Z})) + \int_0^{\mathcal{Z}} \mathcal{G}(\mathcal{T}, \mathcal{Y}(\mathcal{T})) d\mathcal{T}$$

Results (2/2): Oracle

1. The egfs and the ogfs of *constructible* combinatorial classes are **convergent** in the neighborhood of 0;
2. A numerical iteration converging to $\mathbf{Y}(\alpha)$ in the labelled case (α inside the disk);
   $\triangleright$ convergence asymptotically quadratic
3. A numerical iteration converging to the sequence

$$\mathbf{Y}(\alpha), \mathbf{Y}(\alpha^2), \mathbf{Y}(\alpha^3), \ldots$$

   in the unlabeled case (α inside the disk);
   $\triangleright$ hybrid algorithm;

... and also: a criterion to decide if α is inside the disk of cvg of $\mathbf{Y}$.

From Analytic Combinatorics to Species Theory

Analytic Combinatorics:

- Symbolic method to describe recursive combinatorial classes
- A **restricted set of combinatorial constructions**, with a dictionary for gfs.
- Powerful tools for enumeration (singularity analysis,...)
- Automatic methods for random generation (Recursive, Boltzmann)

But no well-founded systems...

Species Theory:

- A more general framework for combinatorial structures
- **Implicit species theorem**
- Labelle's work on the combinatorial derivative
- Combinatorial Newton iteration (Decoste, Labelle, Leroux)
- Combinatorial *differential systems*

But no analytic tools...

II Combinatorics

The key point

Theorem (Implicit Species Theorem (Joyal 81))

Let $\mathcal{H}$ be a vector of multisort species, such that

- $\mathcal{H}(\mathbf{0},\mathbf{0}) = \mathbf{0}$ *and*
- *the Jacobian matrix* $\partial\mathcal{H}/\partial\mathcal{Y}(\mathbf{0},\mathbf{0})$ *is nilpotent.*

The system of equations

$$\mathcal{Y} = \mathcal{H}(\mathcal{Z},\mathcal{Y})$$

admits a vector $\mathcal{S}$ *of species solution such that* $\mathcal{S}(\mathbf{0}) = \mathbf{0}$, *which is unique up to isomorphism.*

- What do the bold symbols mean?
- $\mathcal{H}(\mathbf{0},\mathbf{0}) = \mathbf{0}$?
- What about the other condition?
- Why is it so important?

Short introduction to Species

Definition (Species $\mathcal{F}$)

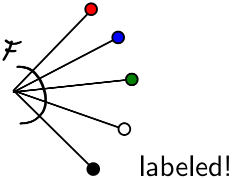
finite set $U \mapsto$ finite set $\mathcal{F}[U]$
bij. $\sigma : U \to V \mapsto$ bij. $\mathcal{F}[\sigma] : \mathcal{F}[U] \to \mathcal{F}[V]$

Examples:

- 0, 1, $\mathcal{Z}$;
- $\textsc{Set}$;
- $\textsc{Seq}$, $\textsc{Cyc}$.

Short introduction to Species

Species $\mathcal{F}$:



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Short introduction to Species

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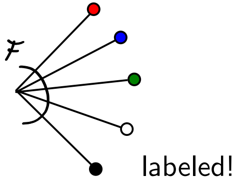
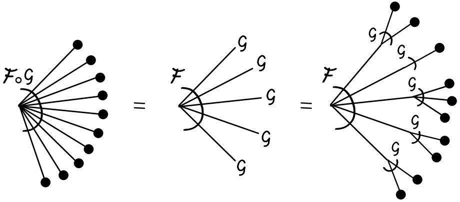


Examples:

- 0, 1, $\mathcal{Z}$;
- SET;
- SEQ, CYC.

- Composition $\mathcal{F} \circ \mathcal{G}$:



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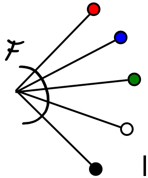

labeled!

Examples:

- 0, 1, $\mathcal{Z}$;
- $\textsc{Set}$;
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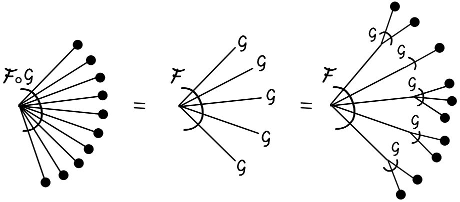
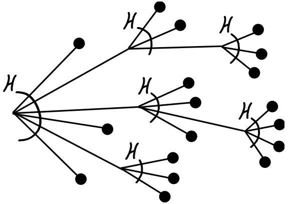
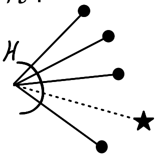


- $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$



Derivative

$\mathcal{H}'$:



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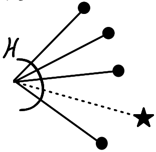
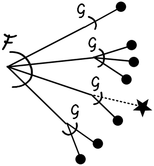


$(\mathcal{F} \circ \mathcal{G})'$:



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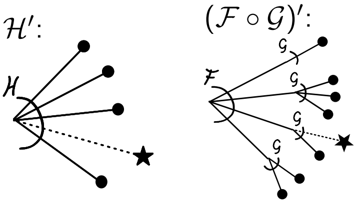

species	derivative
$\mathcal{A}+\mathcal{B}$	$\mathcal{A}'+\mathcal{B}'$
$\mathcal{A}\cdot\mathcal{B}$	$\mathcal{A}'\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{B}'$
$\textsc{Seq}(\mathcal{B})$	$\textsc{Seq}(\mathcal{B})\cdot\mathcal{B}'\cdot\textsc{Seq}(\mathcal{B})$
$\textsc{Cyc}(\mathcal{B})$	$\textsc{Seq}(\mathcal{B})\cdot\mathcal{B}'$
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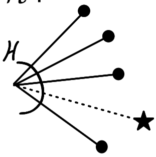

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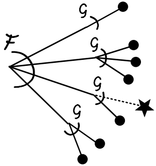

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$\textsc{Set}(\mathcal{B})$	$\textsc{Set}(\mathcal{B})\cdot\mathcal{B}'$

Example:

$$\boldsymbol{\mathcal{H}}(\mathcal{G},\mathcal{S},\mathcal{P}) := (\mathcal{S}+\mathcal{P}, \textsc{Seq}(\mathcal{Z}+\mathcal{P}), \textsc{Set}(\mathcal{Z}+\mathcal{S}))\,.$$

$$\frac{\partial\boldsymbol{\mathcal{H}}}{\partial\boldsymbol{\mathcal{Y}}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & \textsc{Seq}(\mathcal{Z}+\mathcal{P})\cdot 1\cdot\textsc{Seq}(\mathcal{Z}+\mathcal{P}) \\ 0 & \textsc{Set}(\mathcal{Z}+\mathcal{S})\cdot 1 & 0 \end{pmatrix}$$

Back to Joyal's Implicit Species Theorem

Theorem

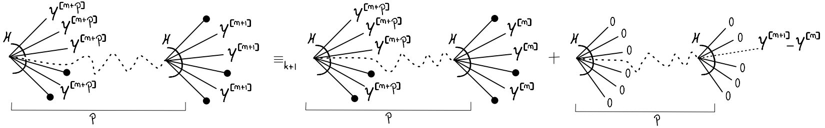
If $\mathcal{H}(\mathbf{0},\mathbf{0}) = \mathbf{0}$ *and* $\partial\mathcal{H}/\partial\mathcal{Y}(\mathbf{0},\mathbf{0})$ *is nilpotent, then* $\mathcal{Y} = \mathcal{H}(\mathcal{Z},\mathcal{Y})$ *has a unique solution, limit of*

$$\mathcal{Y}^{[0]} = \mathbf{0}, \qquad \mathcal{Y}^{[n+1]} = \mathcal{H}(\mathcal{Z},\mathcal{Y}^{[n]}) \quad (n \geq 0).$$

Def. $\mathcal{A} =_k \mathcal{B}$ if they coincide up to size k (*contact* k).

Key Lemma

If $\mathcal{Y}^{[n+1]} =_k \mathcal{Y}^{[n]}$, then $\mathcal{Y}^{[n+p+1]} =_{k+1} \mathcal{Y}^{[n+p]}$, ($p =$ dimension).



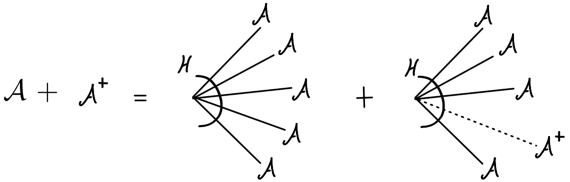
Combinatorial Newton Iteration

Theorem (essentially Labelle)

For any well-founded system $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}})$, if $\boldsymbol{\mathcal{A}}$ has contact k with the solution and $\boldsymbol{\mathcal{A}} \subset \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{A}})$, then

$$\boldsymbol{\mathcal{A}} + \sum_{i \geq 0} \left(\frac{\partial \boldsymbol{\mathcal{H}}}{\partial \boldsymbol{\mathcal{Y}}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{A}}) \right)^i \cdot (\boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{A}}) - \boldsymbol{\mathcal{A}})$$

has contact $2k+1$ with it.



$$\begin{cases} \mathcal{A} =_k \mathcal{Y}, \\ \mathcal{Y} - \mathcal{A}^+ =_k 0, \\ \mathcal{A} + \mathcal{A}^+ =_{2k+1} \mathcal{Y}, \end{cases}$$

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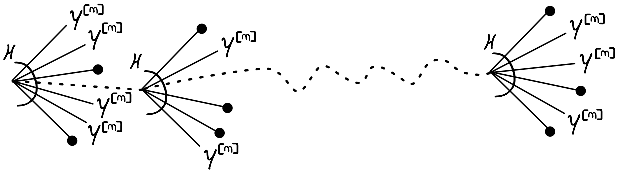

Rmk : Generation by increasing Strahler numbers.

Newton Iteration for Binary Trees

$\mathcal{Y} = 1 + \mathcal{Z} \times \mathcal{Y}^2$

$$\mathcal{Y}_{n+1} = \mathcal{Y}_n + \textsc{Seq}(\mathcal{Z} \times \mathcal{Y}_n \times \star + \mathcal{Z} \times \star \times \mathcal{Y}_n) \times (1 + \mathcal{Z} \times \mathcal{Y}_n^2 - \mathcal{Y}_n).$$

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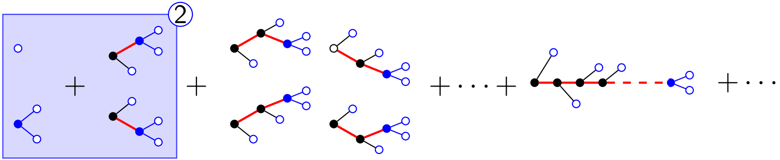
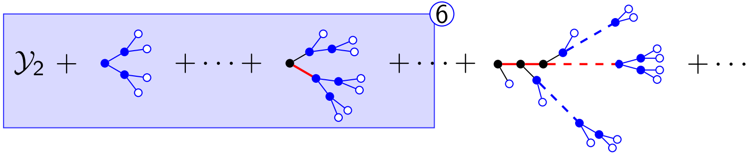
$\mathcal{Y}_0 = 0 \qquad \mathcal{Y}_1 = \circ$

$\mathcal{Y}_2 =$



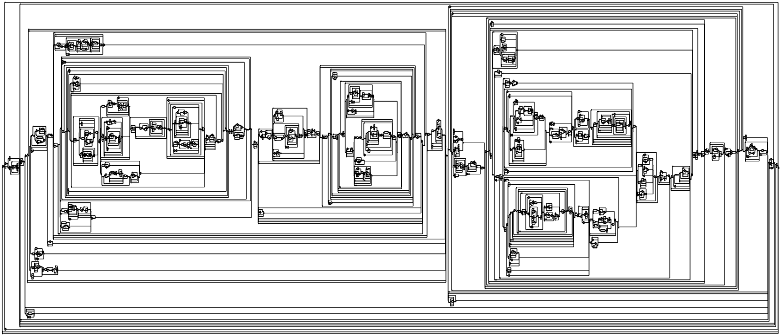
$\mathcal{Y}_3 =$



[Décoste, Labelle, Leroux 1982]

III Algorithms

Example: Series-Parallel Graphs



$$\begin{cases} \mathcal{G} &= \mathcal{S} + \mathcal{P}, \\ \mathcal{S} &= \mathrm{SEQ}(\mathcal{Z} + \mathcal{P}), \\ \mathcal{P} &= \mathrm{SET}_{>0}(\mathcal{Z} + \mathcal{S}). \end{cases} \qquad \frac{\partial \mathcal{H}}{\partial \mathcal{Y}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & \mathrm{SEQ}^2(\mathcal{Z} + \mathcal{P}) \\ 0 & \mathrm{SET}(\mathcal{Z} + \mathcal{S}) & 0 \end{pmatrix}$$

Example: Series-Parallel Graphs (labelled case)

$$\begin{cases} \mathcal{G} & = \mathcal{S} + \mathcal{P}, \\ \mathcal{S} & = \textsc{Seq}(\mathcal{Z} + \mathcal{P}), \\ \mathcal{P} & = \textsc{Set}_{>0}(\mathcal{Z} + \mathcal{S}). \end{cases} \qquad \frac{\partial \boldsymbol{\mathcal{H}}}{\partial \boldsymbol{\mathcal{Y}}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & \textsc{Seq}^2(\mathcal{Z} + \mathcal{P}) \\ 0 & \textsc{Set}(\mathcal{Z} + \mathcal{S}) & 0 \end{pmatrix}$$

$$\begin{cases} G & = S + P, \\ S & = (1 - z - P)^{-1}, \\ P & = \exp(z + S) - 1. \end{cases} \qquad \frac{\partial \mathbf{H}}{\partial \mathbf{Y}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & (1 - z - P)^{-2} \\ 0 & \exp(z + S) & 0 \end{pmatrix}$$

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Newton iteration: $\mathbf{Y}^{[n]} := \begin{pmatrix} G^{[n]} \\ S^{[n]} \\ P^{[n]} \end{pmatrix}$,

$$\mathbf{Y}^{[n+1]} = \mathbf{Y}^{[n]} + \left(\mathrm{Id} - \frac{\partial \mathbf{H}}{\partial \mathbf{Y}} (\mathbf{Y}^{[n]}) \right)^{-1} \cdot \left(\mathbf{H}(\mathbf{Y}^{[n]}) - \mathbf{Y}^{[n]} \right) \bmod z^{2^{n+1}}.$$

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$\Rightarrow$ What about the inverse? And the exponential?

Example: Series-Parallel Graphs (labelled case)

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Newton iteration: $\mathbf{Y}^{[n]} := \begin{pmatrix} G^{[n]} \\ S^{[n]} \\ P^{[n]} \end{pmatrix}$,

$$\begin{cases} U^{[n+1]} &= U^{[n]} + U^{[n]} \cdot \left(\frac{\partial \mathbf{H}}{\partial \mathbf{Y}}(\mathbf{Y}^{[n]}) \cdot U^{[n]} + \mathrm{Id} - U^{[n]} \right) \bmod z^{2^n}, \\ \mathbf{Y}^{[n+1]} &= \mathbf{Y}^{[n]} + U^{[n+1]} \cdot \left(\mathbf{H}(\mathbf{Y}^{[n]}) - \mathbf{Y}^{[n]} \right) \bmod z^{2^{n+1}}. \end{cases}$$

$$\widehat{Y}_2^{[0]}(z) = \mathbf{0} \quad \widehat{Y}_3^{[0]}(z) = \mathbf{0}$$

$$\widehat{Y}_2^{[1]}(z) = \boldsymbol{z^2} + \mathbf{3}\,\boldsymbol{z^3} + \frac{29}{6}\,z^4 + \frac{139}{12}\,z^5 + \frac{3337}{120}\,z^6 + \frac{601}{9}\,z^7 + \frac{808243}{5040}\,z^8 + \cdots$$

$$\widehat{Y}_3^{[1]}(z) = \mathbf{\frac{1}{2}}\,\boldsymbol{z^2} + \mathbf{\frac{7}{6}}\,\boldsymbol{z^3} + \frac{61}{24}\,z^4 + \frac{721}{120}\,z^5 + \frac{10351}{720}\,z^6 + \frac{173867}{5040}\,z^7 + \frac{667957}{8064}\,z^8 + \cdots$$

$$\widehat{Y}_2^{[2]}(z) = \boldsymbol{z^2} + \mathbf{3}\,\boldsymbol{z^3} + \mathbf{\frac{61}{12}}\,\boldsymbol{z^4} + \mathbf{\frac{29}{2}}\,\boldsymbol{z^5} + \mathbf{\frac{15961}{360}}\,\boldsymbol{z^6} + \mathbf{\frac{2841}{20}}\,\boldsymbol{z^7} + \frac{9484021}{20160}\,z^8 + \cdots$$

$$\widehat{Y}_3^{[2]}(z) = \mathbf{\frac{1}{2}}\,\boldsymbol{z^2} + \mathbf{\frac{7}{6}}\,\boldsymbol{z^3} + \mathbf{\frac{73}{24}}\,\boldsymbol{z^4} + \mathbf{\frac{1051}{120}}\,\boldsymbol{z^5} + \mathbf{\frac{19381}{720}}\,\boldsymbol{z^6} + \mathbf{\frac{436087}{5040}}\,\boldsymbol{z^7} + \frac{11584693}{40320}\,z^8 + \cdots$$

$$\widehat{Y}_2^{[3]}(z) = \boldsymbol{z^2} + \mathbf{3}\,\boldsymbol{z^3} + \mathbf{\frac{61}{12}}\,\boldsymbol{z^4} + \mathbf{\frac{29}{2}}\,\boldsymbol{z^5} + \mathbf{\frac{15961}{360}}\,\boldsymbol{z^6} + \cdots + \mathbf{\frac{3665584824929391 01}{108972864000}}\,\boldsymbol{z^{15}} + \cdots$$

$$\widehat{Y}_3^{[3]}(z) = \mathbf{\frac{1}{2}}\,\boldsymbol{z^2} + \mathbf{\frac{7}{6}}\,\boldsymbol{z^3} + \mathbf{\frac{73}{24}}\,\boldsymbol{z^4} + \mathbf{\frac{1051}{120}}\,\boldsymbol{z^5} + \mathbf{\frac{19381}{720}}\,\boldsymbol{z^6} + \mathbf{\frac{386081655546862081}{186810624000}}\,\boldsymbol{z^{15}} + \cdots$$

$$\boldsymbol{y}^{[1]} = (\mathbf{0.1}2305106632099430637 22\ldots,\quad \mathbf{0.0}6462664750711721439535\ldots)$$

$$\boldsymbol{y}^{[2]} = (\mathbf{0.1}6270003893196157969 26\ldots,\quad \mathbf{0.09}2012932660348777349 70\ldots)$$

$$\boldsymbol{y}^{[3]} = (\mathbf{0.17}2433330700324571068 6\ldots,\quad \mathbf{0.09}798441803578338336038\ldots)$$

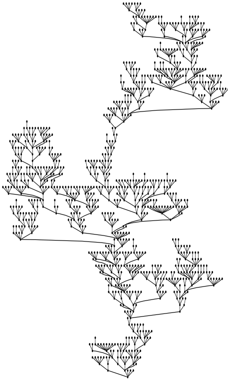
$$\boldsymbol{y}^{[4]} = (\mathbf{0.17304}6096550753535357 4\ldots,\quad \mathbf{0.09836}8315143074664998 45\ldots)$$

$$\boldsymbol{y}^{[5]} = (\mathbf{0.173048639}2973095133433\ldots,\quad \mathbf{0.098369899}17963665326450\ldots)$$

$$\boldsymbol{y}^{[6]} = (\mathbf{0.1730486393408452105}149\ldots,\quad \mathbf{0.0983698992067876912 6}015\ldots)$$

Example: Unlabelled Rooted Trees

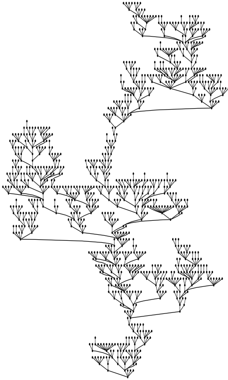
1. Well-founded system: $\mathcal{Y} = \mathcal{Z} \cdot \textsc{Set}(\mathcal{Y}) =: \mathcal{H}(\mathcal{Z}, \mathcal{Y})$;



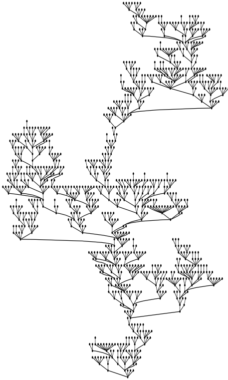
Example: Unlabelled Rooted Trees

1. Well-founded system: $\mathcal{Y} = \mathcal{Z} \cdot \textsc{Set}(\mathcal{Y}) =: \mathcal{H}(\mathcal{Z}, \mathcal{Y})$;
2. Combinatorial Newton iteration:

$$\mathcal{Y}^{[n+1]} = \mathcal{Y}^{[n]} + \textsc{Seq}(\mathcal{H}(\mathcal{Y}^{[n]})) \cdot (\mathcal{H}(\mathcal{Y}^{[n]}) - \mathcal{Y}^{[n]})$$



Example: Unlabelled Rooted Trees



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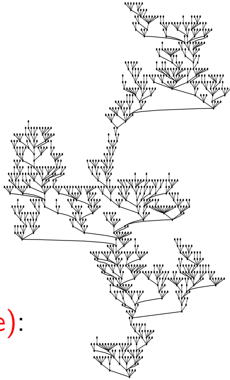
3. OGF equation: $\tilde{Y}(z) = H(z, \tilde{Y}(z))$

$$\tilde{Y}(z) = z \exp(\tilde{Y}(z) + \frac{1}{2}\tilde{Y}(z^2) + \frac{1}{3}\tilde{Y}(z^3) + \cdots)$$

Example: Unlabelled Rooted Trees



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4. Newton for OGF (thanks to the combinatorial derivative):

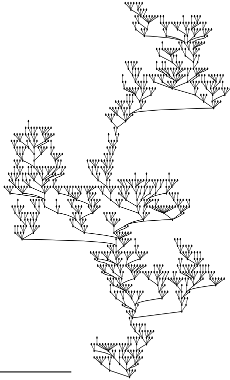
$$\tilde{Y}^{[n+1]} = \tilde{Y}^{[n]} + \frac{H(z, \tilde{Y}^{[n]}) - \tilde{Y}^{[n]}}{1 - H(z, \tilde{Y}^{[n]})}$$

$0,$

$z + z^2 + z^3 + z^4 + \cdots,$

$z + z^2 + 2z^3 + 4z^4 + 9z^5 + 20z^6 + \cdots$

Example: Unlabelled Rooted Trees



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4. Numerical iteration:

n	$\tilde{Y}^{[n]}(0.3)$	$\tilde{Y}^{[n]}(0.3^2)$	$\tilde{Y}^{[n]}(0.3^3)$
0	0	0	0
1	.43021322639	0.99370806338e-1	0.27759817516e-1
2	.54875612912	0.99887132154e-1	0.27770629187e-1
3	.55709557053	0.99887147197e-1	0.27770629189e-1
4	.55713907945	0.99887147198e-1	0.27770629189e-1
5	.55713908064	0.99887147198e-1	0.27770629189e-1

IV Well-founded combinatorial systems

The nature of combinatorial systems...

Joyal's Implicit Species Theorem is **too restrictive**:

- We don't want the condition $\boldsymbol{\mathcal{H}}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.
- To allow equations such as $\mathcal{Y} = 1 + \mathcal{Z}\mathcal{Y}$.
- We want to *characterize precisely* which are the systems that define combinatorial structures ▷ well-founded systems.

Bonus :
A better understanding of the role played by the Jacobian matrix and a better knowledge of the structure of combinatorial systems.

General Implicit Species Theorem

Theorem (General Implicit Species Theorem)

Let $\boldsymbol{\mathcal{H}} = (\mathcal{H}_{1:m})$ *be* **any** *vector of species, such that the system* $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\mathcal{Z}, \boldsymbol{\mathcal{Y}})$ *is* **well-founded**. *Then, this system admits a solution* $\boldsymbol{\mathcal{S}}$ *such that* $\boldsymbol{\mathcal{S}}(\mathbf{0}) = \boldsymbol{\mathcal{H}}^m(\mathbf{0}, \mathbf{0})$, *which is unique up to isomorphism.*

Definition (Well-founded combinatorial system)

$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\mathcal{Z}, \boldsymbol{\mathcal{Y}})$ is said to be *well-founded* when the iteration

$$\boldsymbol{\mathcal{Y}}^{[0]} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\mathcal{Y}}^{[n+1]} = \boldsymbol{\mathcal{H}}(\mathcal{Z}, \boldsymbol{\mathcal{Y}}^{[n]}), \quad n \geq 0 \qquad (\Phi)$$

is well-defined, defines a convergent sequence and the limit $\boldsymbol{\mathcal{S}}$ of this sequence has no zero coordinate.

Algorithmic Characterization

Definition

Companion system of $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}})$:

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{K}}(\mathcal{Z}_1, \boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}}), \quad \text{where} \quad \boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}}) - \boldsymbol{\mathcal{H}}(\mathbf{0}, \mathbf{0}) + \mathcal{Z}_1 \boldsymbol{\mathcal{H}}(\mathbf{0}, \mathbf{0}).$$

Theorem (Characterization of well-founded systems)

Let $\boldsymbol{\mathcal{H}} = (\mathcal{H}_{1:m})$ *be a vector of species. The combinatorial system* $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}})$ *is well-founded* **if and only if**

1. *the companion system* $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{K}}(\mathcal{Z}_1, \boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}})$ *is well-founded at* $\mathbf{0}$
2. *if* $\boldsymbol{\mathcal{S}}_1(\mathcal{Z}_1, \boldsymbol{\mathcal{Z}})$ *is the solution of* $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{K}}(\mathcal{Z}_1, \boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}})$ *with* $\boldsymbol{\mathcal{S}}_1(0, \mathbf{0}) = \mathbf{0}$, *then* $\boldsymbol{\mathcal{S}}_1(\mathcal{Z}_1, \boldsymbol{\mathcal{Z}})$ *is polynomial in* $\mathcal{Z}_1$.

In this case, the limit of (Φ) *is* $\boldsymbol{\mathcal{S}}_1(1, \boldsymbol{\mathcal{Z}})$.

Examples $\mathcal{H}(0,0)=0$

Joyal's conditions:

$\mathcal{Y} = \textsc{Seq}(\mathcal{Z})$ ✓
$\mathcal{H}'(0) = 0$

$\mathcal{Y} = \textsc{Seq}(\mathcal{Z}\,\textsc{Seq}(\mathcal{Z}))$ ✓
$\mathcal{H}'(0) = 0$

$\mathcal{Y} = \textsc{Seq}(\textsc{Seq}(\mathcal{Z}))$ ✗
$\mathcal{H}'(0)$ not defined!

$\mathcal{Y} = \mathcal{Z}\,\mathcal{Y}$ ✓
$\mathcal{H}'(0,0) = 0$

$\mathcal{Y} = \mathcal{Z} + \mathcal{Z}\,\mathcal{Y}$ ✓
$\mathcal{H}'(0,0) = 0$

$\mathcal{Y} = \mathcal{Z} + \mathcal{Y}$ ✗
$\mathcal{H}'(0,0) = 1$

Joyal's conditions:

$\mathcal{Y}=\textsc{Seq}(\mathcal{Z})$ ✓ $\mathcal{H}'(0)=0$

$\mathcal{Y}=\textsc{Seq}(\mathcal{Z}\,\textsc{Seq}(\mathcal{Z}))$ ✓ $\mathcal{H}'(0)=0$

$\mathcal{Y}=\textsc{Seq}(\textsc{Seq}(\mathcal{Z}))$ ✗ $\mathcal{H}'(0)$ not defined!

$\mathcal{Y}=\mathcal{Z}\,\mathcal{Y}$ ✓ $\mathcal{H}'(0,0)=0$

$\mathcal{Y}=\mathcal{Z}+\mathcal{Z}\,\mathcal{Y}$ ✓ $\mathcal{H}'(0,0)=0$

$\mathcal{Y}=\mathcal{Z}+\mathcal{Y}$ ✗ $\mathcal{H}'(0,0)=1$

With our conditions:

$\mathcal{Y}=\mathcal{Z}\,\mathcal{Y}$ ✗ because $\mathcal{Y}=0$.

How to detect 0 coordinates:

Look for 0 in $\boldsymbol{\mathcal{H}}^m(\mathcal{Z},\mathbf{0})$.

Joyal's conditions:

$\mathcal{Y} = \textsc{Seq}(\mathcal{Z})$ ✓
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$\mathcal{H}'(0)$ not defined!

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$\mathcal{Y} = \mathcal{Z} + \mathcal{Y}$ ✗
$\mathcal{H}'(0,0) = 1$

With our conditions:

$\mathcal{Y} = \mathcal{Z}\,\mathcal{Y}$ ✗ because $\mathcal{Y} = 0$.

How to detect 0 coordinates:

Look for 0 in $\boldsymbol{\mathcal{H}}^m(\boldsymbol{\mathcal{Z}},\mathbf{0})$.

Examples:

$$\begin{cases} \mathcal{A} = \mathcal{B} \\ \mathcal{B} = \mathcal{C} \\ \mathcal{A} = \mathcal{Z} \end{cases} \qquad \begin{cases} \mathcal{A} = \mathcal{B} \\ \mathcal{B} = \mathcal{Z} + \mathcal{C} \\ \mathcal{A} = \mathcal{Z}\mathcal{C} \end{cases}$$

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z}\,\mathcal{Y}_2 \\ \mathcal{Y}_2 = \mathcal{Z}\,\mathcal{Y}_1\,\textsc{Seq}(\mathcal{Y}_2) \end{cases} \quad \checkmark \quad \left.\begin{pmatrix} 0 & 0 \\ \mathcal{Z}\,\textsc{Seq}(\mathcal{Y}_2) & \mathcal{Z}\,\mathcal{Y}_1\,\textsc{Seq}(\mathcal{Y}_2)^2 \end{pmatrix}\right|_{0,\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2 \\ \mathcal{Y}_2 = \mathcal{Z}\,\mathcal{Y}_1\,\textsc{Seq}(\mathcal{Y}_2) \end{cases} \quad \checkmark \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2 \\ \mathcal{Y}_2 = \mathcal{Z} + \mathcal{Y}_1\,\textsc{Seq}(\mathcal{Y}_2) \end{cases} \quad ✗ \quad \left.\begin{pmatrix} 0 & 1 \\ \textsc{Seq}(\mathcal{Y}_2) & \mathcal{Y}_1\,\textsc{Seq}(\mathcal{Y}_2)^2 \end{pmatrix}\right|_{0,\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2^2 \\ \mathcal{Y}_2 = \mathcal{Y}_1 \end{cases} \quad \checkmark \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Definition

$\mathcal{F}(\boldsymbol{\mathcal{Z}}_1, \boldsymbol{\mathcal{Z}}_2)$ is polynomial in the sorts $\boldsymbol{\mathcal{Z}}_1$ when, for all $n \geq 0$, the species $\mathcal{F}_{=(.,n)} = \sum_{k\geq 0} \mathcal{F}_{=(k,n)}$ is polynomial.

Examples:

- $\mathrm{SEQ}(\mathcal{Z}_1 + \mathcal{Z}_2)$: not polynomial in $\mathcal{Z}_1$ or $\mathcal{Z}_2$
- $\mathrm{SEQ}(\mathcal{Z}_1 \cdot \mathcal{Z}_2)$: polynomial in $\mathcal{Z}_1$ and $\mathcal{Z}_2$ (but not in $\boldsymbol{\mathcal{Z}}$)
- $\mathcal{Z}_1\mathrm{SEQ}(\mathcal{Z}_2)$: polynomial in $\mathcal{Z}_1$ and not in $\mathcal{Z}_2$.

Well-founded Systems?

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_1\mathcal{Y}_2 \\ \mathcal{Y}_2 = 1 \end{cases} \qquad \begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2\mathcal{Y}_1^2 \\ \mathcal{Y}_2 = 1 \end{cases}$$

Definition

$\mathcal{F}(\boldsymbol{\mathcal{Z}}_1, \boldsymbol{\mathcal{Z}}_2)$ is polynomial in the sorts $\boldsymbol{\mathcal{Z}}_1$ when, for all $n \geq 0$, the species $\mathcal{F}_{=(.,n)} = \sum_{k\geq 0} \mathcal{F}_{=(k,n)}$ is polynomial.

Examples:

- $\mathrm{S}\small{\mathrm{EQ}}(\mathcal{Z}_1 + \mathcal{Z}_2)$: not polynomial in $\mathcal{Z}_1$ or $\mathcal{Z}_2$
- $\mathrm{S}\small{\mathrm{EQ}}(\mathcal{Z}_1 \cdot \mathcal{Z}_2)$: polynomial in $\mathcal{Z}_1$ and $\mathcal{Z}_2$ (but not in $\boldsymbol{\mathcal{Z}}$)
- $\mathcal{Z}_1\mathrm{S}\small{\mathrm{EQ}}(\mathcal{Z}_2)$: polynomial in $\mathcal{Z}_1$ and not in $\mathcal{Z}_2$.

Well-founded Systems?

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_1\mathcal{Y}_2 \\ \mathcal{Y}_2 = \mathcal{Z}_1 \end{cases} \quad ✗ \qquad\qquad \begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2\mathcal{Y}_1^2 \\ \mathcal{Y}_2 = \mathcal{Z}_1 \end{cases} \quad ✓$$

Information given by the Jacobian Matrix

Role of the Jacobian Matrix:

1. Well-founded systems at $\mathbf{0}$: nilpotence of $\partial\boldsymbol{\mathcal{H}}/\partial\boldsymbol{\mathcal{Y}}(\mathbf{0},\mathbf{0})$
2. Implicit polynomial species: nilpotence of $\partial\boldsymbol{\mathcal{H}}/\partial\boldsymbol{\mathcal{Y}}(\boldsymbol{\mathcal{Z}},\boldsymbol{\mathcal{Y}})$
3. Implicit partially polynomial species:
   nilpotence of $\partial\boldsymbol{\mathcal{H}}/\partial\boldsymbol{\mathcal{Y}}(\mathcal{Z}_1,\mathbf{0},\boldsymbol{\mathcal{S}}(\mathcal{Z}_1,\mathbf{0}))$
   (+ conditions on $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{S}}(\mathcal{Z}_1,\mathbf{0})$)
4. Well-founded systems: both 1 and 3.
5. The key for Newton iteration.

But no information on the 0 coordinates.

What's next?

Use this to compute gfs singularities...