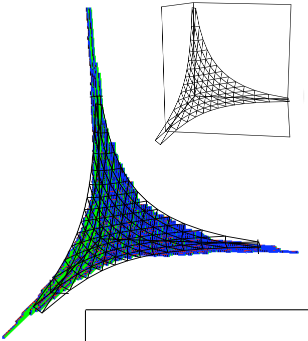
Random sampling of plane partitions

O. Bodini   E. Fusy   C. Pivoteau

Laboratoire d'Informatique de Paris 6 (LIP6)

INRIA Rocquencourt

Introduction



- Well known combinatorial objects (semistandard Young tableaux)
- Statistical physics, mathematics, computer science,...
- Observation of limit properties

Boltzmann sampling technics

$+$

Explicit bijection with a constructible class

$\Downarrow$

Polynomial-time sampler for plane partitions

Plan of the talk

1. **Pak's bijection**
2. **Boltzmann sampler**
3. **Analysis of Complexity**

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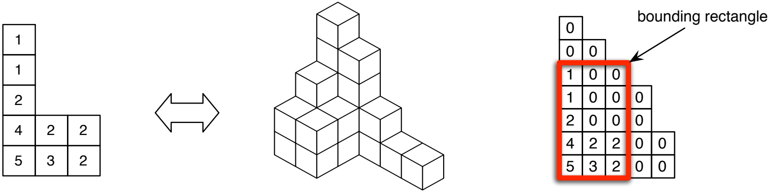
Planes partitions

- **Plane partitions** of n ($\mathcal{P}$)

  $\longrightarrow$ matrix of integers that are decreasing in both dimensions.



- **Bounding rectangle** of a plane partition

  $\longrightarrow$ smallest rectangle such that all the cells outside are empty.

- **$(p \times q)$-boxed plane partitions** ($\mathcal{P}_{p,q}$)

  $\longrightarrow$ the size of the bounding rectangle is at most $(p \times q)$.

Counting plane partitions

Generating function of plane partitions (Mac Mahon, 1912) :

$$P(z) = \prod_{r \geq 1} (1 - z^r)^{-r}$$

- simple expression for the generating function
- combinatorial isomorphism with a constructible class (symbolic methods)

$$\mathcal{P} \simeq \mathcal{M} \quad \text{and} \quad \mathcal{P}_{p,q} \simeq \mathcal{M}_{p,q}$$

- non-trivial bijection
- for long, non constructive proof...

An isomorphic class

- $\mathcal{M} = \mathbf{MSet}(\mathbb{N}^2)$ ~ multiset of pairs of integers

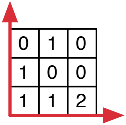
  → example : $\{(0,0),(1,0),(2,0),(2,0),(0,1),(1,2)\}$, size $= 15$

  → size of (i,j) : $(i+j+1)$

- $\mathcal{M}_{p,q} = \mathbf{MSet}(\mathbb{N}_{<p} \times \mathbb{N}_{<q})$

- **Diagram** of an element $\in \mathcal{M}$ or $\mathcal{M}_{p,q}$

{ (0,0), (1,0), (2,0), (2,0), (0,1), (1,2) } ⇒



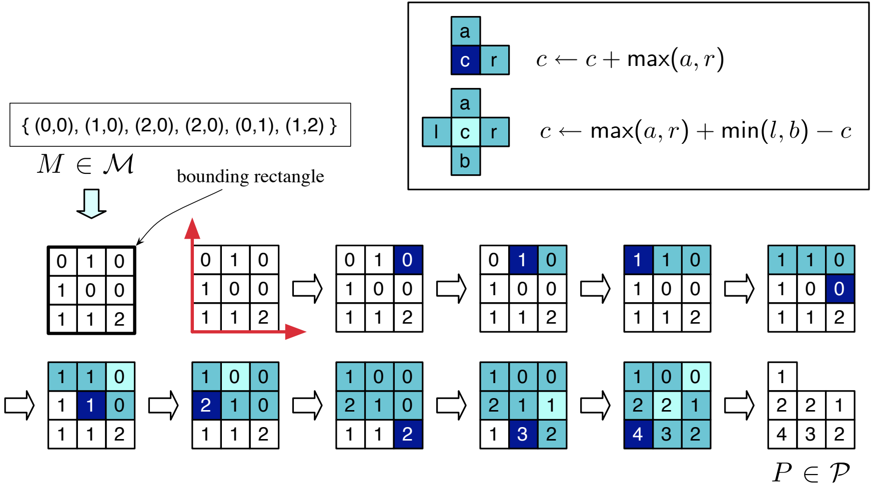
size $= 15$

$$|D| = \sum_{i,j} m_{i,j}(i+j+1)$$

→ sum of the hook lengths weighted by the values of the cells.

Pak's bijection

Pak's bijection (2001)



Application of Pak's algorithm on an example.

Boltzmann sampler

Boltzmann sampling (2003)

Boltzmann sampling basic principles :

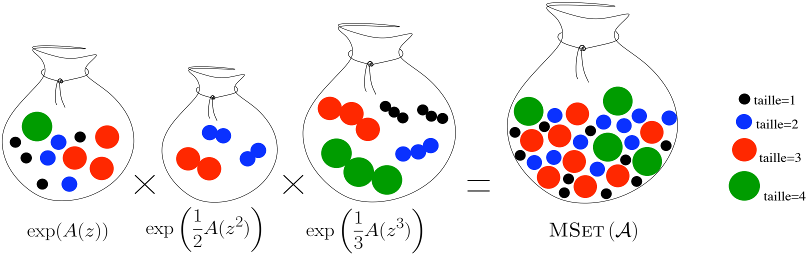
- for any constructible class
- an object γ is drawn proportionally to $x^{|\gamma|}$
- same probability for all objects of the same size
- size distribution spread over the whole combinatorial class

construction	sampler
$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$\Gamma C(x) := \text{Bern } \frac{A(x)}{C(x)} \longrightarrow \Gamma A(x) \mid \Gamma B(x)$
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$\Gamma C(x) := (\ \Gamma A(x); \Gamma B(x)\ )$
$\mathcal{C} = \textsc{Seq}(\mathcal{A})$	$\Gamma C(x) := (\ \text{Geom } A(x) \Longrightarrow \Gamma A(x)\ )$

Generating multisets

$$\mathcal{C} = \text{MSet}(\mathcal{A}) \cong \prod_{\gamma \in \mathcal{A}} \text{Seq}(\gamma) \quad \Rightarrow \quad C(z) = \prod_{n \geq 1} (1 - z^n)^{-C_n}$$

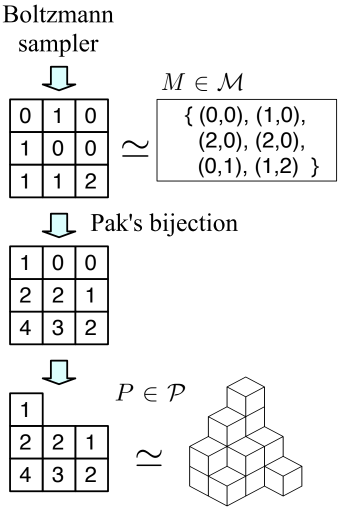
$$C(z) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} A(z^k) \right) = \prod_{k=1}^{\infty} \exp\left(\frac{1}{k} A(z^k) \right)$$



A sampler for plane partitions

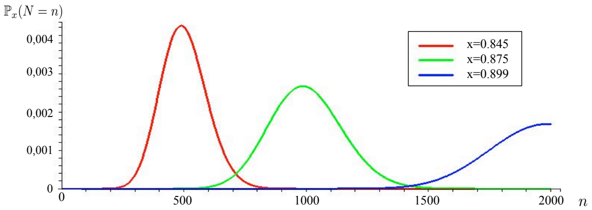
- Boltzmann sampler for
  - $\mathcal{M} = \textsc{MSet}(\mathbb{N}^2)$.
  - $\mathcal{M}_{p,q} = \textsc{MSet}(\mathbb{N}_{<p} \times \mathbb{N}_{<q})$
    $= \prod_{\substack{0 \le i < p \\ 0 \le j < q}} \textsc{Seq}(\mathcal{Z} \times i \times j)$

  Output : a *diagram* D.
- Pak's algorithm transforms D into a plane partition.
- Size of the ouptut plane partition = size of the original diagram.



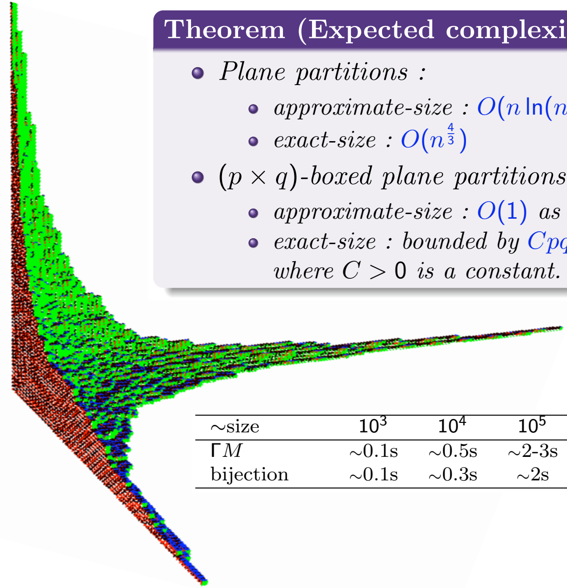
How to choose x such that the size of the output partition is n ?

- Probability of drawing a partition of size n : $\dfrac{P_n x^n}{P(x)}$
- Expectation of the size of a partition : $x\dfrac{P'(x)}{P(x)}$



$\Rightarrow$ **Targetted sampler + rejection.**

Result



Theorem (Expected complexity)

- *Plane partitions :*
  - *approximate-size :* $O(n \ln(n)^3)$
  - *exact-size :* $O(n^{\frac{4}{3}})$
- $(p \times q)$*-boxed plane partitions (for fixed* p, q*) :*
  - *approximate-size :* $O(1)$ *as* $n \to \infty$
  - *exact-size : bounded by* $Cpq.n$*,*
    where $C > 0$ *is a constant.*

~size	10^3	10^4	10^5	10^6	10^7
ΓM	~0.1s	~0.5s	~2-3s	~10s	~60s
bijection	~0.1s	~0.3s	~2s	~20-30s	~8-9min

Analysis of Complexity

General scheme

Generation of a plane partition of size n (resp. $\sim n$), with a targetted sampler, i.e., with a parameter tuned such that $\mathbb{E}(N_x) = n$.

mean cost

=

cost of one call to ΓM $\times$ expected number of calls

+

cost of pak's algorithm

1. cost of one call to ΓM : $O(n^{\frac{2}{3}})$
2. expected number of calls to the sampler :
   - approximate size sampler : $O(1)$
   - exact size sampler : $O(n^{\frac{2}{3}})$
3. expected complexity of Pak's algorithm applied to a diagram of size n : $O(n \ln(n)^3)$

Conclusion

Theorem (Expected complexity)

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Efficiency of Boltzmann samplers combined with results of bijective combinatorics yields a polynomial time sampler for planes partitions.