

Boltzmann oracle for combinatorial systems

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joint work with Bruno Salvy and Michèle Soria

Examples of combinatorial specifications

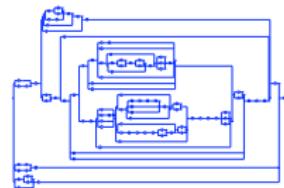
- plane binary trees:

$$\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{B}^2$$



- series-parallel graphs:

$$\begin{aligned}\mathcal{S} &= \text{SEQ}_{\geq 2}(\mathcal{P} + \mathcal{Z}) \\ \mathcal{P} &= \text{MSET}_{\geq 2}(\mathcal{S} + \mathcal{Z})\end{aligned}$$



- an algebraic language:

$$\mathcal{C}_0 = \mathcal{Z} \mathcal{C}_1 \mathcal{C}_2 \mathcal{C}_3 (\mathcal{C}_1 + \mathcal{C}_2)$$

$$\mathcal{C}_1 = \mathcal{Z} + \mathcal{Z} \text{SEQ}(\mathcal{C}_1^2 \mathcal{C}_3^2)$$

$$\mathcal{C}_2 = \mathcal{Z} + \mathcal{Z}^2 \text{SEQ}(\mathcal{Z} \mathcal{C}_2^2 \text{SEQ}(\mathcal{Z})) \text{SEQ}(\mathcal{C}_2)$$

$$\mathcal{C}_3 = \mathcal{Z} + \mathcal{Z}(3\mathcal{Z} + \mathcal{Z}^2 + \mathcal{Z}^2 \mathcal{C}_1 \mathcal{C}_3) \text{SEQ}(\mathcal{C}_1^2)$$

Examples of combinatorial specifications

- plane binary trees:

$$\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{B}^2$$

- series-parallel graphs:

$$\begin{aligned}\mathcal{S} &= \text{SEQ}_{\geq 2}(\mathcal{P} + \mathcal{Z}) \\ \mathcal{P} &= \text{MSET}_{\geq 2}(\mathcal{S} + \mathcal{Z})\end{aligned}$$

- an algebraic language:

$$C_0(z) = zC_1(z)C_2(z)C_3(z)(C_1(z) + C_2(z))$$

$$C_1(z) = z + z/(1 - C_1(z)^2C_3(z)^2)$$

$$C_2(z) = z + z^2/((1 - zC_2(z)^2/(1 - z))(1 - C_2(z)))$$

$$C_3(z) = z + z(3z + z^2 + z^2C_1(z)C_3(z))/(1 - C_1^2(z))$$

Examples of combinatorial specifications

- plane binary trees:

$$\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{B}^2$$

- series-parallel graphs:

$$\begin{aligned}\mathcal{S} &= \text{SEQ}_{\geq 2}(\mathcal{P} + \mathcal{Z}) \\ \mathcal{P} &= \text{MSET}_{\geq 2}(\mathcal{S} + \mathcal{Z})\end{aligned}$$

- an algebraic language: with $z = 0.27$

$$C_0 = C_0(0.27) = 0.27C_1C_2C_3(C_1 + C_2)$$

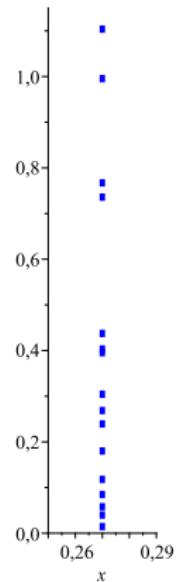
$$C_1 = C_1(0.27) = 0.27 + 0.27/(1 - C_1^2C_3^2)$$

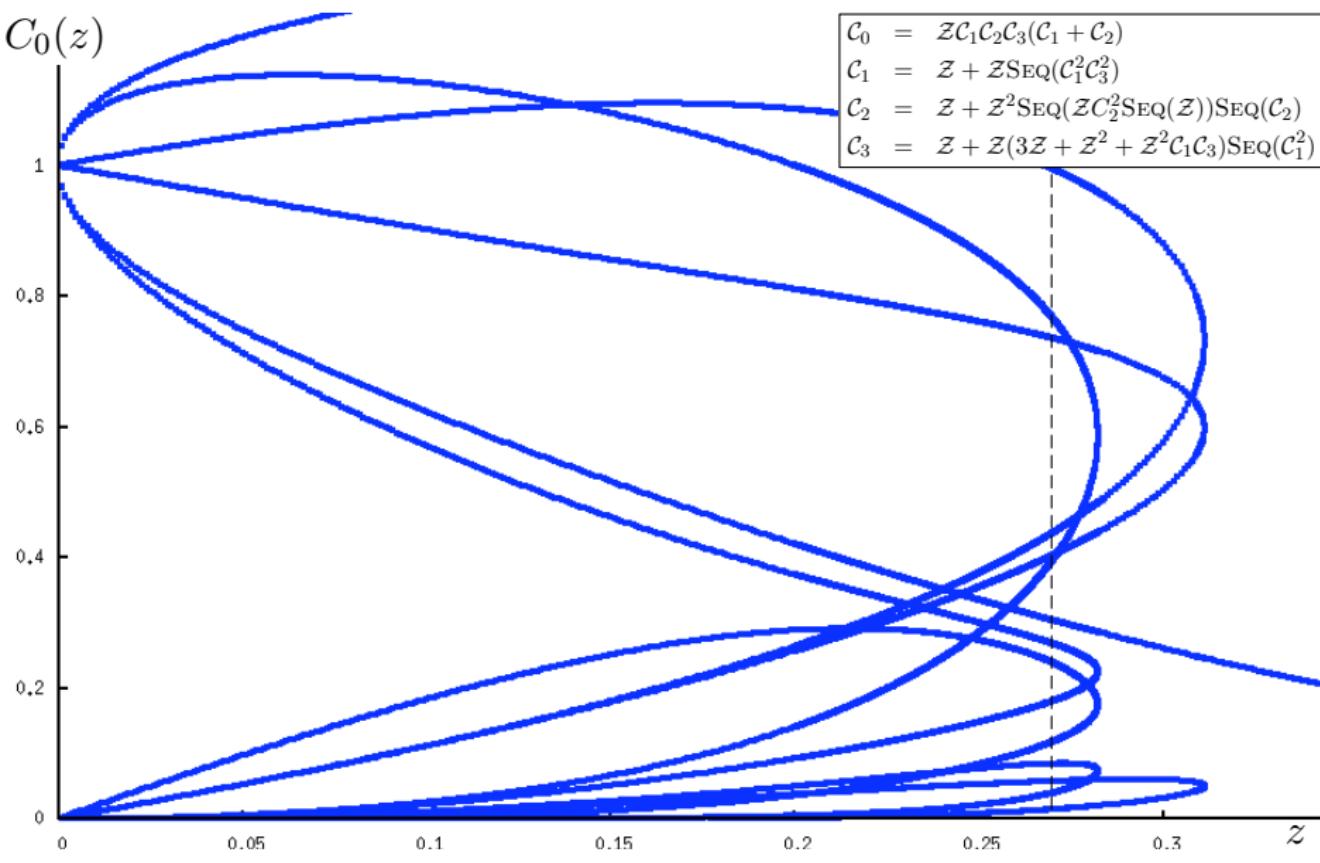
$$C_2 = C_2(0.27) = 0.27 + 0.0729/((1 - 0.2754211138C_2^2)(1 - C_2))$$

$$C_3 = C_3(0.27) = 0.27 + 0.27(0.8829 + 0.0729C_1C_3)/(1 - C_1^2)$$

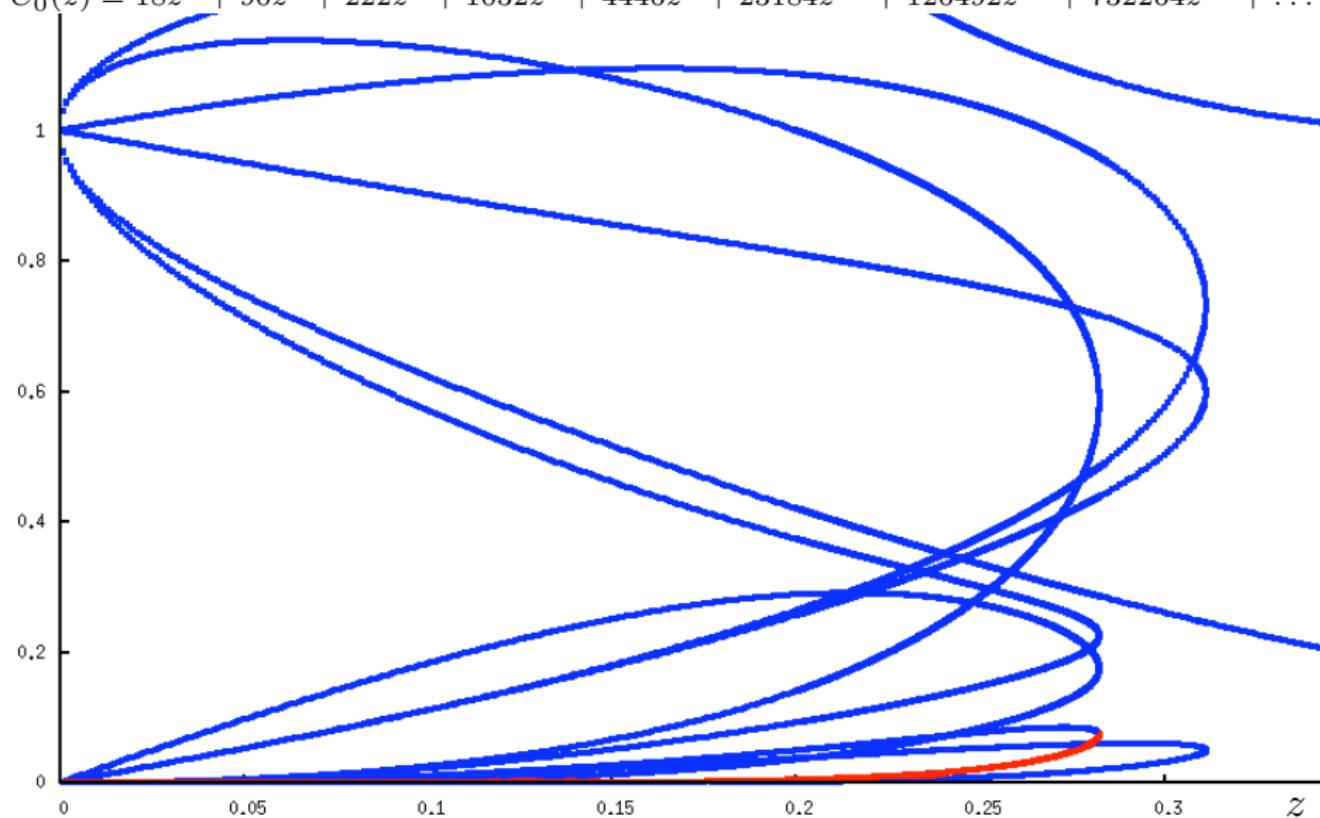
$$\begin{aligned}
 \text{sys} := & \left[C0 = x C3 C1 C2 (C1 + C2), C3 = x + \frac{x(3x + x^2 + C1 C3 x^2)}{1 - C1^2}, C2 = x \right. \\
 & \left. + \frac{x^2}{\left(1 - \frac{x C2^2}{1-x}\right)(1-C2)}, C1 = x + \frac{x}{1 - C1^2 C3^2} \right] \\
 = & > [\text{seq}(\text{subs}(t, C0), t=\text{solve}(\text{subs}(x=0.27, \text{sys})))];
 \end{aligned}$$

[0.03981177932, 0.08483583327, -0.1330967042, 0.4375304567, 0.05798674454,
 -0.08236513610, 0.4033042476, 0.1178894123, 0.2393370428, -0.3529667643,
 0.9953320976, 0.01464289097, -0.06660288853, 0.7359229322, 0.3953535311,
 0.7672908692, -1.061211795, 2.444202876, -0.4183928219, 0.3043493358,
 1.103791701, 0.1801240933, 0.2686181060, -0.2029314456, -0.9333902580,
 -1.261357198, 1.347299205, -2.621596974]

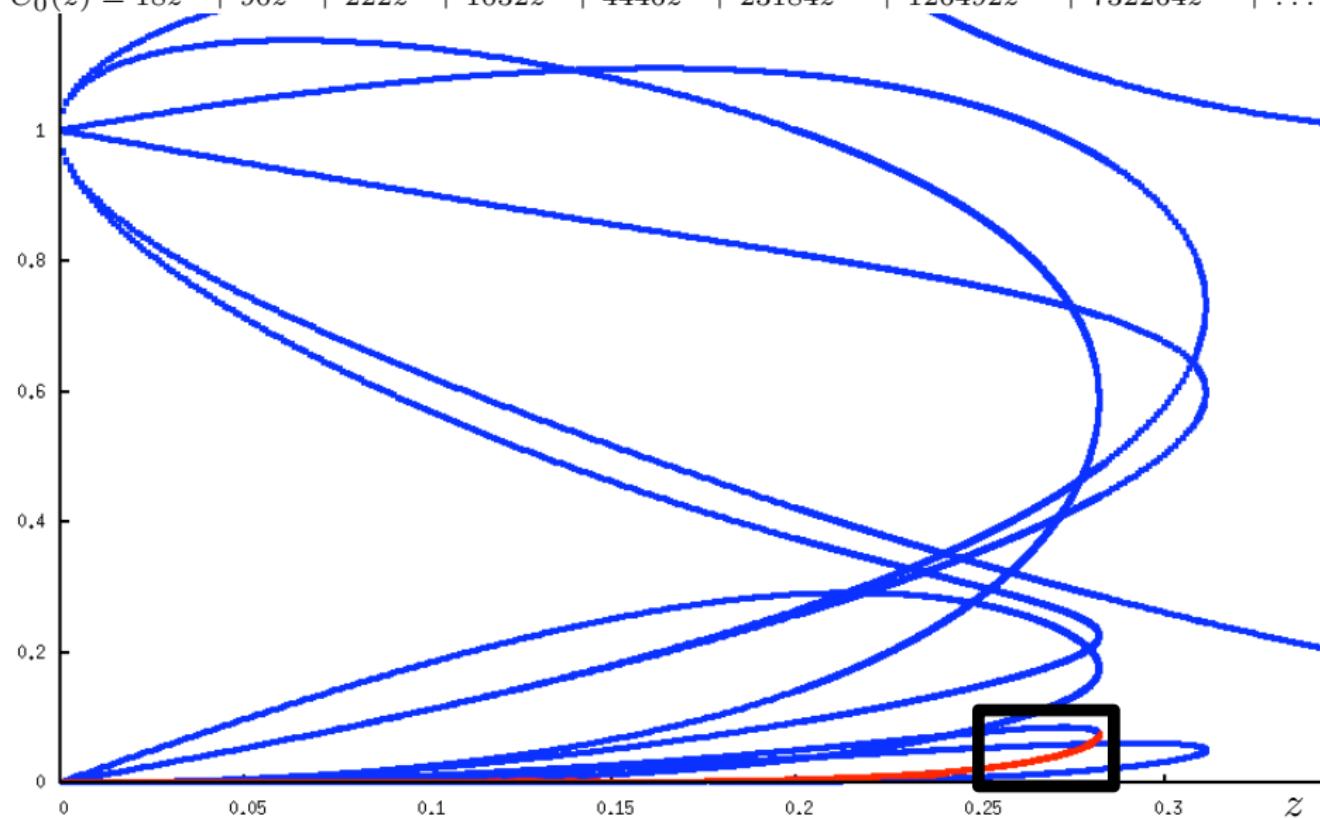
$$\begin{aligned} \text{sys} := & \left[C0 = x C3 C1 C2 (C1 + C2), C3 = x + \frac{x(3x + x^2 + C1 C3 x^2)}{1 - C1^2}, C2 = x \right. \\ & \left. + \frac{x^2}{\left(1 - \frac{x C2^2}{1-x}\right)(1-C2)}, C1 = x + \frac{x}{1 - C1^2 C3^2} \right] \\ = & > [\text{seq}(\text{subs}(t, C0), t=\text{solve}(\text{subs}(x=0.27, \text{sys})))]; \\ & [0.03981177932, 0.08483583327, -0.1330967042, 0.4375304567, 0.05798674454, \\ & -0.08236513610, 0.4033042476, 0.1178894123, 0.2393370428, -0.3529667643, \\ & 0.9953320976, 0.01464289097, -0.06660288853, 0.7359229322, 0.3953535311, \\ & 0.7672908692, -1.061211795, 2.444202876, -0.4183928219, 0.3043493358, \\ & 1.103791701, 0.1801240933, 0.2686181060, -0.2029314456, -0.9333902580, \\ & -1.261357198, 1.347299205, -2.621596974] \end{aligned}$$




$$C_0(z) = 18z^5 + 90z^6 + 222z^7 + 1032z^8 + 4446z^9 + 23184z^{10} + 126492z^{11} + 732264z^{12} + \dots$$



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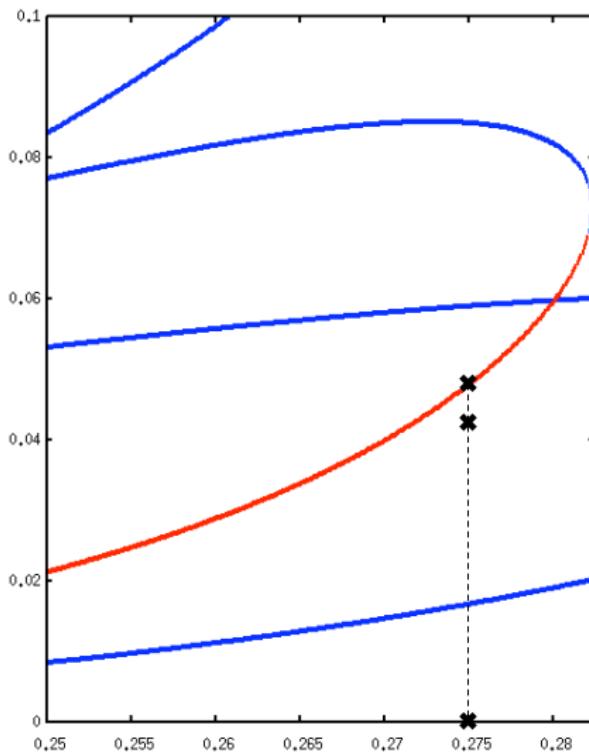


Proposition

Newton iteration converges quadratically to the solution.

Approach

numerical iteration
converges to the solution
↑
counting series
evaluation
↑
combinatorial systems
of equations



Combinatorial structures

Combinatorial specification

A *combinatorial specification* for an m -tuple $\mathcal{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$ of classes is a system

$$\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y}) \equiv \begin{cases} \mathcal{Y}_1 = \mathcal{H}_1(\mathcal{Z}, \mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m), \\ \mathcal{Y}_2 = \mathcal{H}_2(\mathcal{Z}, \mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m), \\ \vdots \\ \mathcal{Y}_m = \mathcal{H}_m(\mathcal{Z}, \mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m), \end{cases}$$

each \mathcal{H}_i denoting a term built from the \mathcal{Y}_i 's and the initial class \mathcal{Z} (atomic) using the classical *constructions*.

construction	notation	generating function
Disjoint union	$\mathcal{A} + \mathcal{B}$	$A(z) + B(z)$
Cartesian product	$\mathcal{A} \times \mathcal{B}$	$A(z) \cdot B(z)$
Sequence	$\text{SEQ}(\mathcal{B})$	$1/(1 - B(z))$
Cycle	$\text{CYC}(\mathcal{B})$	$\log(1/(1 - B(z)))$
Set	$\text{SET}(\mathcal{B})$	$\exp(B(z))$

Binary plane trees:

$$T = \mathcal{H}(\mathcal{Z}, T)$$

$$\mathcal{H}(\mathcal{Z}, T) = \mathcal{Z} + \mathcal{Z}T^2$$

Combinatorial derivative

$\partial\mathcal{H}/\partial T$: *derivative* of $\mathcal{H}(\mathcal{Z}, \mathcal{Y}_1, \dots, \mathcal{Y}_m)$ with respect to T .

- If \mathcal{H} is either \mathcal{E} or \mathcal{Z} then $\partial\mathcal{H}/\partial T = \emptyset$,
- if $\mathcal{H} = T$ then $\partial\mathcal{H}/\partial T = \mathcal{E}$,
- else

construction	notation	derivative
Disjoint union	$\mathcal{H} = \mathcal{A} + \mathcal{B}$	$\mathcal{A} + \mathcal{A}' + \mathcal{B}'$
Cartesian product	$\mathcal{H} = \mathcal{A} \times \mathcal{B}$	$\mathcal{A}' \times \mathcal{B} + \mathcal{A} \times \mathcal{B}'$
Sequence	$\mathcal{H} = \text{SEQ}(\mathcal{B})$	$\text{SEQ}(\mathcal{B}) \times \mathcal{B}' \times \text{SEQ}(\mathcal{B})$
Cycle	$\mathcal{H} = \text{CYC}(\mathcal{B})$	$\text{SEQ}(\mathcal{B}) \times \mathcal{B}'$
Set	$\mathcal{H} = \text{SET}(\mathcal{B})$	$\text{SET}(\mathcal{B}) \times \mathcal{B}'$

$$\begin{aligned}\mathcal{H}(\mathcal{Z}, T) &= \mathcal{Z} + T^2, & \mathcal{H}_T^\bullet(\mathcal{Z}, T) &= \mathcal{Z}T^\bullet T + \mathcal{Z}TT^\bullet, \\ \partial\mathcal{H}/\partial T(\mathcal{Z}, T) &= 2\mathcal{Z}T\end{aligned}$$

$\partial\mathcal{H}/\partial Y$: the *Jacobian matrix* of $\mathcal{H}(\mathcal{Z}, \mathbf{Y})$ with respect to \mathbf{Y} ,
its entries are the partial derivatives $\partial\mathcal{H}_i(\mathcal{Z}, \mathbf{Y})/\partial Y_j$.

Nilpotent matrix : one of its powers is \emptyset (all its entries are \emptyset).

Well founded specification

Definition

The combinatorial specification $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ is *well founded* if and only if, for all $n \geq 0$, it derives only *finitely many* structures of size n .

Proposition

A combinatorial specification $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ such that $\mathcal{H}(\emptyset, \emptyset) = \emptyset$ is *well founded* if and only if the Jacobian matrix $\partial\mathcal{H}/\partial\mathcal{Y}(\emptyset, \emptyset)$ is *nilpotent*.

★ conditions of Joyal's Implicit Species theorem.

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2 \\ \mathcal{Y}_2 = \mathcal{Z} \text{ } \mathcal{Y}_1 \text{ SEQ}(\mathcal{Y}_2) \end{cases} \quad \checkmark \quad \mathcal{Y} = \mathcal{Z} + \mathcal{Y}^2 \quad \checkmark \quad \mathcal{Y} = \mathcal{Z} + \mathcal{Y} \quad \times$$

$$\frac{\partial\mathcal{H}}{\partial\mathcal{Y}}(\emptyset, \emptyset) = \begin{pmatrix} \emptyset & \mathcal{E} \\ \emptyset & \emptyset \end{pmatrix} \quad \mathcal{H}'(\emptyset, \emptyset) = \emptyset \quad \mathcal{H}'(\emptyset, \emptyset) = \mathcal{E}$$

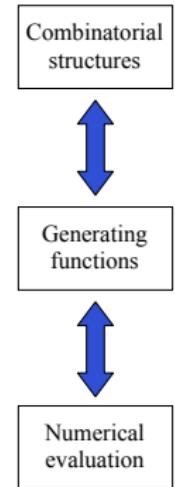
Iteration and Oracle

Result

Theorem (Transfer of Convergence)

Let $\mathcal{Y} = \mathcal{F}(\mathcal{Z}, \mathcal{Y})$ be well founded and $\mathcal{F}(\emptyset, \emptyset) = \emptyset$.

- ➊ The iteration $\mathbf{y}_{n+1} = \mathcal{F}(\mathcal{Z}, \mathbf{y}_n)$, with $\mathbf{y}_0 = \emptyset$, converges to the combinatorial class \mathcal{Y} , solution of $\mathcal{Y} = \mathcal{F}(\mathcal{Z}, \mathcal{Y})$.
- ➋ The iteration $\mathbf{Y}_{n+1}(z) = \mathbf{F}(z, \mathbf{Y}_n(z))$, with $\mathbf{Y}_0(z) = 0$, converges to the generating series $\mathbf{Y}(z)$ of the class \mathcal{Y} .
- ➌ If \mathcal{F} is an analytic specification, then \mathbf{Y} has positive radius of convergence ρ and for all α such that $|\alpha| < \rho$, the iteration $\mathbf{y}_{n+1} = \mathbf{F}(\alpha, \mathbf{y}_n)$, with $\mathbf{y}_0 = 0$, converges to $\mathbf{Y}(\alpha)$.



$\mathcal{F}(\mathcal{Z}, \mathcal{Y})$ is called *analytic* when the generating series $\mathbf{F}(z, \mathbf{Y})$ is analytic in (z, \mathbf{Y}) in the neighborhood of $(0, \mathbf{0})$, with nonnegative coefficients.

Example: binary trees

$$\mathcal{Y} = \mathcal{Z} + \mathcal{Z} \times \mathcal{Y}^2$$

$$\mathcal{Y} = F(\mathcal{Z}, \mathcal{Y})$$

$$\mathcal{Y}_{k+1} = \mathcal{Z} + \mathcal{Z} \times \mathcal{Y}_k^2$$

Iteration: $\mathcal{Y}_{k+1} = F(\mathcal{Z}, \mathcal{Y}_k)$

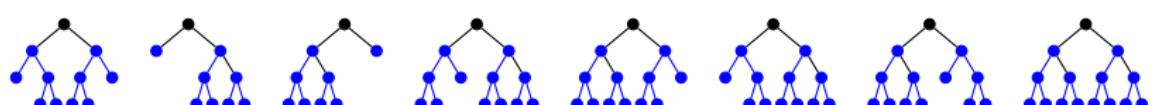
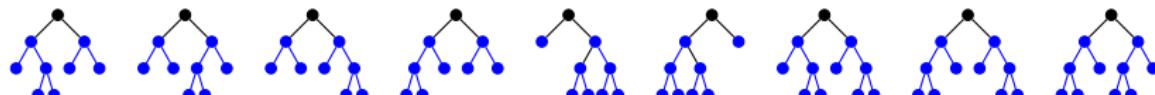
$$\mathcal{Y}_0 = \emptyset$$

$$\mathcal{Y}_1 = \boxed{\bullet}$$

$$\mathcal{Y}_2 = \boxed{\bullet + \text{tree}}$$

$$\mathcal{Y}_3 = \boxed{\bullet + \text{trees}} + \boxed{\text{trees}} + \boxed{\text{trees}}$$

$$\mathcal{Y}_4 = \boxed{\bullet + \text{trees}} + \boxed{\text{trees}} + \boxed{\text{trees}} + \boxed{\text{trees}}$$



Generating functions

$$Y(z) = z + zY^2(z)$$

$$Y(z) = F(z, Y(z))$$

$$Y_{k+1}(z) = z + zY_k(z)^2$$

Iteration: $Y_{k+1}(z) = F(z, Y_k(z))$

$$Y_0(z) = \mathbf{0}$$

$$Y_1(z) = z$$

$$Y_2(z) = z + z^3$$

$$Y_3(z) = z + z^3 + 2z^5 + z^7$$

$$Y_4(z) = z + z^3 + 2z^5 + 5z^7 + 6z^9 + 6z^{11} + 4z^{13} + z^{15}$$

$$Y_5(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + 26z^{11} + 44z^{13} + 69z^{15}$$

$$Y(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + 42z^{11} + 132z^{13} + 429z^{15} + \dots$$

convergence for structures \Rightarrow convergence for series

Numerical iteration

$$Y(\alpha) = \alpha + \alpha Y^2(\alpha)$$

$$Y(\alpha) = F(\alpha, Y(\alpha))$$

$$Y_{k+1} = 0.2 + 0.2Y_k^2$$

Iteration: $Y_{k+1}(\alpha) = F(\alpha, Y_k(\alpha))$

$$Y_0 = Y_0(0.2) = 0$$

$$Y_1 = Y_1(0.2) = 0.2$$

$$Y_2 = Y_2(0.2) = 0.208$$

$$Y_3 = Y_3(0.2) = 0.2086528$$

$$Y_4 = Y_4(0.2) = 0.208707198189568\dots$$

$$Y_5 = Y_5(0.2) = 0.2087117389152279\dots$$

$$Y = Y(0.2) = 0.2087121525220799\dots$$

- for any value of $\alpha < \rho$
- numerical iteration \Leftrightarrow evaluation of the series at α .

Proof idea

① Combinatorial convergence

Part of Joyal's proof of his Implicit Species theorem

② Power series

Symbolic method. $\mathbf{Y}_n(z)$ are the generating series of the classes $\mathbf{y}_n : \text{val}(\mathbf{Y}_n(z) - \mathbf{Y}(z)) \rightarrow \infty$.

③ Numerical values ($|\alpha| < \rho$)

\mathbf{Y} is analytic at 0 (implicit function theorem).

- $\mathbf{Y}_n(\alpha)$ converges to $\mathbf{Y}(\alpha)$.
- $\mathbf{y}_n = \mathbf{Y}_n(\alpha)$.

Newton iteration

Result

Newton iteration to find roots of $f(x)$: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Theorem (Newton Oracle)

Let $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ be a *well-founded analytic specification* with $\mathcal{H}(\emptyset, \emptyset) = \emptyset$. Let α be *inside the disk of convergence* of the generating series $\mathbf{Y}(z)$ of \mathcal{Y} . Then the iteration

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \left(\mathbf{I} - \frac{\partial \mathbf{H}}{\partial \mathbf{Y}}(\alpha, \mathbf{y}_n) \right)^{-1} \cdot (\mathbf{H}(\alpha, \mathbf{y}_n) - \mathbf{y}_n), \quad \mathbf{y}_0 = \mathbf{0}$$

converges to $\mathbf{Y}(\alpha)$.

★ bonus: optimized Newton iteration.

Numerical convergence

$$Y(\alpha) = \alpha + \alpha Y^2(\alpha)$$

$$Y(\alpha) = H(\alpha, Y(\alpha))$$

Iteration: $Y_{k+1} = Y_k + (I - \frac{\partial H}{\partial Y}(\alpha, Y_k))^{-1}(H(\alpha, Y_k) - Y_k)$

for $\alpha = 0.48$, $Y_{k+1} = Y_k + \frac{1}{1-0.96Y_k}(0.48 + 0.48Y_k^2 - Y_k)$

$$Y_0 = \mathbf{0}$$

$$Y_1 = \mathbf{0.48}$$

$$Y_2 = \mathbf{0.68510385756676557863501483679525\dots}$$

$$Y_3 = \mathbf{0.74409429531735785069315411659589\dots}$$

$$Y_4 = \mathbf{0.74994139686483588184679391778624\dots}$$

$$Y_5 = \mathbf{0.74999999411376420459420080511077\dots}$$

$$Y_4 = \mathbf{0.74999999999999994060382090306852\dots}$$

$$Y_5 = \mathbf{0.74999999999999999999999999999999999997\dots}$$

asymptotically quadratic convergence

Newton on series

$$Y = z + zY^2(z)$$

$$Y(z) = H(z, Y)$$

Iteration: $Y_{k+1} = Y_k + (I - \frac{\partial H}{\partial Y}(z, Y_k))^{-1}(H(z, Y_k) - Y_k)$

$$Y_{k+1} = Y_k + \frac{1}{1-2zY_k}(z + zY_k^2 - Y_k)$$

$$Y_0 = \mathbf{0}$$

$$Y_1 = \mathbf{z}$$

$$Y_2 = \mathbf{z} + \mathbf{z}^3 + 2\mathbf{z}^5 + 4z^7 + 8z^9 + 16z^{11} + 32z^{13} + 64z^{15} + \dots$$

$$Y_3 = \mathbf{z} + \mathbf{z}^3 + 2\mathbf{z}^5 + 5\mathbf{z}^7 + 14\mathbf{z}^9 + 42\mathbf{z}^{11} + 132\mathbf{z}^{13} + 428z^{15} + \dots$$

$$Y_4 = \mathbf{z} + \mathbf{z}^3 + 2\mathbf{z}^5 + 5\mathbf{z}^7 + \dots + 2674440\mathbf{z}^{29} + 9694844z^{31} + \dots$$

Combinatorial Newton for a single equation

(Bergeron, Décoste, Labelle, Leroux)

Iteration: $\mathcal{Y}_{k+1} = \mathcal{Y}_k + \text{SEQ}(\mathcal{H}'(\mathcal{Z}, \mathcal{Y}_k))(\mathcal{H}(\mathcal{Z}, \mathcal{Y}_k) - \mathcal{Y}_k)$

$$\mathcal{Y}_{k+1} = \mathcal{Y}_k + \text{SEQ}(2\mathcal{Z}\mathcal{Y}_k)(\mathcal{Z} + \mathcal{Z}\mathcal{Y}_k^2 - \mathcal{Y}_k)$$

$$\mathcal{Y}_0 = \emptyset \quad \mathcal{Y}_1 = \bullet \quad \mathcal{H}(\mathcal{Y}_1) - \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Z}\mathcal{Y}_1^2 - \mathcal{Y}_1 = \times \bullet$$

$$\mathcal{Y}_2 = \boxed{\bullet + \bullet} + \bullet + \bullet + \dots + \bullet + \dots$$

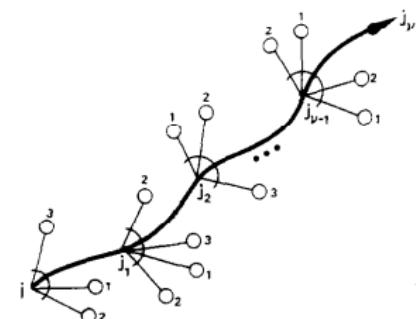
$$\mathcal{Y}_3 = \mathcal{Y}_2 + \boxed{\bullet + \dots + \bullet} + \dots + \bullet + \dots + \bullet + \dots$$

Newton for a system

$$\text{Iteration: } \mathcal{Y}_{k+1} = \mathcal{Y}_k + \boxed{\left(I - \frac{\partial \mathcal{H}}{\partial \mathcal{Y}}(\mathcal{Z}, \mathcal{Y}_k) \right)^{-1} (\mathcal{H}(\mathcal{Z}, \mathcal{Y}_k) - \mathcal{Y}_k)}$$

- one single equation → sequence
- many equations → combinatorial bloomings (Labelle 85)

$$\left(\mathbf{I} - \frac{\partial \mathcal{H}}{\partial \mathcal{Y}}(\mathcal{Z}, \mathcal{Y}) \right)^{-1} = \sum_{k \geq 0} \left(\frac{\partial \mathcal{H}}{\partial \mathcal{Y}}(\mathcal{Z}, \mathcal{Y}) \right)^k$$



Convergence for vectors is componentwise convergence.

Proof idea

Proposition (Combinatorial Newton iteration)

Let $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ be a *well-founded* specification with $\mathcal{H}(\emptyset, \emptyset) = \emptyset$. Let $\mathcal{N}_{\mathcal{H}}$ be the operator defined by

$$\mathcal{N}_{\mathcal{H}}(\mathcal{Z}, \mathcal{Y}) = \mathcal{Y} + \left(\mathbf{I} - \frac{\partial \mathcal{H}}{\partial \mathcal{Y}}(\mathcal{Z}, \mathcal{Y}) \right)^{-1} \times (\mathcal{H}(\mathcal{Z}, \mathcal{Y}) - \mathcal{Y}).$$

Then the sequence defined by $\mathcal{Y}_0 = \emptyset$, $\mathcal{Y}_{n+1} = \mathcal{N}_{\mathcal{H}}(\mathcal{Z}, \mathcal{Y}_n)$ ($n \geq 0$) converges to \mathcal{Y} . Moreover this convergence is quadratic.

1. The iteration is well defined
2. The iteration is not ambiguous
3. Convergence and its quadratic behaviour

Proof of Newton Oracle Theorem

$\mathcal{Y} = \mathcal{N}_{\mathcal{H}}(\mathcal{Z}, \mathcal{Y})$ is well founded and analytic + Transfer theorem

Optimized Newton

★ Optimization:

- combinatorial Newton to compute $\mathbf{U} = (I - \frac{\partial H}{\partial Y}(\mathcal{Z}, Y_k))^{-1}$

$$\mathbf{U}_{k+1} = \mathbf{U}_k + \mathbf{U}_k \mathbf{T}_{k+1}$$

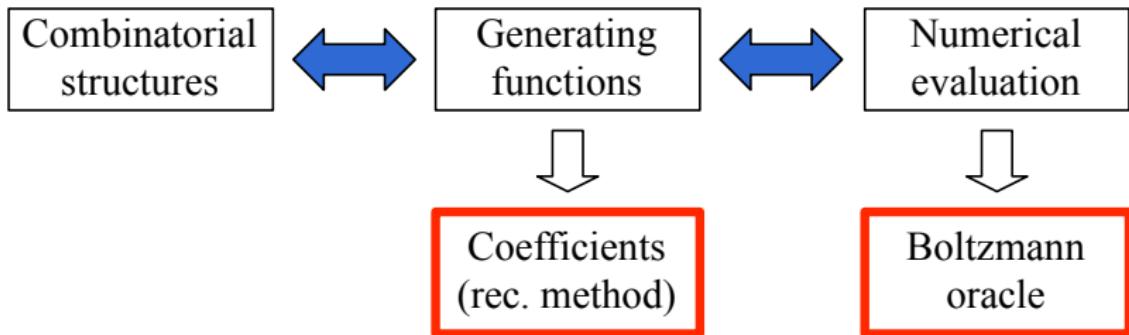
$$\mathbf{T}_{k+1} = \boldsymbol{\beta}_k \mathbf{U}_k + \mathbf{T}_k^2$$

$$\boldsymbol{\beta}_k = \frac{\partial H}{\partial Y}(\mathcal{Z}, Y_k) - \frac{\partial H}{\partial Y}(\mathcal{Z}, Y_{k-1})$$

- at iteration Y_k , perform a single step of the calculation of \mathbf{U} .

★ gain: the inversion is twice as fast.

Summary



Implementation

Maple prototype:

- → library (maple and/or other language)
- random grammars,
- XML grammars, $\sim 10^3$ equations (A. Darrasse),
- software random testing (J. Oudinet),
- others?

# equations	4	10	50		100		500
# constructions/eqn	10	10	10	50	10	50	50
avg size largest scc	2.47	3.42	7.95	18.62	10.93	67.18	339.1
time (0.99ρ)	0.05	0.11	0.17	0.47	0.23	7.29	61.73
time (0.999999ρ)	0.08	0.16	0.19	0.56	0.25	8.11	61.86
avg expected size	$4.1 \cdot 10^{14}$	$1.4 \cdot 10^7$	$2.2 \cdot 10^5$	$1.0 \cdot 10^5$	$1.2 \cdot 10^6$	$5.0 \cdot 10^4$	$3.3 \cdot 10^4$

in seconds, using Maple 11 on an Intel processor at 3.2 GHz with 2 GB of memory.

Future work

- work in progress...
 - unlabelled set and cycles,
 - extension to $\mathcal{H}(\emptyset, \emptyset) \neq \emptyset$,
 - substitution.
- next steps
 - convergence acceleration,
 - singularities,
 - tuning of Boltzmann parameter according to expected size.