Algebraic Combinatorics with MAPLE and ACE

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We illustrate the use of *Maple* in algebraic combinatorics on three examples : double Schubert polynomials, ribbon tableaux and noncommutative symmetric functions. most of the calculations presented in this paper were done by means of **ACE**, a collection of *Maple* packages developed at the University of Marne-la-Vallée.

Introduction

Experimentation in Algebraic Combinatorics, as in any other branch of Discrete Mathematics, requires large samples of data obtained by means of computer programs. Due to the algebraic aspects of the field, these data can then be processed in many different ways, e.g. one can compute generating polynomials for different statistics on a given set of combinatorial objects, try to factorize these polynomials, use them as coefficients of symmetric functions, specialize the variables, and so on. A computer algebra system like *Maple* is well suited for this kind of manipulations. Specialized *Maple* packages for handling the usual objects of Algebraic Combinatorics (permutations, words, Young tableaux, symmetric functions, Schubert polynomials etc.) have been written in recent years and regrouped in a coherent environment called **ACE** (Algebraic Combinatorics Environment), developed at the University of Marne-la-Vallée.

Here, we illustrate the use of *Maple* and **ACE** on three problems rather representative of the field. The first one is the search for a "Pieri formula" for double Schubert polynomials. Experimental data were easily obtained by means of the **SP** package of **ACE**, and we explain how it was analyzed and eventually led to a correct conjecture. The second one is a problem on symmetric functions, formulated in terms of generalizations of Young tableaux called ribbon (or rim-hook) tableaux. We explain how the generating functions (spin polynomials) of these objects have been implemented. Taking the spin polynomials as coefficients of symmetric functions and specializing the parameter at a root of unity, we arrived at an interesting conjecture in representation theory (which is still unproved at the time of writing). Finally, we explain how the same kind of manipulations with noncommutative symmetric functions (using **NCSF**) led to the discovery of a one-parameter family of idempotents in the group algebra of the symmetric group, interpolating between all known examples of Lie idempotents, *i.e.* idempotents projecting a homogeneous component of the free associative algebra onto the free Lie algebra.

The choice of *Maple* for developing **ACE** was motivated by the fact that this system, which is widely used among combinatorists, provides a good interface and can be used on all classical computer systems (*Unix*, *MsDos* or *Macintosh*).

Schubert polynomials

Computations in the ring $\mathbb{Z}[\mathbf{x}]$ of polynomials in several variables $\mathbf{x} := \{x_1, \ldots, x_n\}$ become rapidly intractable since dimensions grow very fast with the number of variables and with the degree. For certain specific problems, appropriate linear bases allow to compact the data, but we need to keep track of the multiplicative structure. Geometry motivates the definition of such bases ([3], [1]).

Lascoux and Schützenberger [15] defined *the Schubert polynomials*, which form a basis of $\mathbb{Z}[\mathbf{x}]$ as a free module over the ring of symmetric polynomials. This basis is indexed by permutations of $\mathfrak{S}_n$ (and thus is finite) and algebraic manipulations with Schubert polynomials mostly reduce to combinatorial constructions on permutations. These polynomials are defined as follows. Let ∂_i be the divided difference operator:

$$\partial(x_1, \ldots, x_n) = \frac{f - \sigma_i(f)}{x_i - x_{i+1}} ,$$

where σ_i is the transposition exchanging x_i and x_{i+1}. These operators satisfy the braid relations:

$$\begin{aligned} \partial_i \partial_{i+1} \partial_i &= \partial_{i+1} \partial_i \partial_{i+1} , \\ \partial_i \partial_j &= \partial_j \partial_i \quad \text{for } |i-j| > 1 . \end{aligned}$$

So that for any permutation $w \in \mathfrak{S}_n$, there is a well defined operator $\partial_w = \partial_{i_1} \ldots \partial_{i_r}$, where $w = \sigma_{i_1} \ldots \sigma_{i_r}$ is any reduced decomposition of w. Let $w_0 = (n, n-1, \ldots, 1)$ be the maximal permutation of $\mathfrak{S}_n$.

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The Schubert polynomial $\mathbb{X}_w$ is then defined as $\mathbb{X}_w = \partial w^{-1} w_0(\mathbb{X}_{w_0})$, where $\mathbb{X}_{w_0} = x_1^{n-1} \cdots x_{n-1}^1$. For example, $\mathbb{X}_{321} = x_1^2 + x_2$, $\mathbb{X}_{3,1,2} = x_1^2$ and $\mathbb{X}_{2,1,3} = x_1$.

These polynomials are canonical representatives of the classes of Schubert cycles int the cohomology of the variety of complete flags of $\mathbb{C}^n$, which is isomorphic to the quotient ring of $\mathbb{Z}[\mathbf{x}]$ by the ideal generated by symmetric polynomials without constant term (*cf.* [1], [3], [17]).

Computations with Schubert polynomials can be simplified by introducing a second set of parameters $\mathbf{y} := \{y_1, \ldots, y_n\}$ and defining the corresponding (double) *Schubert polynomials* [12] as follows. The maximal Schubert polynomial $\mathbb{X}_{w_0}$ is now:

$$\mathbb{X}_{w_0} := \prod_{i+j \leq n} (x_i - y_j) \ ,$$

and the general Schubert polynomials are as above, its images by divided differences [16] acting only on $\mathbf{x}$. More precisely:

$$\mathbb{X}_w := \partial_{w^{-1} w_0}(\mathbb{X}_{w_0}) \ .$$

Simple Schubert polynomials $\mathbf{X}_w$ are obtained by specializing all y_i to 0 in $\mathbb{X}_w$. They contain as a subfamily the Schur functions, which form a fundamental linear basis of the ring of symmetric functions. The combinatorics of symmetric functions mostly involves partitions and Young tableaux. Schubert polynomials allow the extension of this combinatorics to nonsymmetric polynomials, the basic combinatorial objects being now permutations.

For example, the multiplicative structure of $\mathbb{Z}[\mathbf{y}][\mathbf{x}]$ in the basis of Schubert polynomials is described by Monks formula [17], [9]. This formula gives the decomposition, in the Schubert basis, of the product of a Schubert polynomial by a single variable. The remarkable fact is that there is no multiplicity in this multiplication, *e.g.* $x_4\, \mathbb{X}_{5,2,7,3,6,4,1} = -\mathbb{X}_{5,3,7,2,6,4,1} + \mathbb{X}_{5,2,7,6,3,4,1} + \mathbb{X}_{5,2,7,4,6,3,1} + y_3 \mathbb{X}_{5,2,7,3,6,4,1}$.

On the other hand, the multiplicative structure of the ring of symmetric functions is given by Pieri's formula, which gives the decomposition on the Schur basis of the product of a Schur function by a complete or elementary symmetric function. Lascoux and Schützenberger have given the corresponding Pieri formula [15] for simple Schubert polynomials, *i.e.* the product of a simple Schubert polynomial by a complete or elementary symmetric function. This Pieri formula still has no multiplicity.

We were naturally led to investigate a Pieri formula for Schubert polynomials in $\mathbf{x}$ and $\mathbf{y}$. Numerical experiments where conducted with *Maple*, and more precisely with the **ACE** package **SP**, devoted to Schubert polynomials and symmetrization operators.

An efficient way to apply Monk's formula to this computation is to observe that the elementary symmetric polynomials can be defined by the recursion $e_i(x_1, \ldots, x_n) = x_n\ e_{i-1}(x_1, \ldots, x_{n-1}) \ + \ e_i(x_1, \ldots, x_{n-1})$, implemented as follows:

```
> e2x:=proc(i, n)
>   if (i=0) then
>     1
>   elif (i>n) then
>     0
>   else
>     x.n*e2x(i-1, n-1) + e2x(i, n-1)
>   fi
> end ;
```

The package **SP** provides the function `ToXX` which transforms any algebraic expression in $\mathbf{x}$, $\mathbf{y}$, and Schubert polynomials into a linear combination of double Schubert polynomials (by iterating Monk's formula). The following *Maple* procedure computes the decomposition on the general Schubert basis of the product $e_i(x_1, \ldots, x_n)\mathbb{X}_w$ on the double Schubert basis.

```
> Pieri:=proc(i, n, w)
>   ToXX(e2x(i, n)*XX[op(w)], 'collect');
> end ;
```

Now, a sample computation such as:

```
> w:=[4,7,1,5,6,2,3]:
> Pieri(3, 5, w);
```

$(y_5y_4y_7 + y_6y_5y_1 + y_6y_1y_7 + y_6y_5y_7 + y_4y_1y_7 + y_6y_5y_4 + y_5y_1y_7 + y_6y_1y_4 + y_6y_4y_7 + y_5y_1y_4)\mathbb{X}_{4,7,1,5,6,2,3} + (y_5y_7 + y_1y_7 + y_5y_4 + y_4y_7 + y_1y_4 + y_5y_1)\mathbb{X}_{4,7,1,5,8,2,3,6} + (y_1 + y_4 + y_7)\mathbb{X}_{4,7,1,6,8,2,3,5} + (y_4y_7 + y_5y_7 + y_6y_5 + y_5y_4 + y_6y_7 + y_6y_4)\mathbb{X}_{4,7,2,5,6,1,3} + \mathbb{X}_{4,8,1,6,7,2,3,5} + (y_6y_1 + y_5y_1 + y_6y_5 + y_1y_4 + y_5y_4 + y_6y_4)\mathbb{X}_{4,8,1,5,6,2,3,7} + (y_4 + y_5 + y_7)\mathbb{X}_{4,7,2,5,8,1,3,6} + \mathbb{X}_{5,7,1,6,8,2,3,4} + \mathbb{X}_{4,7,2,6,8,1,3,5} + (y_4 + y_5 + y_1)\mathbb{X}_{4,8,1,5,7,2,3,6} + (y_6 + y_4 + y_5)\mathbb{X}_{4,8,2,5,6,1,3,7} + \mathbb{X}_{4,8,2,5,7,1,3,6}$

shows that coefficients in the expansion are symmetric polynomials (in fact elementary symmetric functions) in some subsets of $\mathbf{y}$. The Pieri's formula for simple Schubert polynomials is obtained by specializing all y_k to 0. All coefficients e_i are sent to 0 except $e_0 = 1$. Thus, the coefficients "1" in the Pieri's formula for simple Schubert polynomials should be interpreted as elementary symmetric functions of degree 0.

To further describe the previous output $\sum c_\nu \mathbb{X}_\nu$ of the computation of $e_i(x_1, \ldots, x_n)\mathbb{X}_w$, we shall rather write $\sum c_\nu$ cycle(νw^{-1}) where cycle(νw^{-1}) is the cycle decomposition of νw^{-1}:

```
> XX2Cycle:=proc(expr, w)
>   if (type(expr, `+`) or type(expr, `*`)) then
>     map(XX2Cycle, expr, w)
>   elif (type(expr, indexed)) then
>     Perm2Cycle(MultPerm([op(expr)],
>                 InvPerm(w)))
>   else
>     expr
>   fi
> end :
> XX2Cycle(Pieri(3, 5, w), w);
```

$(y_5y_4y_7 + y_6y_5y_1 + y_6y_1y_7 + y_6y_5y_7 + y_4y_1y_7 + y_6y_5y_4 + y_5y_1y_7 + y_6y_1y_4 + y_6y_4y_7 + y_5y_1y_4)[] + (y_5y_7 + y_1y_7 + y_5y_4 + y_4y_7 + y_1y_4 + y_5y_1)[[6,8]] + (y_1 + y_4 + y_7)[[5,6,8]] + [[1,2],[5,6,8]] + (y_4 + y_5 + y_7)[[1,2],[6,8]] + (y_4y_7 + y_5y_7 + y_6y_5 + y_5y_4 + y_6y_7 + y_6y_4)[[1,2]] + [[5,6,7,8]] + (y_4 + y_5 + y_1)[[6,7,8]] + (y_6y_1 + y_5y_1 + y_6y_5 + y_1y_4 + y_5y_4 + y_6y_4)[[7,8]] + [[4,5,6,8]] + [[1,2],[6,7,8]] + (y_6 + y_4 + y_5)[[1,2],[7,8]]$

This type of experiments led us to an explicit Pieri formula for double Schubert polynomials. More precisely, let w be a permutation in $\mathfrak{S}_n$ and i, n two positive integers. We denote by $e_i(w \to \nu)$ the image of the symmetric function e_i under the specialization $x_\alpha \longrightarrow y_\beta$ in which β takes its values in the entries of w not involved in the cycle ζ, to the left of position n and such that $\nu = w\zeta$. Then, double Pieri formula can be formulated as follows:

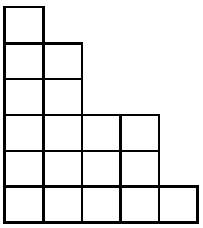
$$e_i(x_1, \ldots, x_n)\, \mathbb{X}_w \;=\; \sum_{\nu} e_{i-\ell}(w \to \nu)\, \mathbb{X}_\nu \;,$$

in which $\ell = l(w) - l(\nu)$, and where the sum is over all permutations ν obtained from w by a sequence of left (k,n)-cycles ζ_j such that $l(\nu) \leq l(w) + \sum l(\zeta_j)$. The combinatorial description of the left (k,n)-cycles appearing in this formula is very simple and is obtained from the case of simple Schubert polynomials (see [15]).

Generating functions of ribbon tableaux

The investigation of certain problems in representation theory leads to the consideration of *k-ribbon tableaux*, which are generalizations of Young tableaux.

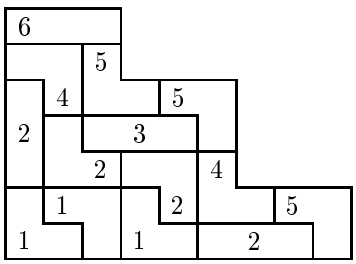
Recall that a partition is a non-increasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$. A partition is usually represented by its Ferrers diagram, which is a left justified diagram of boxes in the plane, such that the i-th row contains λ_i boxes. For example, the partition $\lambda = (5,4,4,2,2,1)$ is represented as



A Young tableau of shape λ is a filling of a Ferrers diagram with integers $1, 2, \ldots$ such that each row is weakly increasing from left to right, and each column is strictly increasing from bottom to top. For example

10				
7	9			
6	8			
4	4	5	5	
2	2	2	3	
1	1	1	1	2

is a Young tableau of shape $(5,4,4,2,2,1)$ (see *e.g.* [8] for more on this subject). Now, a ribbon tableau is a tiling of a Ferrers diagram with k-ribbons (or rim-hooks) instead of boxes, filled with numbers weakly increasing along rows and strictly increasing along columns [21, 13]. A k-ribbon is a connected diagram of boxes, containing no 2×2 square, and proceeding only by "down" and "right" moves from its top left most cell. The precise condition for columns is that the lowest and rightmost cell (called the root) of a ribbon labelled i should not be on the top of a ribbon labelled j if $j \geq i$. A typical 3-ribbon tableau is shown below:



With a ribbon tableau T, one associates a monomial x^T by associating with each ribbon labelled i the variable x_i and taking the product. For example, the monomial corresponding to the above example is $x_1^3x_2^4x_3x_4^2x_5^3x_6$. The sequence $\mu = (3,4,1,2,3,1)$ of exponents of x^T is called the *weight* of T, while the sequence $\lambda = (9,9,6,6,6,3,3)$ of lengths of rows of T is the *shape* of T.

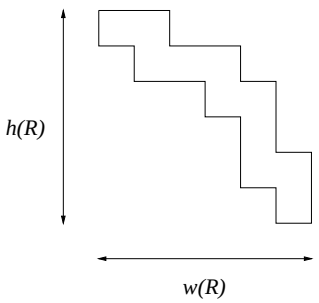
Let $\mathrm{Tab}_k(\lambda, \mu)$ denote the set of all k-ribbon tableaux of shape λ and weight μ. The 1-ribbon tableaux are just ordinary Young tableaux, and the combinatorial expansion of Schur polynomials:

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{Tab}_1(\lambda,.)} x^T \;,$$

can be generalized to:

$$G_\lambda^{(k)}(\mathbf{x}) = \sum_{T \in \mathrm{Tab}_k(\lambda,.)} x^T \ .$$

It is not difficult to show that when this polynomial is not zero (*i.e.* when λ has empty k-core), $G_\lambda^{(k)}(\mathbf{x})$ is equal to a product of Schur functions $s_{\lambda^{(0)}}(\mathbf{x}) \dots s_{\lambda^{(k-1)}}(\mathbf{x})$, where $(\lambda^{(0)}, \dots, \lambda^{(k-1)})$ is the k-quotient of λ. However, for $k \geq 2$, one can read on ribbon tableaux an information which is not available on ordinary Young tableaux, namely the *heights* of the ribbons:



Define the spin of a ribbon tableau as:

$$s(T) = \frac{1}{2} \sum_{R \in T} (h(R) - 1) \ ,$$

and introduce the q-generating function:

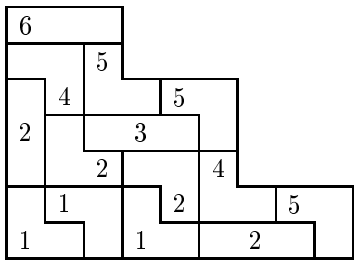
$$G_\lambda^{(k)}(\mathbf{x}; q) = \sum_{T \in \mathrm{Tab}_k(\lambda,.)} q^{s(T)} x^T \ .$$

It is no longer obvious that $G_\lambda^{(k)}(\mathbf{x}; q)$ is actually symmetric in the x_i's. This was proved quite recently using techniques from the theory of quantum affine algebras [13]. Assuming that this is true we could write:

$$G_\lambda^{(k)}(\mathbf{x}; q) = \sum_\mu \left(\sum_{T \in \mathrm{Tab}_k(\lambda,\mu)} q^{s(T)} \right) m_\mu(\mathbf{x}) \ ,$$

where m_μ is the monomial symmetric polynomial indexed by the partition μ.

It appeared on small examples that the symmetric functions $H_\lambda^{(k)}(\mathbf{x}; q) = G_{k\lambda}^{(k)}(\mathbf{x}; q)$ form for each $k \geq 1$ a basis of k-ribbons (or rim-hooks) instead of boxes, and filling them with numbers weakly increasing along rows and strictly increasing along columns [21, 13]. The precise condition for columns is that the lowest and rightmost cell (called the root) of a ribbon labelled i should not be on the top of a ribbon labelled j if $j \geq i$. A typical 3-ribbon tableau is shown below:



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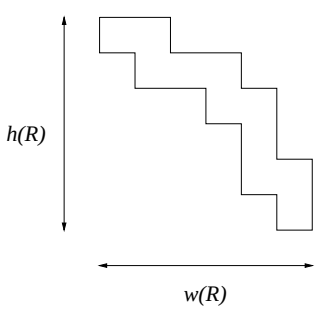

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where m_μ is the monomial symmetric polynomial indexed by the partition μ.

It appeared on small examples that the symmetric functions $H_\lambda^{(k)}(\mathbf{x};q) = G_{k\lambda}^{(k)}(\mathbf{x};q)$ form for each $k \geq 1$ a basis of the ring of symmetric functions interpolating between the Schur functions $s_\lambda = H_\lambda^{(1)}$ and the Hall-Littlewood functions $Q_\lambda^{'} = H_\lambda^{(N)}(\mathbf{x})$ (N sufficiently large). By analogy with the results of [14] and [2], we were led to conjecture that in the case $\lambda = \nu^k := (\nu_1,\ldots,\nu_1,\nu_2,\ldots,\nu_2,\ldots)$ (each part repeated k times), $H_\lambda^{(k)}(\mathbf{x};q)$ has interesting properties when q is specialized to a primitive k-th root of unity. To get some confidence in this conjecture we needed some large examples. The first really significant ones being $H_{222111}^{(3)}(\mathbf{x},e^{2\pi i/3})$ and $H_{22221111}^{(4)}(\mathbf{x},i)$.

The resulting conjecture in representation theory can be stated as follows. Let V_ν be a polynomial irreducible representation of $GL(n,\mathbb{C})$ with highest weight ν, and let γ be the endomorphism of $V_\nu^{\otimes k}$ such that $\gamma(v_1 \otimes \cdots \otimes v_k) = v_2 \otimes \cdots \otimes v_k \otimes v_1$. Let $E^{(r)}$ be the eigenspace of γ with eigenvalue ω^r, where $\omega = e^{2ri\pi/k}$. $E^{(r)}$ is $GL(n)$-stable, and the multiplicity of the irreducible V_μ in $E^{(r)}$ is equal to the coefficient of s_μ in the reduction modulo $1-q^k$ of $H_\lambda^{(k)}(\mathbf{x};q)$ with $\lambda = \nu^k$. In terms of symmetric functions this amounts to say that:

$$H_\lambda^{(k)}(\mathbf{x};\omega) = (-1)^{(k-1)|\nu|} s_\nu(x_1^k, x_2^k, \ldots) \ .$$

To compute the spin polynomials:

$$\sigma_{\lambda\mu}^{(k)} = \sum_{T \in \mathrm{Tab}_k(\lambda,\mu)} q^{s(T)} \ ,$$

we used a *Maple* program based on the following properties. First, $\sigma_{\lambda\mu}^{(k)} = q^{|\mu|(k-1)/2} F_{\alpha\mu}^{(k)}(q^{-1/2})$, where $\alpha = \lambda' + \rho_n$, $\rho_n = (n-1, n-2, \ldots, 1, 0)$, $n \geq \lambda_1$, and the polynomials $F_{\alpha\mu}^{(k)}$ are defined by:

$$F_{\alpha\mu}^{(k)} = \sum_{T \in \mathrm{Tab}_k(\lambda,\mu)} \prod_{R \in T} q^{w(R)-1} \ ,$$

$w(R)$ being the width of R. These polynomials satisfy the recurrence relation:

$$F_{\alpha\mu}^{(k)} = \sum_p q^{I_p} F_{\alpha_p,\mu^*}^{(k)}(q) \ ,$$

where $\mu = (\mu_1,\ldots,\mu_r)$, $\mu^* = (\mu_1,\ldots,\mu_{r-1})$ and the sequences α_p are obtained in the following way. Subtract to α all the distinct permutations of the vector $[0,\ldots,0,k,\ldots,k]$ (n components, k repeated μ_r times). Among the resulting vectors, discard all those having at least two identical components, and sort the other ones in decreasing order, storing the inversion number of the sorting permutation. The α_p are the resulting sorted vectors, and the I_p are the corresponding inversion numbers.

The F polynomials are computed by the *Maple* program `F` given below, which is the core of the H-function program. The conversion to the Schur basis can be performed either by Stembridge's package **SF** (in the share library) or by the **SYMF** package of **ACE**. Note the use of the `option remember` which allows the simple implementation of a complicated recursion. The auxiliary function:

```
> q_inv:=proc(perm) ... end
```

returns q^I where I is the inversion number of a permutation `perm`.

For example, the two domino tableaux of shape $(4,4)$ and weight $(3,1)$ and the five standard domino tableaux of the same shape have respective generating functions given by:

```
> F(2, [2,3,4,5], [3,1]);
```

$$q^2 + 1$$

```
> F(2, [2,3,4,5], [1,1,1,1]);
```

$$2q^4 + 3q^2 + 1$$

Similarly, a call to `F(4,[4,5,6,7,12,13,14,15], [1$12])` gives the generating function of the 5913600 standard 4-ribbon tableaux of shape (88884444) needed to compute the coefficient of $M_{(1^{12})}$ in $H_{22221111}^{(4)}$, that is:

$$\begin{aligned}
&275\,q^{36} + 2596\,q^{34} + 12881\,q^{32} + \\
&44648\,q^{30} + 119438\,q^{28} + 258259\,q^{26} + \\
&462332\,q^{24} + 694552\,q^{22} + 883474\,q^{20} + \\
&956690\,q^{18} + 883474\,q^{16} + 694552\,q^{14} + \\
&462332\,q^{12} + 258259\,q^{10} + 119438\,q^8 + \\
&44648\,q^6 + 12881\,q^4 + 2596\,q^2 + \\
&275
\end{aligned}$$

```
> ## computes the polynomial F^(k)_{alpha, mu}
> ## alpha is entered as an increasing sequence
>
> F:=proc(k, alpha, mu)
>   local
>     r,weight,i,length,j,perm,fact,res,ll;
>   option remember ;
>   r:=nops(mu); length:=nops(alpha);
>   for i to length while alpha[i]=i-1 do od ;
>   if i=length+1 then
>     res:=1
>   elif r=0 then
>     res:=0
>   else
>     weight:=[seq(0, i=1..length-mu[r]),
>              seq(k, i=1..mu[r])];
>     ll:=combinat['permute'](weight);
>     res:=0;
>     for i in ll do
>       perm:=[seq(alpha[j]-i[j], j=1..length)];
>       fact:=invers(perm);
>       if fact=0 then
>         next
>       else
>         res:=expand(res+fact*
>                     F(k,sort(perm),
>                       [op(1..r-1,mu)]))
>       fi
>     od
>   fi;
>   res
> end ;
```

Noncommutative symmetric functions

Noncommutative symmetric functions (*cf.* [7, 10, 4, 11]) provide an algebraic theory related to the combinatorics of compositions, in such the same way as ordinary symmetric functions are related to the combinatorics of partitions.

The so called "fundamental theorem of the theory of symmetric functions" states that the elementarysymmetric polynomials $e_1, \ldots, e_n$ in n variables $x_1, \ldots, x_n$ are algebraically independent. Thus, the algebraic of symmetric functions (that is, polynomials in an infinite subset of variables) is just a polynomial algebra $\mathbb{Q}[e_1, e_2, \ldots]$, graded by the weight function $w(e_k) = k$. One can as well take as generators the complete homogeneous functions h_k (sum of all monomial of degree k) or the powersums $p_k = \sum x_i^k$.

By analogy, ones defines the algebra of formal *noncommutative symmetric functions* as the free associative algebra $\mathbf{Sym} = \mathbb{Q}\langle S_1, S_2, \ldots\rangle$ generated by an infinite sequence of noncommutative indeterminates $(S_n)_{n\geq 1}$, called *complete* symmetric functions. The elementary symmetric functions Λ_k are defined as in the commutative case by

$$\lambda(-t) := \sum_{k\geq 0} \Lambda_k (-t)^k = [\sigma(t)]^{-1}$$

where $\sigma(t) = \sum_{k\geq 0} S_k t^k$. On the other hand, there is no canonical definition of the noncommutative power sums. Noncommutative analogues of their various expressions in terms of complete or elementary functions lead to different noncommutative analogues. Two of them called power sums of first and second kind, are especially important :

$$\Psi_n = \sum_{i_1+\cdots+i_r=n} (-1)^{r-1}\ i_r\ S_{i_1}\ \ldots\ S_{i_r}\ , \tag{1}$$

$$\Phi_n = \sum_{i_1+\cdots+i_r=n} (-1)^{r-1}\ \frac{n}{r}\ S_{i_1}\ \ldots\ S_{i_r}\ . \tag{2}$$

A composition of n is a sequence I of positive integers whose sum $|I|$ is equal to n. For any composition $I = (i_1, \ldots, i_r)$, one defines $S^I = S_{i_1}\ \ldots\ S_{i_r}$. The homogeneous component $\mathbf{Sym}_n$ of degree n of $\mathbf{Sym}$ is the subspace of $\mathbf{Sym}$ generated by all S^I with $|I| = n$. Note that the family (S^I) is a homogeneous basis of $\mathbf{Sym}$. The set of all compositions of an integer n can be equipped with the *reverse refinement order*, denoted $\preceq$. One has for example $(5, 2, 3) \preceq (1, 4, 2, 1, 2)$. The noncommutative *ribbon Schur* functions (R_I) are then defined by:

$$R_I = \sum_{J\preceq I} (-1)^{\ell(I)-\ell(J)}\ S^J\ , \tag{3}$$

where $\ell(I)$ denotes the *length* of I. The family (R_I) is also a homogeneous basis of $\mathbf{Sym}$.

An integer $i \in [1, n-1]$ is said to be a *descent* of a permutation $w \in \mathfrak{S}_n$ iff $w(i) > w(i+1)$. The set $\mathrm{Des}(w)$ of those integers is called the *descent set* of w. We also associate with any composition $I = (i_1, \ldots, i_r)$ of n the subset $D(I) = \{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_r\}$ of $[1, n-1]$. Solomon showed that the subspace of $\mathbb{Q}[\mathfrak{S}_n]$ generated by the elements:

$$D_I = \sum_{\mathrm{Des}(w)=D(I)} w\ ,$$

is a subalgebra of $\mathbb{Q}[\mathfrak{S}_n]$ (see [20]). This subalgebra is called the *descent algebra* of $\mathfrak{S}_n$ and is denoted by Σ_n. One can now define a linear isomorphism:

$$\alpha :\ \mathbf{Sym} = \bigoplus_{n=0}^{\infty} \mathbf{Sym}_n\ \longrightarrow\ \Sigma = \bigoplus_{n=0}^{\infty} \Sigma_n\ ,$$

by setting $\alpha(R_I) = D_I$ for every composition I. It is interesting to notice that the image of S^I by α is equal to:

$$D_{\subseteq I} = \sum_{\mathrm{Des}(w)\subseteq D(I)} w\ .$$

A *Lie idempotent* is an idempotent of the group algebra of $\mathfrak{S}_n$ which projects a homogeneous component of the free associative algebra onto the free Lie algebra (see *e.g.* [6, 19]). It turns out that several classical Lie idempotents belong to Solomon's descent algebra. This allows us to interpret them as noncommutative symmetric functions (using α^{-1}). It can for instance be proved that the image under α^{-1} of Dynkin's Lie idempotent $\frac{1}{n}\,\theta_n$, where:

$$\begin{aligned}\theta_n &= [[\ldots[[1,2],3],\ldots],n] \\ &= \sum_{k=0}^{n-1} (-1)^k\ D_{=\{1,2,\ldots,k\}}\ ,\end{aligned}$$

(and $[x,y]=xy-yx$), is exactly $\frac{1}{n}\,\Psi_n$. The noncommutative symmetric function $\frac{1}{n}\,\Phi_n$ can be interpreted in the same way as another classical Lie idempotent, *i.e.* Solomon's Eulerian idempotent (see [5, 7, 18]).

The complete symmetric functions of the (virtual) alphabet $A/(1-q)$ are the noncommutative symmetric functions $S_n(\frac{A}{1-q})$ defined by:

$$\begin{aligned}\sigma(t;\frac{A}{1-q}) &= 1+\sum_{n=1}^{+\infty} S_n(\frac{A}{1-q})\,t^n \\ &= \prod_{k\geq 0}^{\leftarrow} \sigma(q^k\,t;A) \\ &= \ldots\ \sigma(q^2\,t;A)\ \sigma(q\,t;A)\ \sigma(t;A)\ ,\end{aligned}$$

Ones define:

$$\varphi_n(q) = \frac{1-q^n}{n}\ \Psi_n\left(\frac{A}{1-q}\right)\ ,$$

where $\Psi_n(\frac{A}{1-q})$ is obtained by substituting $S_n(\frac{A}{1-q})$ in place of S_n in (1). The theory tells us that this element encodes a Lie idempotent of Σ_n It turns out that it interpolates among all known examples. Indeed, it can be shown that $\varphi_n(0)=\Psi_n/n$ which encodes Dynkin's idempotent and that $\varphi_n(1)=\Phi_n/n$ which corresponds to Solomon's Eulerian idempotent. Moreover, when q is a primitive n-th root of unity, $\varphi_n(q)$ specializes to Klyachko's idempotent (*cf.* [19, 10]). This motivated the search of an explicit formula for $\varphi_n(q)$.

The **ACE** package **NCSF** (described in [23]) is devoted to noncommutative symmetric functions. The following *Maple* program was used to compute the decomposition of $\varphi_n(q)$ on the basis of ribbon functions:

```
> varphi := proc(n)
> local tmp ;
>   tmp := SpDirect(Ps[n], q) ;
>   map(normal,ToR((1-q^n)*tmp/n, 'collect')) ;
> end ;
```

It is based on the **NCSF** functions:

`ToR`	#	change to basis `R`
`SpDirect`	#	transformation of alphabet A→A/(1-q)

For $n=3,4$, the procedure `varphi` gives:

$$\begin{aligned}\varphi_2(q) &= \frac{R_2}{2}-\frac{R_{1,1}}{2} \\ \varphi_3(q) &= \frac{R_3}{3}-\frac{qR_{2,1}}{3\,q+3}-\frac{R_{1,2}}{3\,q+3}+\frac{R_{1,1,1}}{3} \\ \varphi_4(q) &= \frac{R_4}{4}-\frac{q^2R_{3,1}}{4\,q^2+4\,q+4}-\frac{qR_{2,2}}{4\,q^2+4\,q+4} \\ &\quad+\frac{q^2R_{2,1,1}}{4\,q^2+4\,q+4}-\frac{R_{1,3}}{4\,q^2+4\,q+4} \\ &\quad+\frac{qR_{1,2,1}}{4\,q^2+4\,q+4}+\frac{R_{1,1,2}}{4\,q^2+4\,q+4}-\frac{R_{1,1,1,1}}{4}\end{aligned}$$

We were thus led to look for a formula of the form:

$$\varphi_n(q) = \frac{1}{n}\ \sum_{|I|=n} (-1)^{\ell(I)-1}\,\frac{q^{f(I)}}{g(I;q)}\,R_I\ ,$$

where $f(I)$ is an integer and $g(I;q)$ a polynomial in q. When $|I|=5$, the first values of the polynomials $g(I;q)$ are:

$$\begin{aligned}g([5];q) &= 1\ , \\ g([4,1];q) &= q^3+q^2+q+1=\begin{bmatrix}4\\1\end{bmatrix}_q\ , \\ g([3,2];q) &= q^3+q^2+q+1=\begin{bmatrix}4\\1\end{bmatrix}_q\ , \\ g([3,1,1];q) &= (q^2+1)(q^2+q+1)=\begin{bmatrix}4\\2\end{bmatrix}_q\ , \\ g([2,3];q) &= q^3+q^2+q+1=\begin{bmatrix}4\\1\end{bmatrix}_q\ , \\ g([2,2,1];q) &= (q^2+1)(q^2+q+1)=\begin{bmatrix}4\\2\end{bmatrix}_q\ , \\ g([2,1,2];q) &= (q^2+1)(q^2+q+1)=\begin{bmatrix}4\\2\end{bmatrix}_q\ , \\ g([2,1,1,1];q) &= q^3+q^2+q+1=\begin{bmatrix}4\\3\end{bmatrix}_q\ .\end{aligned}$$

Here, $\begin{bmatrix}n\\p\end{bmatrix}_q$ is the q-binomial coefficient (or Gaussian polynomial) defined by:

$$\begin{bmatrix}n\\p\end{bmatrix}_q = \frac{(q)_n}{(q)_p(q)_{n-p}}\ ,$$

with $(q)_n=(1-q)\ldots(1-q^n)$. From other similar observations, it was easy to conjecture that:

$$g(I;q) = \begin{bmatrix}n-1\\ \ell(I)-1\end{bmatrix}_q\ .$$

Before stating the result obtained for $f(I)$, we need a definition. Let I be a composition of n and let $D(I) = \{d_1, d_2, \ldots, d_{r-1}\}$ be the associated subset of $[1, n-1]$. The major index of I is defined by:

$$\mathrm{maj}\,(I) = d_1 + \cdots + d_{r-1} \ .$$

For $n = 5$, explicit computations led to the table:

I	[5]	[4,1]	[3,2]	[3,1,1]
$f(I)$	0	3	2	4
$\mathrm{maj}\,(I)$	0	4	3	7
$\mathrm{maj}\,(I) - f(I)$	0	1	1	3

I	[2,2,1]	[2,1,2]	[2,1,1,1]	[1,4]
$f(I)$	3	2	3	0
$\mathrm{maj}\,(I)$	6	5	9	1
$\mathrm{maj}\,(I) - f(I)$	3	3	6	1

We see that $\mathrm{maj}\,(I) - f(I)$ depends only on the length of I. Furthermore, the corresponding values are here $\{0, 1, 3, 6\}$ which are the first four triangular numbers. Similar experimental results showed that:

$$f(I) = \mathrm{maj}\,(I) - \binom{\ell(I)}{2} \ ,$$

up to the order 8. Hence we were led to the experimental formula:

$$\varphi_n(q) = \frac{1}{n} \sum_{|I|=n} \frac{(-1)^{\ell(I)-1}}{\left[\begin{smallmatrix} n-1 \\ \ell(I)-1 \end{smallmatrix}\right]_q} \, q^{\mathrm{maj}\,(I) - \binom{\ell(I)}{2}} \, R_I \ ,$$

which was proved three months later [10].

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