MPRI
TD 3

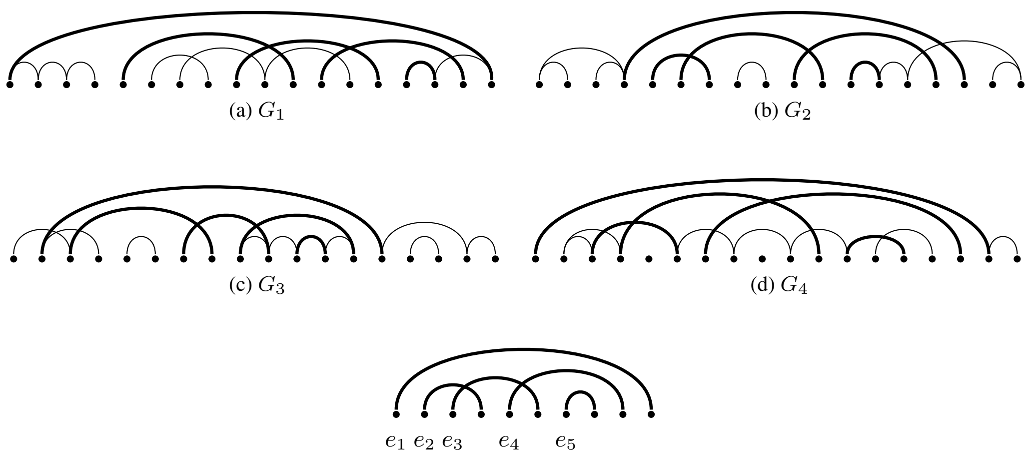
Stéphane Vialette

October 17, 2016

1 Definitions

Definition 1 (MCSP). *Given a family of linear graphs $\mathcal{G} = G_1, \ldots, G_n$ and a subset $\mathcal{R} \subseteq \{<, \sqsubset, \between\}$, $\mathcal{R} \neq \emptyset$, the Maximum Common Structured Pattern (MCSP) problem asks to find a maximum-size common $\mathcal{R}$-structured pattern of $\mathcal{G}$.*

See Figure 1 for an illustration.



(a) G_1 (b) G_2 (c) G_3 (d) G_4 (e) G_{sol}

Figure 1 – Four linear graphs G_1, G_2, G_3 and G_4 and a $\{<, \sqsubset, \between\}$-comparable common structured pattern (depicted also as G_{sol} in the bottom part). The occurrence of the structured pattern G_{sol} in each graph is emphasized in bold. Edges e_2, e_3, e_4 and e_5 are nested in edge e_1 ; edges e_2 and e_3 precede edge e_5 ; edge e_2 precedes edge e_4 and crosses edge e_3, while edge e_3 crosses both edges e_2 and e_4.

We will use the following terminology to describe special edge-disjoint linear graphs. A linear graph is called a *sequence* if it is $\{<\}$-comparable, a *tower* if it is $\{\sqsubset\}$-comparable, and a *staircase* if it is $\{\between\}$-comparable. We define the *width* (resp. *height* and *depth*) of a linear graph to be the size of the maximum cardinality sequence (resp. tower and staircase) subgraph of the graph. A $\{<, \sqsubset\}$-comparable linear graph with the additional property that any two maximal towers in it do not share an edge is called a *sequence of towers*. Similarly, a $\{<, \between\}$-comparable linear graph is a *sequence of staircases* if any two maximal staircases do not share an edge. A *tower of staircases* is a $\{\sqsubset, \between\}$-comparable linear graph where any pair of maximal staircases do not share an edge, and a *staircase of towers* is a $\{\sqsubset, \between\}$-comparable linear graph where any pair of maximal towers do not share an edge. A sequence of towers (resp.

sequence of staircases, tower of staircases, and staircase of towers) is *balanced* if all of its maximal towers (resp. staircases, staircases, staircases, and towers) are of equal size. Figure 2 illustrates an example of the above types of linear graphs.

2 Questions

2.1 $\{<, \between\}$-structure

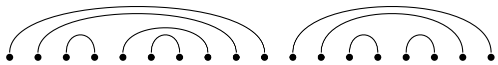
1. Show that the MCSP problem for $\{<, \between\}$-structured patterns is NP-complete.
2. Let $\mathcal{M}$ be a $\{<, \between\}$-structured linear matching. Show that $\mathcal{M}$ is the disjoint union of two sequences of staircases.
3. Let $\mathcal{M}$ be a sequence of staircases containing k edges. Show that $\mathcal{M}$ contains a balanced sequence of staircases with at least $\frac{k}{H_k}$ edges, where H_k is the k-th harmonic number:
$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}.$$
4. Show that any $\{<, \between\}$-structured linear matching with k edges contains balanced sequence of staircases with at least $\frac{k}{2H_k}$ edges.
5. Propose a polynomial-time algorithm for computing the largest balanced sequence of staircases in a $\{<, \between\}$-structured linear matching. Prove correctness and running time of your algorithm.
6. Show that the MCSP problem for $\{<, \between\}$-structured patterns is approximable within ratio $\frac{k}{2H_k}$.

2.2 $\{\sqsubset, \between\}$-structure

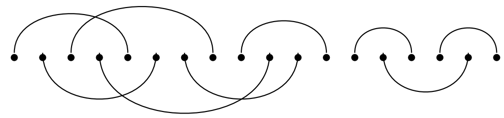
1. Show that the MCSP problem for $\{<, \sqsubset\}$-structured patterns is NP-complete.
2. Show that the MCSP problem for $\{<, \sqsubset\}$-structured patterns is approximable within ration $\sqrt{\text{opt}}$, where opt is the size of an optimal solution.
3. Show that for every family of permutations $\Pi_k \subseteq S_k$, $k \in \mathbb{N}$ and $|\Pi_k| \leq 2^k$, there exists a permutation $\pi \in S_K$, $K = \Omega(k^2)$, which avoids all permutations in Π_k.

2.3 $\{<, \sqsubset, \between\}$-structure

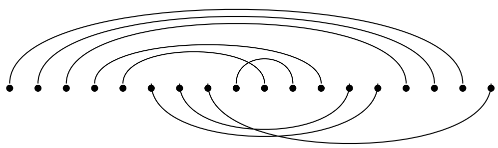
1. Let H be a $\{<, \sqsubset, \between\}$-comparable linear graph of size k, width $w(H)$, and height $h(H)$. Also, let $hd(H)$ and $wd(H)$ be the sizes of the maximum $\{\sqsubset, \between\}$-comparable and $\{<, \between\}$-comparable subsets of $E(H)$. Show that $k \leq w(H) \cdot hd(H)$ and $k \leq h(H) \cdot wd(H)$.
2. Let H be a $\{<, \sqsubset, \between\}$-comparable linear graph of size k. Show that H contains a simple structured pattern of size at least $k^{1/3}$.
3. Show that MCSP problem for $\{<, \sqsubset, \between\}$-structured patterns is approximable within ratio $O(\text{opt}^{2/3})$, where opt is the size of an optimal solution.
4. Let k be an integer such that $k^{1/3}$ is also integer. Show that there exists a $\{<, \sqsubset, \between\}$-comparable linear graph of size k that does not contain a simple structured pattern of size $\varepsilon\, k^{1/3}$ for any $\varepsilon > 1$.
5. $(*)$ Let H be a $\{<, \sqsubset, \between\}$-comparable linear graph of size k. Show that H contains either a tower or a balanced sequence of staircases of size $\Omega(\sqrt{k/\log k})$. Conclude that the MCSP problem for $\{<, \sqsubset, \between\}$-structured patterns is approximable within ratio $O(\sqrt{k \log k})$



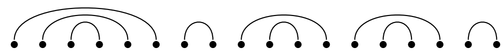
(a) A $\{<, \sqsubset\}$-structured pattern of width 4 and height 4



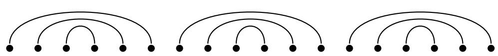
(b) A $\{<, \between\}$-structured pattern of width 4 and depth 4.



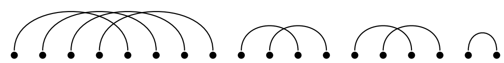
(c) A $\{\sqsubset, \between\}$-structured pattern of height 6 and depth 3



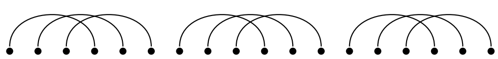
(d) A sequence of towers of width 5 and height 3.



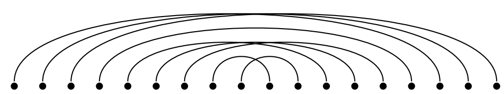
(e) A sequence of balanced towers of width 3 and height 3.



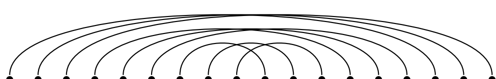
(f) A sequence of staircases of width 4 and depth 4.



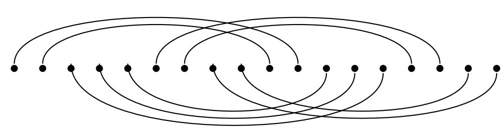
(g) A sequence of balanced staircases of width 3 and depth 3.



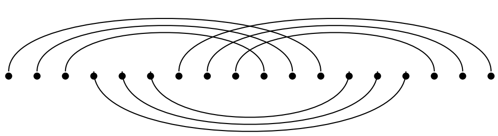
(h) A tower of staircases of height 4 and depth 3.



(i) A tower of balanced staircases of height 3 and depth 3.



(j) A staircase of towers of height 3 and depth 4.



(k) A staircase of balanced towers of height 3 and depth 3.

Figure 2 – Some restricted structured patterns. Edges are drawn above or below the vertices with no particular signification.