



Transverse Approach to Geometric Algebra Models for Manipulating Quadratic Surfaces

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Abstract. Quadratic surfaces gain more and more attention in the geometric algebra community and some frameworks to represent, transform, and intersect these quadratic surfaces have been proposed. To the best of our knowledge, however, no framework has yet proposed that supports all the operations required to completely handle these surfaces. Some existing frameworks do not allow the construction of quadratic surfaces from control points while some do not allow to transform these quadratic surfaces. Although a framework does not exist that covers all the required operations, if we consider all already proposed frameworks together, then all the operations over quadratic surfaces are covered there. This paper presents an approach that transversely uses different frameworks for covering all the operations on quadratic surfaces. We employ a framework to represent any quadratic surfaces either using control points or the coefficients of its implicit form and then map the representation into another framework so that we can transform them and compute their intersection. Our approach also allows us to easily extract some geometric properties.

Keywords: Geometric algebra · Quadratic surfaces · Conformal geometric algebras

1 Introduction

Geometric algebra provides convenient and intuitive tools to represent, transform, and intersect geometric objects. Deeply explored by physicists, it has been used in quantum mechanics and electromagnetism [12] as well as in classical mechanics [13]. Geometric algebra has also found some interesting applications in geographic data manipulations [16, 21]. Among them, geometric algebra is

used within the computer graphics community. More precisely, it is used not only in basis geometric primitive manipulations [20] but also in complex illumination processes as in [17] where spherical harmonics are substituted by geometric algebra entities. For image data analysis, on the other hand, we can find the usefulness of geometric algebra in mathematical morphology [6] and in neural networking [3,14]. In the geometric algebra community, quadratic surfaces gain more and more attention, and some frameworks to represent, transform, and intersect these quadratic surfaces have been proposed.

There exist three main approaches to deal with quadratic surfaces in geometric algebra. The first one, introduced in [9], is called double conformal geometric algebra (DCGA), $\mathbb{G}_{8,2}$. It is capable of representing quadratic surfaces from the coefficients of their implicit form. The second one is double projective geometric algebra (DPGA), $\mathbb{G}_{4,4}$, whose definition was firstly introduced in [11] and has been further developed in [8]. This approach is based on a duplication of \mathbb{R}^4 and it represents quadratic surfaces from the coefficients of their implicit form, as bivectors. However, it cannot construct quadratic surfaces from control points. The third one was introduced in [2] and is denoted as quadric conformal geometric algebra (QCGA), $\mathbb{G}_{9,6}$. QCGA allows to define any general quadratic surface from 9 control points, and to represent objects by only 1 or 2-vectors. QCGA is capable of constructing quadratic surfaces either using control points or implicit equations as 1-vector. QCGA also allows to efficiently intersect quadratic surfaces. However, it does not yet allow all geometric transformations over quadratic surfaces. In order to enhance the usefulness of geometric algebra for the geometry and the computer graphics communities, a new framework that allows to represent and manipulate quadratic surfaces has to be developed. It is the main purpose of this paper.

1.1 Contributions

We propose a new approach that transversely uses the three above mentioned geometric algebra models to compensate drawbacks of each model and show that it is possible to not only represent quadratic surfaces using either control points or implicit coefficients but also transform these quadratic surfaces using versors. More precisely, we employ a model that allows us to represent quadratic surfaces, and convert them into another model that allows us to transform them using versors, and then convert the result back into the original model. With our approach, the tangent planes to a quadratic surfaces and intersection of quadratic surfaces can be computed.

1.2 Notations

Following the state-of-the-art usage in [5] and [19], upper-case bold letters denote blades (blade \mathbf{A}) whose grade is higher than 1. Multivectors and k -vectors are denoted with upper-case non-bold letters (multivector A). Lower-case bold letters refer to vectors and lower-case non-bold to multivector coordinates. The

vector space dimension is denoted by 2^d , where d is the number of basis blades \mathbf{e}_i of grade 1.

2 Geometric Algebra Models for Quadratic Surfaces

2.1 Measure for Evaluating Complexity of Models

This paper focuses on the most common operations on quadratic surfaces and their intersections, and aims at determining the most efficient geometric algebra model for each operation. These operations can be related to computer graphics or more general geometry and will mainly consist in:

- checking whether a point lies in a quadratic surface,
- intersecting quadratic surface and line,
- computing the normal vector (and the tangent plane) of a surface at a given point.

Indeed, these operations are precisely the minimal tools required to set up a ray-tracer [10]. According to these targeted operations, evaluating their complexity mostly consists in the estimation of the number of operations required for both the outer product of multivectors and the inner product between vectors and bivectors.

First, let us consider the outer product between two homogeneous multivectors whose numbers of components are u and v respectively, $u, v \in \mathbb{N}$. We then assume that an upper bound to the number of required products is at most uv products of scalars, as shown in the definition of the outer product [15].

Second, according to the target operations, we need to use the formula for inner products between 1-vector and 2-vector as well as the inner product between two 1-vectors. Considering that the first multivector has u non-zero components, and the second has v non-zero components, then the inner product between two 1-vectors will result in uv products. The inner product between 1-vectors and 2-vectors, on the other hand, requires two inner products for each pair of components of the two multivectors, that is to say $2uv$ products.

There exist three main geometric algebra frameworks to manipulate general quadratic surfaces: DCGA of $\mathbb{G}_{8,2}$ [9], DPGA of $\mathbb{G}_{4,4}$ [8, 18], and QGCA of $\mathbb{G}_{9,6}$ [2]. The following sections present their specificities as well as their complexity for the targeted operations.

2.2 DCGA of $\mathbb{G}_{8,2}$

DCGA was presented by Hitzer and Easter [9] and aims at having entities representing both quartic surfaces and quadratic surfaces. In more details, DCGA of $\mathbb{G}_{8,2}$ is defined over a 10-dimensional vector space. The base vectors of the space are basically divided into two groups: $\{\mathbf{e}_{o1}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{\infty1}\}$, corresponding to the CGA vectors, and a copy of this basis $\{\mathbf{e}_{o2}, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_{\infty2}\}$. A point of DCGA whose Euclidean coordinates are (x, y, z) is defined as the outer product of two CGA points with coordinates (x, y, z) .

Quadratic Surfaces: A general quadratic surface merely consists of defining some operators that extract the components of \mathbf{x} . A general quadratic surface is defined as:

$$ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + iz + j = 0. \tag{1}$$

In DCGA, 10 extraction operators $\{\mathbf{T}_{x^2}, \mathbf{T}_{y^2}, \mathbf{T}_{z^2}, \mathbf{T}_{xy}, \mathbf{T}_{xz}, \mathbf{T}_{yz}, \mathbf{T}_x, \mathbf{T}_y, \mathbf{T}_z, \mathbf{T}_1\}$ are defined (see [9]) such that the inner product of these operators and a point results in Eq. 1. DCGA not only supports the definition of general quadratic surfaces but also some quartic surfaces like Torus, cyclides (Dupin cyclides, etc.).

Complexity of Some Major Operations: Let us first evaluate the computational cost of checking whether a point is on a quadratic surface using the measure given in Sect. 2.1. \mathbf{Q}_{DCGA} has 10 basis bivector components in total. For each basis bivector, at most 3 inner products (bivector \wedge bivector) are required. The number of point components is 25. Thus, the product $\mathbf{Q}_{\text{DCGA}} \cdot \mathbf{X}$ requires $25 \times 3 \times 10 = 750$ products.

Now we detail the cost of the computation of the tangent plane to a quadratic surface, defined in [9] as:

$$\mathbf{II} = (\mathbf{n}_1 + d\mathbf{e}_{\infty 1}) \wedge (\mathbf{n}_2 + d\mathbf{e}_{\infty 2}). \tag{2}$$

The normal vector is defined as the commutator product of some differential operators and the quadratic surface resulting in a 7-component bivector. Each inner product with \mathbf{X} then has the cost of $7 \times 25 = 175$ products. This latter computation is repeated for each axis; thus, this results in $175 \times 3 = 525$ products. The computation of the distance d , on the other hand, consists of merely 3 inner products. Both operands in this equation are 4-components 1-vector. Thus, the computational cost of the outer product is $4 \times 4 = 16$. Hence, the total cost of the computation of the tangent plane is $525 + 16 = 541$ products. The third operation is the intersection between a quadratic surface and a line. This computation is, unfortunately, not defined in DCGA.

2.3 DPGA of $\mathbb{G}_{4,4}$

DPGA was adapted from the approach of Parkin [18] in 2012 and firstly introduced in 2015 by Goldman and Mann [11] and further developed by Du and Goldman and Mann [8]. DPGA is defined over a 8-dimensional vector space. Similarly to DCGA, the base vectors of the space are divided into two groups: $\{\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, corresponding to the projective geometric algebra vectors, and a copy of this basis $\{\mathbf{w}_0^*, \mathbf{w}_1^*, \mathbf{w}_2^*, \mathbf{w}_3^*\}$ such that $\mathbf{w}_i \mathbf{w}_i^* = 0.5 + \mathbf{w}_i \wedge \mathbf{w}_i^*$, $\forall i \in \{0, 1, 2, 3\}$. In DPGA, the entity representing a point whose Euclidean coordinates are (x, y, z) has two definitions, namely, primal and dual. Both definitions are the base to construct quadratic surfaces by means of the sandwiching product. The definitions of the points are:

$$\mathbf{p} = x\mathbf{w}_0 + y\mathbf{w}_1 + z\mathbf{w}_2 + w\mathbf{w}_3, \quad \mathbf{p}^* = x\mathbf{w}_0^* + y\mathbf{w}_1^* + z\mathbf{w}_2^* + w\mathbf{w}_3^*. \tag{3}$$

Note that the dual definition denotes the fact that

$$\mathbf{w}_i \cdot \mathbf{w}_j^* = \frac{1}{2} \delta_{i,j} \quad (\forall i, j = 0, \dots, 3), \tag{4}$$

where $\delta_{i,j} = 1$ if $i = j$, 0 otherwise. This corresponds to the condition of the dual stated in Sect. 11 of [4].

Quadratic Surfaces: A quadratic surface in DPGA is the bivector Q_{DPGA} defined as follows:

$$\begin{aligned} Q_{\text{DPGA}} = & 4a\mathbf{w}_0^* \wedge \mathbf{w}_0 + 4b\mathbf{w}_1^* \wedge \mathbf{w}_1 + 4c\mathbf{w}_2^* \wedge \mathbf{w}_2 + 4j\mathbf{w}_3^* \wedge \mathbf{w}_3 \\ & + 2d(\mathbf{w}_0^* \wedge \mathbf{w}_1 + \mathbf{w}_1^* \wedge \mathbf{w}_0) + 2e(\mathbf{w}_0^* \wedge \mathbf{w}_2 + \mathbf{w}_2^* \wedge \mathbf{w}_0) \\ & + 2f(\mathbf{w}_1^* \wedge \mathbf{w}_2 + \mathbf{w}_2^* \wedge \mathbf{w}_1) + 2g(\mathbf{w}_0^* \wedge \mathbf{w}_3 + \mathbf{w}_3^* \wedge \mathbf{w}_0) \\ & + 2h(\mathbf{w}_1^* \wedge \mathbf{w}_3 + \mathbf{w}_3^* \wedge \mathbf{w}_1) + 2i(\mathbf{w}_2^* \wedge \mathbf{w}_3 + \mathbf{w}_3^* \wedge \mathbf{w}_2). \end{aligned} \tag{5}$$

A point (x, y, z) is in the quadratic surface Q_{DPGA} if and only if

$$\mathbf{p} \cdot Q_{\text{DPGA}} \cdot \mathbf{p}^* = 0. \tag{6}$$

Table 1 summarises the computations involved in three main operations used for computer graphics.

Table 1. Formulas of DPGA involved in the main computations for computer graphics

Feature	DPGA
Point is on a quadratic surface	$\mathbf{p} \cdot Q_{\text{DPGA}} \cdot \mathbf{p}^*$
Tangent plane	$Q_{\text{DPGA}} \cdot \mathbf{p}^*$
Quadratic surface-line intersection	$(\mathbf{L}^* \wedge Q_{\text{DPGA}} \wedge \mathbf{L}) \cdot \mathbf{I}$

Complexity of Some Major Operations: Q_{DPGA} has a total of 16 basis bivector components. For each basis bivector, 2 inner products are required. Thus, the first product $\mathbf{p} \cdot Q_{\text{DPGA}}$ requires $4 \times 2 \times 16 = 128$ inner products. As previously seen, the resulting entity is a vector with 4 components. Hence, the second inner product requires $4 \times 4 = 16$ products. This results in 144 products in total.

Let us now evaluate the cost of the intersection between a quadratic surface Q_{DPGA} and a line \mathbf{L} and \mathbf{L}^* . The line \mathbf{L}^* is obtained by the outer product of two points \mathbf{x}_1 and \mathbf{x}_2 whose number of components is 4. Thus, a line \mathbf{L} has 6 components. The number of components of the quadratic surface is 16 and the number of components of the line is 6. Then, the computational cost of the outer product $\mathbf{L}^* \wedge Q_{\text{DPGA}}$ is $6 \times 16 = 96$ outer products. The result is a 4-vector and the resulting entity has 16 components. Furthermore, the line \mathbf{l} has 6 components. Hence, the cost of the final outer product is $16 \times 6 = 96$ outer products. The total operation cost is thus $96 + 96 = 192$ products.

Considering the fact that the number of components of \mathbf{p}^* is 4 and the number of components of Q_{DPGA} is 16, the computational cost of the computation of the tangent plane is $16 \times 4 = 64$ products.

2.4 QCGA of $\mathbb{G}_{9,6}$

QCGA was presented by Breuils et al. [2]. The base vectors are composed of Euclidean basis vectors and the 12 null basis vectors $\{\mathbf{e}_{oi}, \mathbf{e}_{\infty i}\}, i = 1 \dots 6$. Firstly, a point is defined as the 12-component vector:

$$\mathbf{x} = \mathbf{x}_\epsilon + \frac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2} + z^2\mathbf{e}_{\infty 3}) + xy\mathbf{e}_{\infty 4} + xz\mathbf{e}_{\infty 5} + yz\mathbf{e}_{\infty 6} + \mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3}. \tag{7}$$

Secondly, the definition of a quadratic surface in QCGA is a 1-vector, called \mathbf{Q}^* , which has also a total of 12 basis vector components as:

$$\begin{aligned} \mathbf{q}^* = & -(2a\mathbf{e}_{o1} + 2b\mathbf{e}_{o2} + 2c\mathbf{e}_{o3} + d\mathbf{e}_{o4} + e\mathbf{e}_{o5} + f\mathbf{e}_{o6}) \\ & + (g\mathbf{e}_1 + h\mathbf{e}_2 + i\mathbf{e}_3) - \frac{j}{3}(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 3}). \end{aligned} \tag{8}$$

Let us evaluate the computational cost of checking whether a point is on a quadratic surface. Furthermore, the number of point component is 12. Thus, the product $\mathbf{x} \cdot \mathbf{Q}^*$ requires at most $12 \times 12 = 144$ products.

The computation of the tangent plane is performed by firstly computing the normal vector. This computation requires the inner product between a vector with 12 components and another vector with 4 components. This is repeated for each Euclidean basis vector; thus, the computation of the normal vector requires $3 \times 4 \times 12 = 144$ inner products.

Then, the tangent plane is computed using the normal vector as follows:

$$\boldsymbol{\pi}^* = \mathbf{n}_\epsilon + \frac{1}{3}(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 3})\sqrt{-2(\mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3}) \cdot \mathbf{x}}. \tag{9}$$

This computation requires the computation of an inner product of a vector with 3 components ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) with a 12 component-vector. This means $12 \times 3 = 36$ products. Thus, the total number of inner products required for computing the tangent plane is $144 + 36 = 180$ products.

The final computational feature is the quadratic surface-line intersection. In QCGA, this simply consists of computing the outer product:

$$\mathbf{C}^* = \mathbf{Q}^* \wedge \mathbf{L}^* \tag{10}$$

The number of components of \mathbf{Q}^* is 12 as already seen. In QCGA, a line with the 6 Plücker coefficients is defined as:

$$\mathbf{L}^* = 3\mathbf{m}\mathbf{I}_\epsilon + (\mathbf{e}_{\infty 3} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1}) \wedge \mathbf{n}\mathbf{I}_\epsilon. \tag{11}$$

The number of components of both \mathbf{m} and \mathbf{n} is 3. The outer product $(\mathbf{e}_{\infty 3} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1}) \wedge \mathbf{n}\mathbf{I}_\epsilon$ yields a copy of the 3 components of \mathbf{n} along $\mathbf{e}_{\infty 1}, \mathbf{e}_{\infty 2}, \mathbf{e}_{\infty 3}$ basis vectors. Thus, the number of components of \mathbf{L}^* is $3 \times 3 + 3 = 12$. The cost of the outer product between \mathbf{Q}^* and \mathbf{L}^* is thus $12 \times 12 = 144$ products.

Table 2 summarises the complexity of DPGA, DCGA, and QCGA for computing features. We remark that the computation of the tangent plane is more efficient if we use DPGA whereas the intersection between a quadratic surface and a line requires less computations if we use QCGA. Furthermore, geometric transformations are not yet defined in QCGA.

Table 2. Numbers of operations required for computation in DPGA, DCGA, and QCGA.

Feature	DPGA	DCGA	QCGA
Point is on a quadratic surface	144	750	144
Tangent plane	64	541	180
Quadratic surface-line intersection	192	—	144

3 Mapping Between the Three Models for Quadratic Surfaces

As one of practical applications, we consider constructing a quadratic surface from 9 points then rotating this quadratic surface. To the best of our knowledge, QCGA is the only approach that can construct a quadratic surface from 9 points. But QCGA does not yet support all the transformations. Furthermore, as seen above, it is more computationally efficient to perform the quadratic surface-line intersection in the QCGA model whereas the computation of the tangent plane or the normal vector at a point of quadratic surface is more efficient in the DPGA model. Moreover, if we represent both a Dupin cyclide and a quadratic surface in a same way as [7], then we need DCGA. The above observation is our motivation for defining new operators that convert quadratic surfaces between the three models, see Fig. 1.

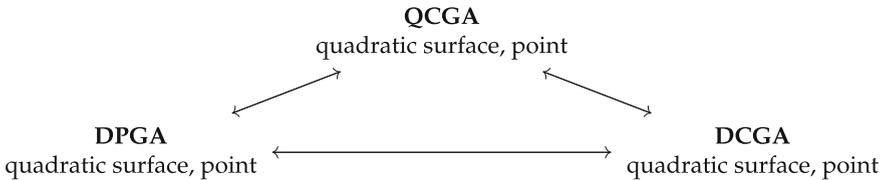


Fig. 1. Encapsulation of the three models of points and quadratic surfaces.

The key idea is that for any entities representing a quadratic surface in QCGA, DCGA, and DPGA, we convert the entity such that all the coefficients of the quadratic surface:

$$ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + iz + j = 0 \tag{12}$$

can be extracted easily.

3.1 DCGA Reciprocal Operators

We start by defining reciprocal operators for DCGA:

$$\mathbf{T}^{x^2} = \mathbf{e}_1 \wedge \mathbf{e}_4, \quad \mathbf{T}^{y^2} = \mathbf{e}_2 \wedge \mathbf{e}_5, \quad \mathbf{T}^{z^2} = \mathbf{e}_3 \wedge \mathbf{e}_6, \quad \mathbf{T}^1 = \mathbf{e}_{o1} \wedge \mathbf{e}_{o2}, \tag{13}$$

along with the 6 following:

$$\begin{aligned} \mathbf{T}^x &= \left(\mathbf{e}_1 \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_4 \right), \quad \mathbf{T}^y = \left(\mathbf{e}_2 \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_5 \right), \\ \mathbf{T}^z &= \left(\mathbf{e}_3 \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_6 \right), \quad \mathbf{T}^{xy} = \left(\mathbf{e}_1 \wedge \mathbf{e}_5 + \mathbf{e}_2 \wedge \mathbf{e}_4 \right), \\ \mathbf{T}^{xz} &= \left(\mathbf{e}_1 \wedge \mathbf{e}_6 + \mathbf{e}_3 \wedge \mathbf{e}_4 \right), \quad \mathbf{T}^{yz} = \left(\mathbf{e}_3 \wedge \mathbf{e}_5 + \mathbf{e}_2 \wedge \mathbf{e}_6 \right). \end{aligned} \tag{14}$$

Given the DCGA extraction operators presented in Sect. 2.2, our defined reciprocal operators verify the following properties:

$$\begin{aligned} \mathbf{T}^{x^2} \cdot \mathbf{T}_{x^2} &= 1, \quad \mathbf{T}^{y^2} \cdot \mathbf{T}_{y^2} = 1, \quad \mathbf{T}^{z^2} \cdot \mathbf{T}_{z^2} = 1, \quad \mathbf{T}^{xy} \cdot \mathbf{T}_{xy} = 1, \quad \mathbf{T}^{xz} \cdot \mathbf{T}_{xz} = 1, \\ \mathbf{T}^{yz} \cdot \mathbf{T}_{yz} &= 1, \quad \mathbf{T}^x \cdot \mathbf{T}_x = 1, \quad \mathbf{T}^y \cdot \mathbf{T}_y = 1, \quad \mathbf{T}^z \cdot \mathbf{T}_z = 1, \quad \mathbf{T}^1 \cdot \mathbf{T}_1 = 1. \end{aligned} \tag{15}$$

Then, given \mathbf{Q}_{DCGA} , the entity representing a quadratic surface of DCGA, any coefficients of this quadratic surface (12) can be extracted as:

$$\begin{aligned} \mathbf{T}^{x^2} \cdot \mathbf{Q}_{\text{DCGA}} &= a, \quad \mathbf{T}^{y^2} \cdot \mathbf{Q}_{\text{DCGA}} = b, \quad \mathbf{T}^{z^2} \cdot \mathbf{Q}_{\text{DCGA}} = c, \quad \mathbf{T}^{xy} \cdot \mathbf{Q}_{\text{DCGA}} = d, \\ \mathbf{T}^{xz} \cdot \mathbf{Q}_{\text{DCGA}} &= e, \quad \mathbf{T}^{yz} \cdot \mathbf{Q}_{\text{DCGA}} = f, \quad \mathbf{T}^x \cdot \mathbf{Q}_{\text{DCGA}} = g, \quad \mathbf{T}^y \cdot \mathbf{Q}_{\text{DCGA}} = h, \\ \mathbf{T}^z \cdot \mathbf{Q}_{\text{DCGA}} &= i, \quad \mathbf{T}^1 \cdot \mathbf{Q}_{\text{DCGA}} = j. \end{aligned} \tag{16}$$

The construction of a DCGA point is explained in Sect. 2.2 and defined in [9]. The reciprocal operation requires the computation of the normalized point $\hat{\mathbf{X}}$ of DCGA that we define as:

$$\hat{\mathbf{X}} = -\frac{\mathbf{X}}{\mathbf{X} \cdot (\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2})}. \tag{17}$$

The extraction of the Euclidean components (x, y, z) of a normalized point $\hat{\mathbf{X}}$ of DCGA can be performed as follows:

$$x = \hat{\mathbf{X}} \cdot (\mathbf{e}_1 \wedge \mathbf{e}_{\infty 2}), \quad y = \hat{\mathbf{X}} \cdot (\mathbf{e}_2 \wedge \mathbf{e}_{\infty 2}), \quad z = \hat{\mathbf{X}} \cdot (\mathbf{e}_3 \wedge \mathbf{e}_{\infty 2}). \tag{18}$$

3.2 DPGA Reciprocal Operators

Let us denote by \mathbf{W} reciprocal operators for DPGA:

$$\begin{aligned} \mathbf{W}^{x^2} &= \mathbf{w}_0^* \wedge \mathbf{w}_0, \quad \mathbf{W}^{y^2} = \mathbf{w}_1^* \wedge \mathbf{w}_1, \quad \mathbf{W}^{z^2} = \mathbf{w}_2^* \wedge \mathbf{w}_2, \quad \mathbf{W}^{xy} = 2\mathbf{w}_1^* \wedge \mathbf{w}_0, \\ \mathbf{W}^{xz} &= 2\mathbf{w}_2^* \wedge \mathbf{w}_0, \quad \mathbf{W}^{yz} = 2\mathbf{w}_2^* \wedge \mathbf{w}_1, \quad \mathbf{W}^x = 2\mathbf{w}_3^* \wedge \mathbf{w}_0, \quad \mathbf{W}^y = 2\mathbf{w}_3^* \wedge \mathbf{w}_1, \\ \mathbf{W}^z &= 2\mathbf{w}_3^* \wedge \mathbf{w}_2, \quad \mathbf{W}^1 = \mathbf{w}_3^* \wedge \mathbf{w}_3. \end{aligned} \tag{19}$$

Given Q_{DPGA} , the entity representing a quadratic surface of DPGA, any coefficients of this quadratic surface (12) can be extracted as:

$$\begin{aligned} \mathbf{W}^{x^2} \cdot Q_{\text{DPGA}} &= a, \quad \mathbf{W}^{y^2} \cdot Q_{\text{DPGA}} = b, \quad \mathbf{W}^{z^2} \cdot Q_{\text{DPGA}} = c, \quad \mathbf{W}^{xy} \cdot Q_{\text{DPGA}} = d, \\ \mathbf{W}^{xz} \cdot Q_{\text{DPGA}} &= e, \quad \mathbf{W}^{yz} \cdot Q_{\text{DPGA}} = f, \quad \mathbf{W}^x \cdot Q_{\text{DPGA}} = g, \quad \mathbf{W}^y \cdot Q_{\text{DPGA}} = h, \\ \mathbf{W}^z \cdot Q_{\text{DPGA}} &= i, \quad \mathbf{W}^1 \cdot Q_{\text{DPGA}} = j. \end{aligned} \tag{20}$$

As in projective geometry, the construction of a finite point of DPGA requires to add a homogeneous component 1 to the Euclidean components. The normalization of a point merely consists of dividing all the components by its \mathbf{w}_3 components (or \mathbf{w}_3^* for the dual form) if it is a non-zero component.

3.3 QCGA Reciprocal Operators

For QCGA, quadratic surfaces can be represented using either the primal form or the dual form. We define the reciprocal operators for the dual form. When considering the primal form, we have only to compute the dual of the primal and then apply the following reciprocal operators:

$$\begin{aligned}
 \mathbf{Q}^{x^2} &= \frac{1}{2}\mathbf{e}_{o1}, & \mathbf{Q}^{y^2} &= \frac{1}{2}\mathbf{e}_{o2}, & \mathbf{Q}^{z^2} &= \frac{1}{2}\mathbf{e}_{o3}, & \mathbf{Q}^{xy} &= \mathbf{e}_{o4}, \\
 \mathbf{Q}^{xz} &= \mathbf{e}_{o5}, & \mathbf{Q}^{yz} &= \mathbf{e}_{o6}, & \mathbf{Q}^x &= \mathbf{e}_1, & \mathbf{Q}^y &= \mathbf{e}_2, \\
 \mathbf{Q}^z &= \mathbf{e}_3, & \mathbf{Q}^1 &= \mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3}.
 \end{aligned}
 \tag{21}$$

Given a general quadratic surface \mathbf{Q}^* whose coefficients are (a, b, c, \dots, j) , the properties of these operators are as follows:

$$\begin{aligned}
 \mathbf{Q}^{x^2} \cdot \mathbf{Q}^* &= a, & \mathbf{Q}^{y^2} \cdot \mathbf{Q}^* &= b, & \mathbf{Q}^{z^2} \cdot \mathbf{Q}^* &= c, & \mathbf{Q}^{xy} \cdot \mathbf{Q}^* &= d, & \mathbf{Q}^{xz} \cdot \mathbf{Q}^* &= e, \\
 \mathbf{Q}^{yz} \cdot \mathbf{Q}^* &= f, & \mathbf{Q}^x \cdot \mathbf{Q}^* &= g, & \mathbf{Q}^y \cdot \mathbf{Q}^* &= h, & \mathbf{Q}^z \cdot \mathbf{Q}^* &= i, & \mathbf{Q}^1 \cdot \mathbf{Q}^* &= j.
 \end{aligned}
 \tag{22}$$

The reciprocal operation requires the computation of the normalized point $\hat{\mathbf{x}}$ of QCGA, which is missing in [2].

Proposition 3.1. *For a QCGA point \mathbf{x} , the normalization is merely computed through an averaging of $\mathbf{e}_{o1}, \mathbf{e}_{o2}, \mathbf{e}_{o3}$ components of \mathbf{e}_o component as:*

$$-\frac{\mathbf{x}}{\mathbf{x} \cdot \mathbf{e}_{\infty}}.
 \tag{23}$$

Proof. A scale α on \mathbf{x} acts the same way on all null basis vectors of \mathbf{x} :

$$\begin{aligned}
 \alpha\mathbf{x} &= \alpha\mathbf{x}_{\epsilon} + \frac{1}{2}\alpha(x^2\mathbf{e}_{\infty1} + y^2\mathbf{e}_{\infty2} + z^2\mathbf{e}_{\infty3}) + xy\alpha\mathbf{e}_{\infty4} + xz\alpha\mathbf{e}_{\infty5} + yz\alpha\mathbf{e}_{\infty6} \\
 &\quad + \alpha\mathbf{e}_{o1} + \alpha\mathbf{e}_{o2} + \alpha\mathbf{e}_{o3}.
 \end{aligned}
 \tag{24}$$

The metric of QCGA indicates (see [2]):

$$\alpha\mathbf{x} \cdot \mathbf{e}_{\infty1} = -\alpha, \quad \alpha\mathbf{x} \cdot \mathbf{e}_{\infty2} = -\alpha, \quad \alpha\mathbf{x} \cdot \mathbf{e}_{\infty3} = -\alpha.
 \tag{25}$$

Thus, if $\alpha \neq 0$:

$$\begin{aligned}
 \frac{-3\alpha\mathbf{x}}{\alpha\mathbf{x} \cdot (\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3})} \cdot \mathbf{e}_{\infty1} &= \frac{-3\alpha\mathbf{x}}{-3\alpha} \cdot \mathbf{e}_{\infty1} \\
 &= \mathbf{x} \cdot \mathbf{e}_{\infty1} = -1.
 \end{aligned}
 \tag{26}$$

A similar result is obtained with $\mathbf{e}_{\infty2}$ and $\mathbf{e}_{\infty3}$:

$$\frac{-3\alpha\mathbf{x}}{\alpha\mathbf{x} \cdot (\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3})} \cdot \mathbf{e}_{\infty2} = \mathbf{x} \cdot \mathbf{e}_{\infty2} = -1.
 \tag{27}$$

$$\frac{-3\alpha\mathbf{x}}{\alpha\mathbf{x} \cdot (\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3})} \cdot \mathbf{e}_{\infty3} = \mathbf{x} \cdot \mathbf{e}_{\infty3} = -1.
 \tag{28}$$

Thus, we check that for any scaled points $\mathbf{x}_1, \mathbf{x}_2$:

$$\frac{\mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{e}_\infty} \cdot \frac{\mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{e}_\infty} = -\frac{1}{2} \|\mathbf{x}_{1\epsilon} - \mathbf{x}_{2\epsilon}\|^2. \tag{29}$$

The extraction of the Euclidean components (x, y, z) of a normalized point $\hat{\mathbf{x}}$ of QCGA can be performed as follows:

$$x = \hat{\mathbf{x}} \cdot \mathbf{e}_1, \quad y = \hat{\mathbf{x}} \cdot \mathbf{e}_2, \quad z = \hat{\mathbf{x}} \cdot \mathbf{e}_3. \tag{30}$$

3.4 How to Choose the Right Model?

Given a geometric operation and this general framework, a question arises that which model we should choose among QCGA, DPGA, and DCGA. To answer this question, we merely consider two criteria, namely, (1) if the operation is defined and (2) on which model it is the most computationally efficient. This is illustrated in Tables 2 and 3.

Table 3. Geometric operations allowed in either QCGA, DPGA or DCGA, where \checkmark means possible and \times means not.

Opération	DPGA	DCGA	QCGA
Quadratic surface from control points	\times	\times	\checkmark
Point \in quadratic surface	\checkmark	\checkmark	\checkmark
Tangent plane	\checkmark	\checkmark	\checkmark
Quadratic surface-line intersection	\checkmark	\times	\checkmark
Quadratic surface-quadratic surface intersection	\times	\times	\checkmark
Transformations	\checkmark	\checkmark	\times
Quartic surfaces	\times	\checkmark	\times

3.5 Example

We test our approach by defining an ellipsoid from 9 points using QCGA. Then we rotate it using DPGA and back-convert the rotated ellipsoid into QCGA. In terms of geometric algebra computations, first we compute the quadratic surface:

$$\mathbf{Q}^* = (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \dots \wedge \mathbf{x}_9 \wedge \mathbf{I}_o^{\triangleright})^*. \tag{31}$$

Then, we apply the extraction operators of QCGA to convert the QCGA quadratic surface to its corresponding DPGA quadratic surface.

$$\begin{aligned}
 Q_{\text{DPGA}} = & 4(\mathbf{Q}^{x^2} \cdot \mathbf{Q}^*)\mathbf{w}_0^* \wedge \mathbf{w}_0 + 4(\mathbf{Q}^{y^2} \cdot \mathbf{Q}^*)\mathbf{w}_1^* \wedge \mathbf{w}_1 + 4(\mathbf{Q}^{z^2} \cdot \mathbf{Q}^*)\mathbf{w}_2^* \wedge \mathbf{w}_2 \\
 & + 4(\mathbf{Q}^1 \cdot \mathbf{Q}^*)\mathbf{w}_3^* \wedge \mathbf{w}_3 + 2(\mathbf{Q}^{xy} \cdot \mathbf{Q}^*)(\mathbf{w}_0^* \wedge \mathbf{w}_1 + \mathbf{w}_1^* \wedge \mathbf{w}_0) \\
 & + 2(\mathbf{Q}^{xz} \cdot \mathbf{Q}^*)(\mathbf{w}_0^* \wedge \mathbf{w}_2 + \mathbf{w}_2^* \wedge \mathbf{w}_0) + 2(\mathbf{Q}^{yz} \cdot \mathbf{Q}^*)(\mathbf{w}_1^* \wedge \mathbf{w}_2 + \mathbf{w}_2^* \wedge \mathbf{w}_1) \\
 & + 2(\mathbf{Q}^x \cdot \mathbf{Q}^*)(\mathbf{w}_0^* \wedge \mathbf{w}_3 + \mathbf{w}_3^* \wedge \mathbf{w}_0) + 2(\mathbf{Q}^y \cdot \mathbf{Q}^*)(\mathbf{w}_1^* \wedge \mathbf{w}_3 + \mathbf{w}_3^* \wedge \mathbf{w}_1) \\
 & + 2(\mathbf{Q}^z \cdot \mathbf{Q}^*)(\mathbf{w}_2^* \wedge \mathbf{w}_3 + \mathbf{w}_3^* \wedge \mathbf{w}_2).
 \end{aligned}
 \tag{32}$$

The rotation is now performed as follows:

$$Q_{\text{DPGA}} = \mathbf{R}Q_{\text{DPGA}}\mathbf{R}^{-1}.
 \tag{33}$$

The rotor \mathbf{R} is defined as:

$$\mathbf{R} = \exp\left(\frac{1}{2}\theta\mathbf{w}_i\mathbf{w}_j^*\right),
 \tag{34}$$

where $i \neq j$. The final step is to convert the resulting quadratic surface back into QCGA. It is merely computed using the QCGA extraction operators as follows:

$$\begin{aligned}
 \mathbf{Q}^* = & -(2(\mathbf{W}^{x^2} \cdot Q_{\text{DPGA}})\mathbf{e}_{o1} + 2(\mathbf{W}^{y^2} \cdot Q_{\text{DPGA}})\mathbf{e}_{o2} + 2(\mathbf{W}^{z^2} \cdot Q_{\text{DPGA}})\mathbf{e}_{o3} \\
 & + (\mathbf{W}^{xy} \cdot Q_{\text{DPGA}})\mathbf{e}_{o4} + (\mathbf{W}^{xz} \cdot Q_{\text{DPGA}})\mathbf{e}_{o5} + (\mathbf{W}^{yz} \cdot Q_{\text{DPGA}})\mathbf{e}_{o6}) \\
 & + ((\mathbf{W}^x \cdot Q_{\text{DPGA}})\mathbf{e}_1 + (\mathbf{W}^y \cdot Q_{\text{DPGA}})\mathbf{e}_2 + (\mathbf{W}^z \cdot Q_{\text{DPGA}})\mathbf{e}_3) \\
 & - \frac{(\mathbf{W}^1 \cdot Q_{\text{DPGA}})}{3}(\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3}).
 \end{aligned}
 \tag{35}$$

Note that the program can be found in the plugin folder of the git repository <https://git.renater.fr/garamon.git>.

4 Conclusion

In this paper, we focused on an approach to deal with quadratic surfaces. After presenting the main geometric algebras to represent and manipulate quadratic surfaces, we introduced an approach that transversely uses the main geometric algebras. This approach unifies all the models of geometric algebra into one more general approach that allows to represent and manipulate any quadratic surface either using control points or from the coefficients of its implicit form. For the following, we seek for a generalisation of this approach for the representation of quadratic and cubic surfaces. A potential drawback of the proposed approach is that the used algebras are all of high dimensions and, thus, are difficult to implement efficiently. The algebra generator Garamon [1] allows such efficient implementation and so, it will be interesting to compare the proposed approach to the usual way to represent and manipulate quadratic surfaces.

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4 Conclusion

In this paper we have described a technique that can be used to adapt the Sphere Tracing algorithm to render isosurfaces of elements in arbitrary Geometric Algebras. Avoiding parametrizations or algebra specific ray intersection formulas, the algorithm is readily applicable to even the most challenging algebras, and capable of producing high quality images in real time. It represents a significant increase in flexibility and applicability compared to the visualization methods in use today.

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