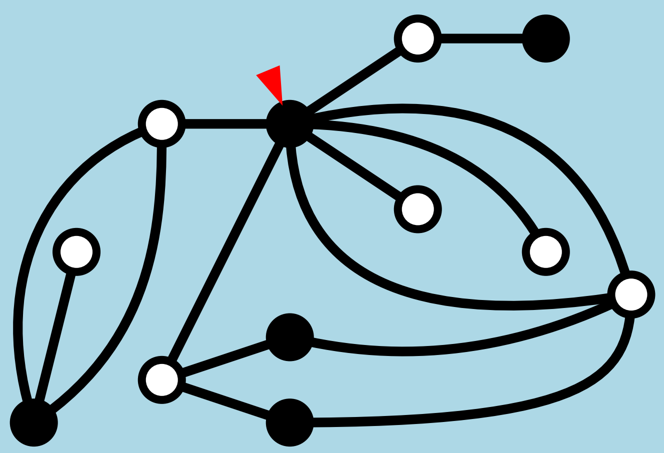
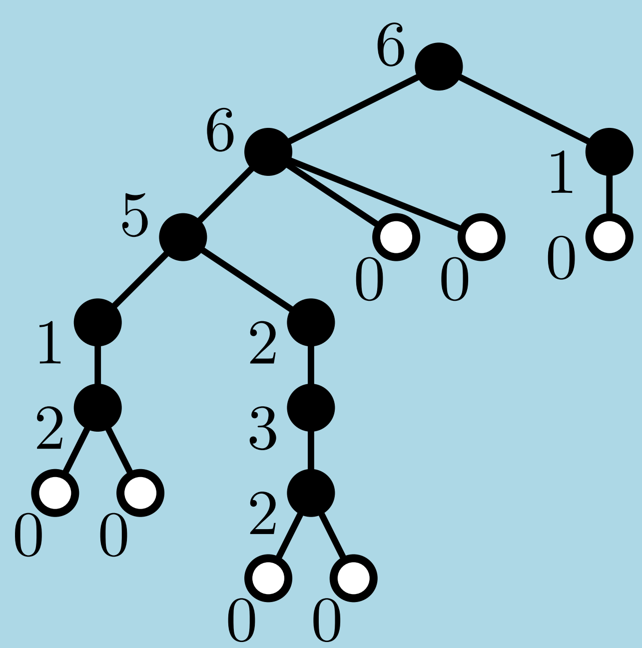
Bijective link between Chapoton's new intervals and bipartite planar maps

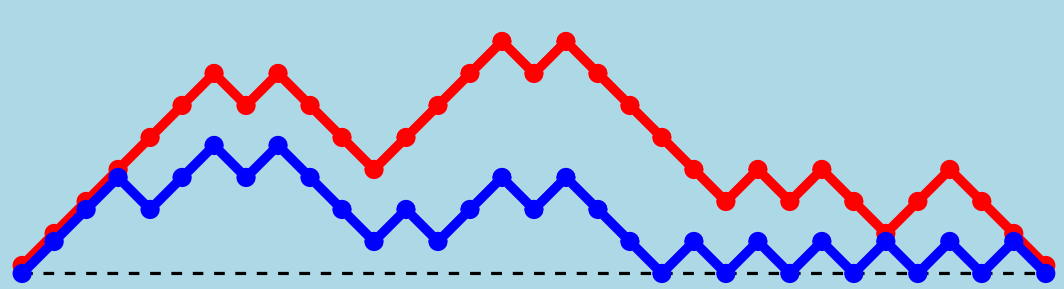
Wenjie Fang, LIGM, Univ. Gustave Eiffel, CNRS, ESIEE Paris, F-77454 Marne-la-Vallée, France



bipartite planar maps with n edges



degree trees with n edges



Chapoton's new interval of length $2n+2$

Introduction

In *Sur le nombre d'intervalles dans les treillis de Tamari* (*Sém. Lothar. Combin.*, B55f, 2006), Chapoton defined new intervals in the Tamari lattice, and gave the following counting formula:

$$\frac{3 \cdot 2^{n-2}(2n-2)!}{(n-1)!(n+1)!},$$

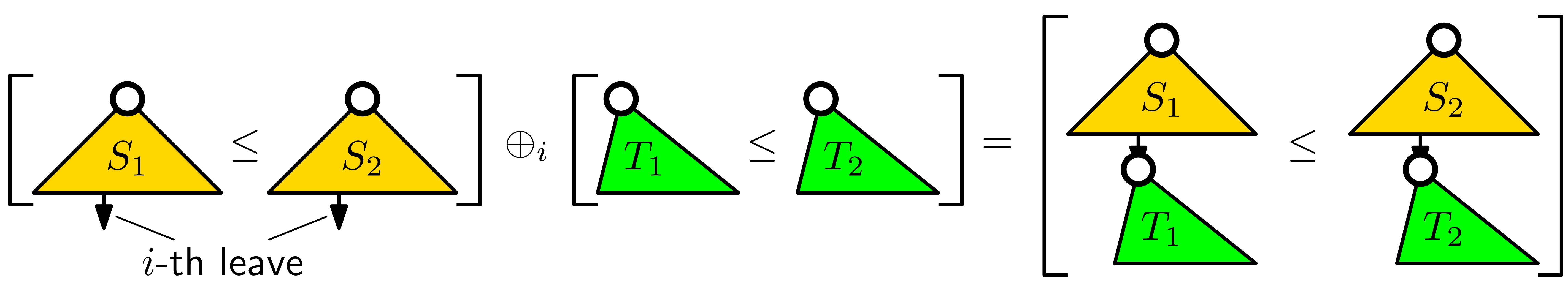
which also counts the number of bipartite planar maps with $n-1$ edges. See also OEIS A000257.

Chapoton and Fusy (unpublished) found a symmetry in three statistics on new intervals. They are equi-distributed as the number of black vertices, white vertices and faces in bipartite planar maps, three statistics well-known to be symmetric.

We found a bijection that naturally shows such correspondence, and also some further ones.

Chapoton's new intervals

As pairs of binary trees, some Tamari intervals are "compositions" of smaller Tamari intervals:



Tamari intervals without such "composition" are called new intervals.

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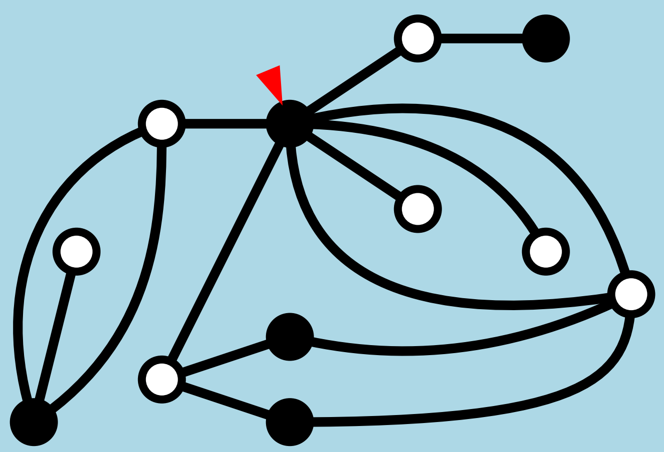

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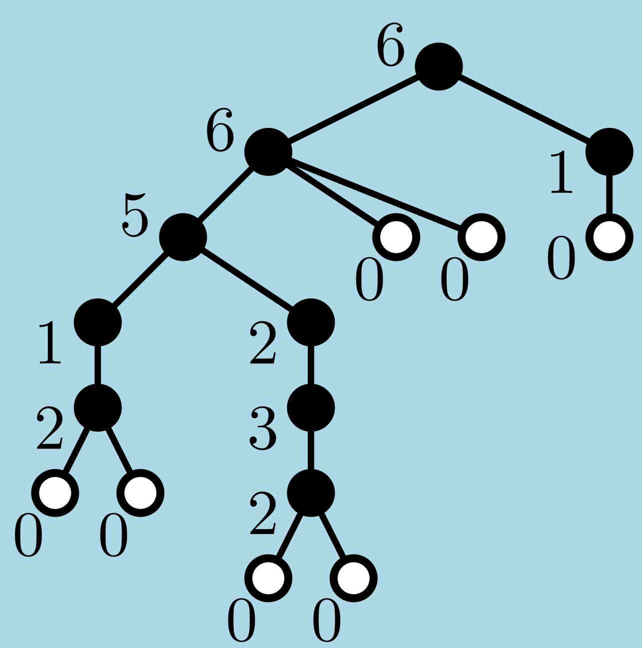

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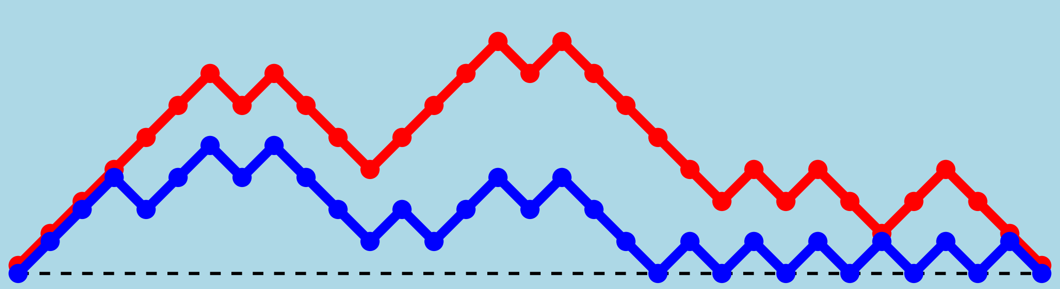

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Chapoton's new intervals (Dyck path version)

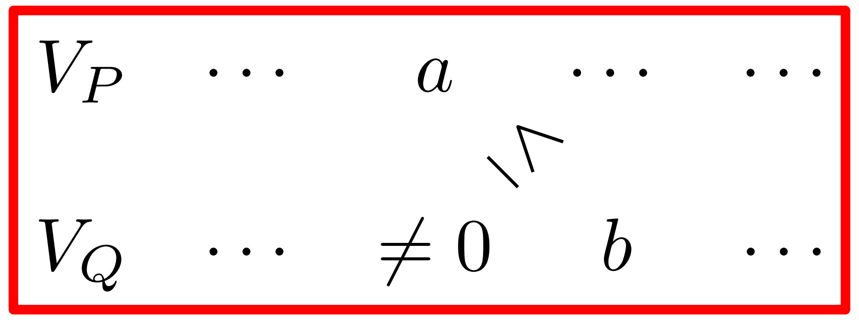
Bracket vector V_P of a Dyck path P:
$V_P(i)$ = half-distance of the i-th up-step to its matching down-step

$P \leq Q$ in the Tamari lattice $\Leftrightarrow V_P \leq V_Q$ componentwise

A Tamari interval $[P,Q]$ is a **new interval** if

(i) $V_Q(1) = n$;

(ii) For all $1 \leq i \leq n$, if $V_Q(i) > 0$, then $V_P(i) \leq V_Q(i+1)$.

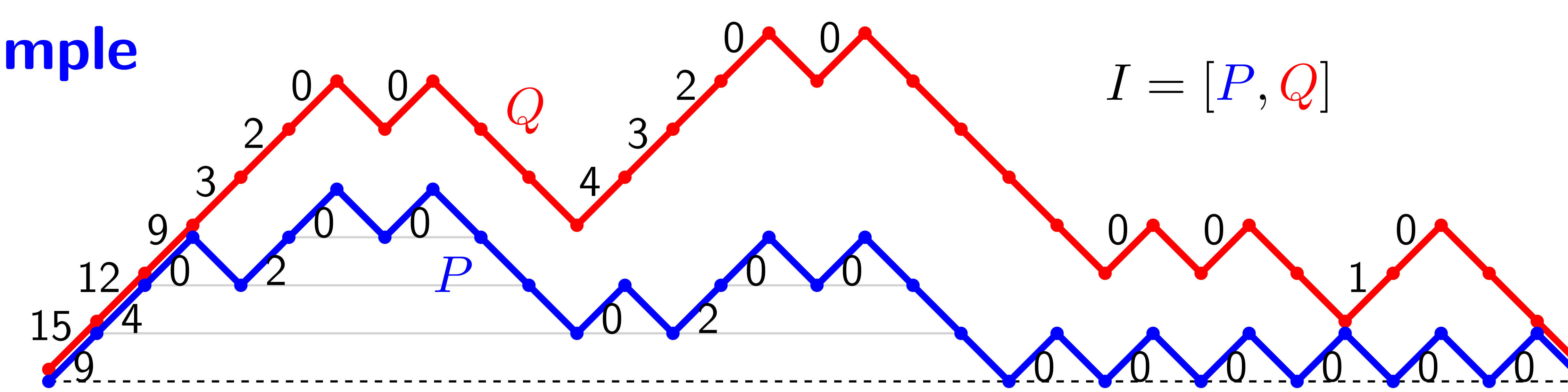


Three statistics for an interval $I = [P,Q]$:

$$c_{0,0}(I) = \# \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad c_{0,1}(I) = \# \begin{bmatrix} 0 \\ \neq 0 \end{bmatrix}, \quad c_{1,1}(I) = \# \begin{bmatrix} \neq 0 \\ \neq 0 \end{bmatrix}$$

Symmetry between $c_{0,0}(I), c_{0,1}(I), 1 + c_{1,1}(I)$ when summing over all new intervals of size n

Example



$I = [P, Q]$

$c_{0,0}(I) = 7$

$c_{0,1}(I) = 5$

$c_{1,1}(I) = 4$

$V_P = 9, 4, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0$

$V_Q = 15, 12, 9, 3, 2, 0, 0, 4, 3, 2, 0, 0, 0, 0, 1, 0$

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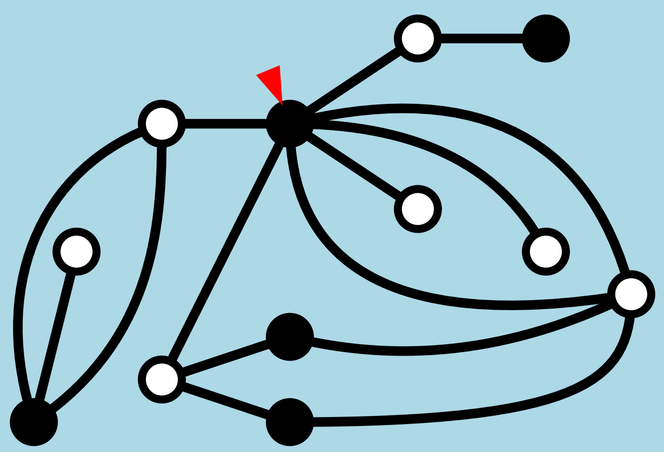
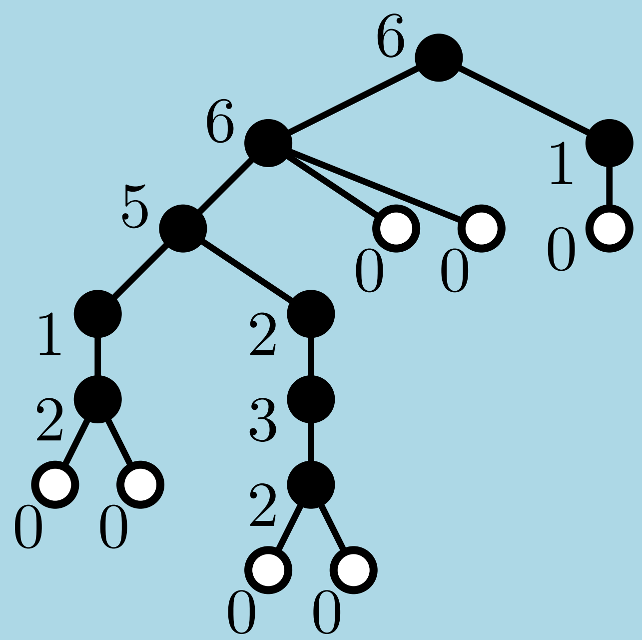
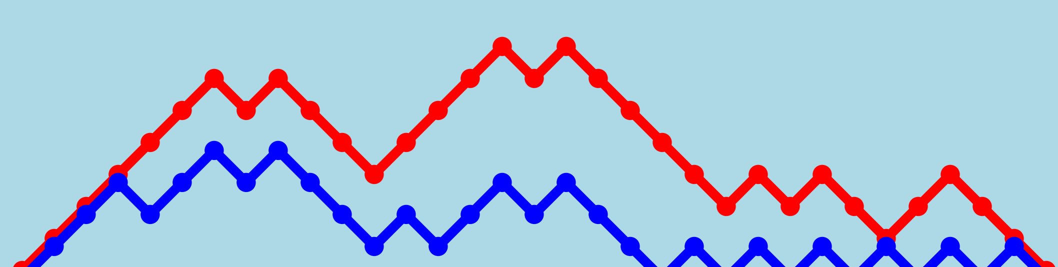


bipartite planar maps with n edges



degree trees with n edges



Chapoton's new interval of length $2n+2$

Bipartite planar maps

Drawings of bipartite graphs on the plane, **rooted** by choosing a corner on the outer face

Three statistics: black, white, face, symmetric on the set of bipartite maps with n edges

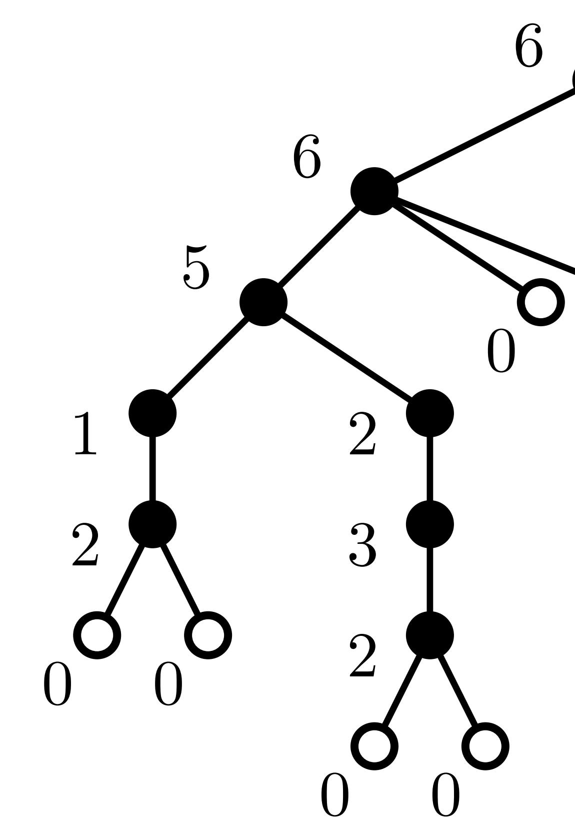
Degree trees

A **degree tree** is a pair (T, ℓ), with

- T: a rooted plane tree,
- ℓ: a labeling on nodes of T,

such that for any node v in T,

- v is a leaf $\Rightarrow \ell(v) = 0$;
- v not a leaf, with children $v_1, v_2, \ldots, v_k$
  $\Rightarrow \ell(v) = k - a + \sum_i \ell(v_i)$ for some $0 \leq a \leq \ell(v_1)$.

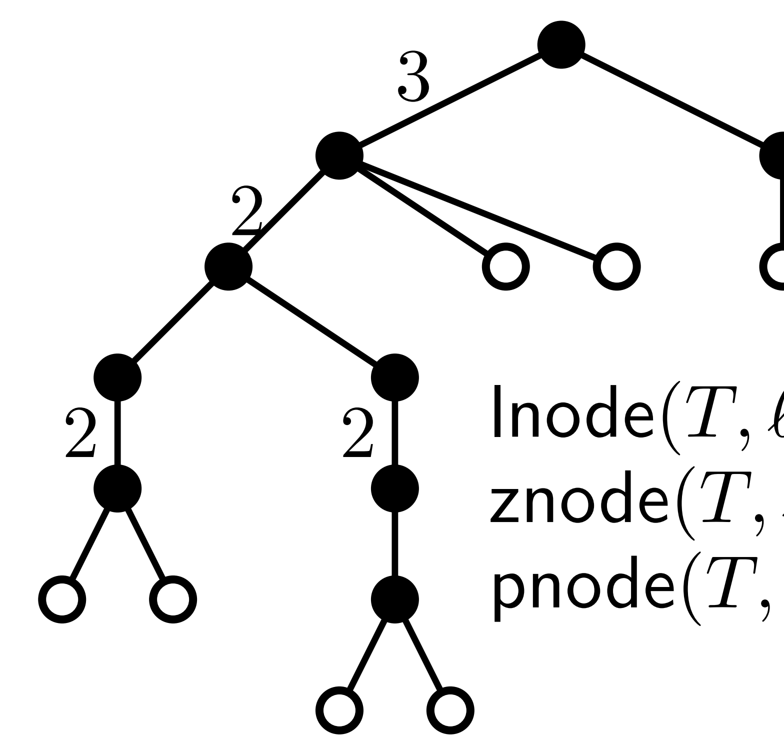


(T, ℓ)

Edge labeling ℓ_Λ of (T, ℓ): on the first descending edge of every node v, with value a used to obtain $\ell(v)$. Clear bijection $\ell \Leftrightarrow \ell_\Lambda$

Three statistics:

- $\mathrm{lnode}(T, \ell)$: #leaves,
- $\mathrm{znode}(T, \ell)$: #nodes with $\ell_\Lambda(e) = 0$ for its first down edge e,
- $\mathrm{pnode}(T, \ell)$: #nodes with $\ell_\Lambda(e) \neq 0$ for its first down edge e.



$\mathrm{lnode}(T, \ell) = 7$
$\mathrm{znode}(T, \ell) = 5$
$\mathrm{pnode}(T, \ell) = 4$

Edge labeling ℓ_Λ of (T, ℓ)

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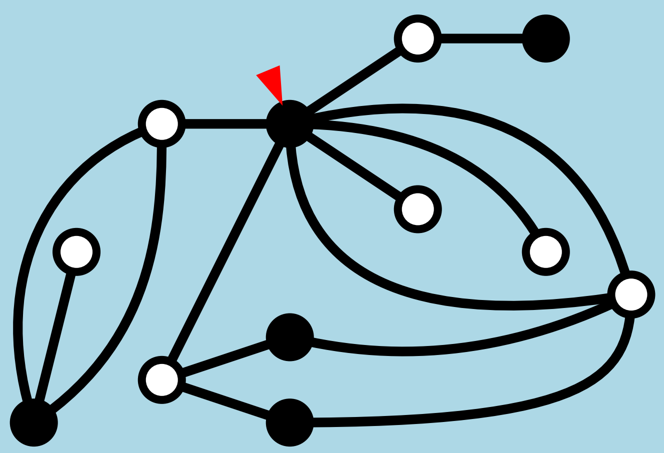
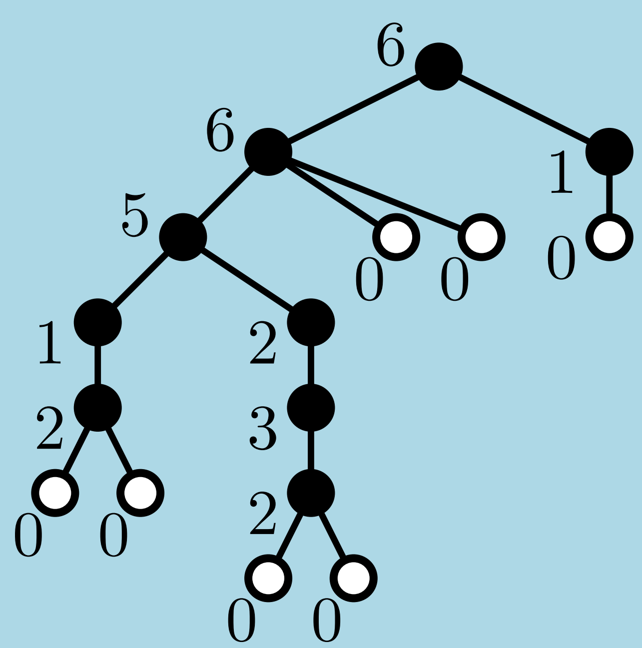
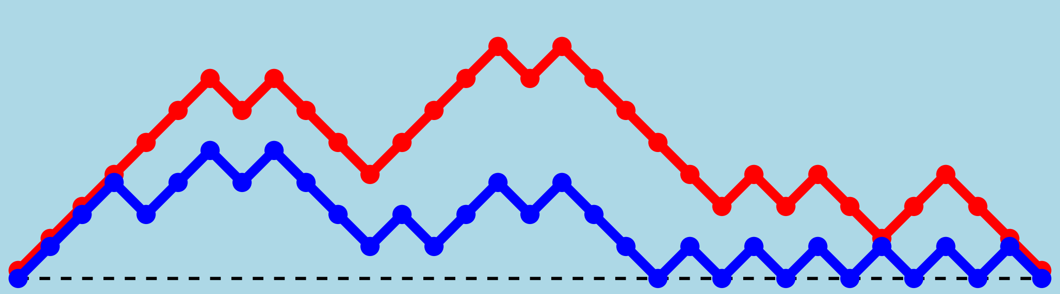


bipartite planar maps with n edges



degree trees with n edges

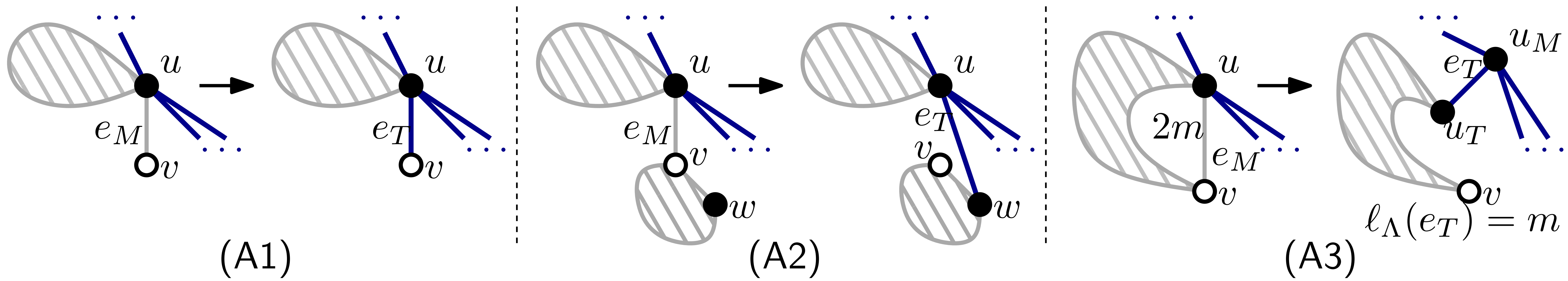


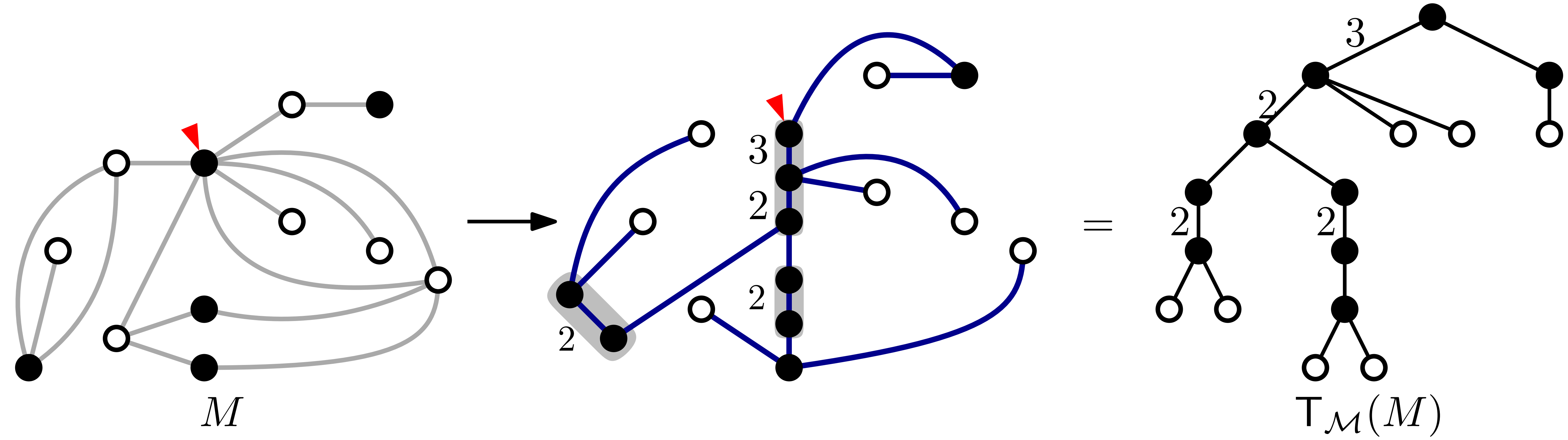
Chapoton's new interval of length $2n+2$

From maps to trees: an exploration

Depth-first exploration of edges, clockwise, starting from the root corner

Turning edges in M into edges in (T, ℓ). Walking only on black vertices, except for leaves.





Statistics correspondence

$$\mathrm{white}(M) = \mathrm{lnode}(T, \ell), \quad \mathrm{black}(M) = \mathrm{znode}(T, \ell), \quad \mathrm{face}(M) = 1 + \mathrm{pnode}(T, \ell).$$

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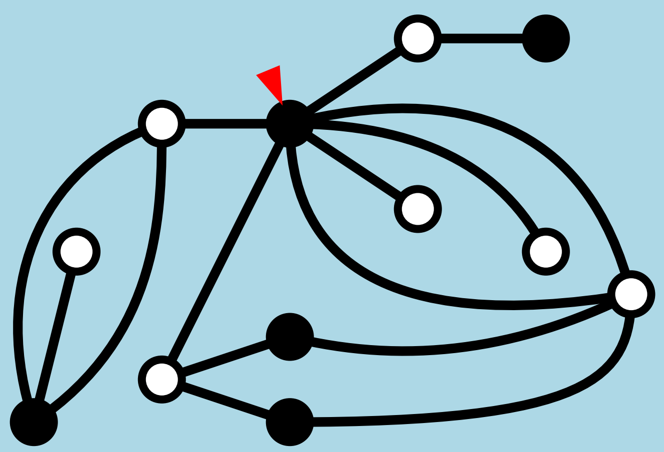
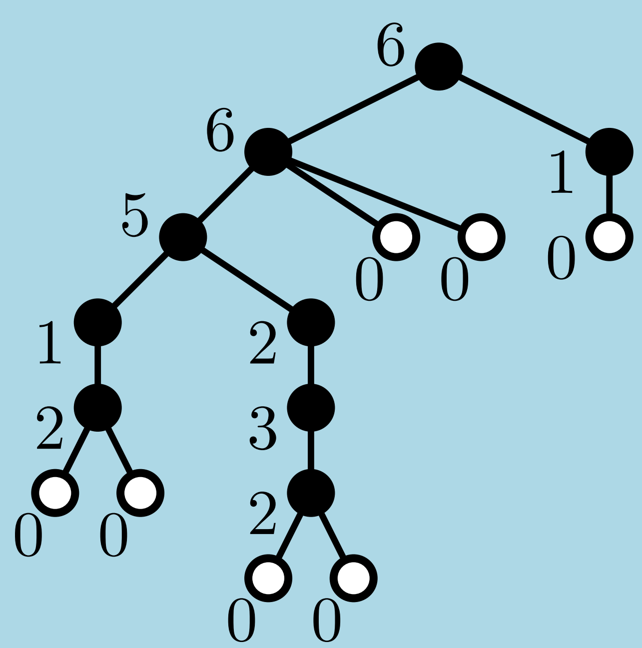


bipartite planar maps with n edges



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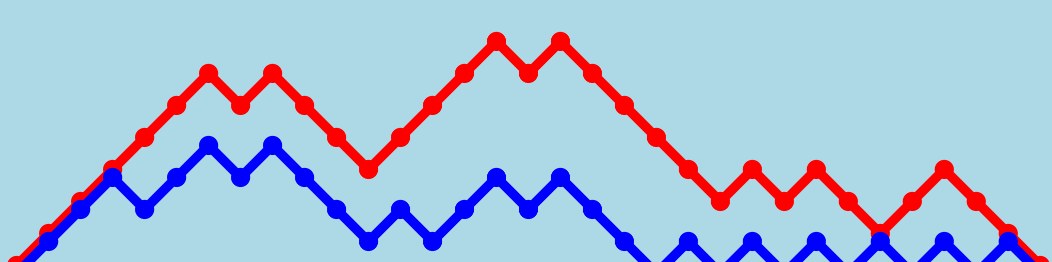

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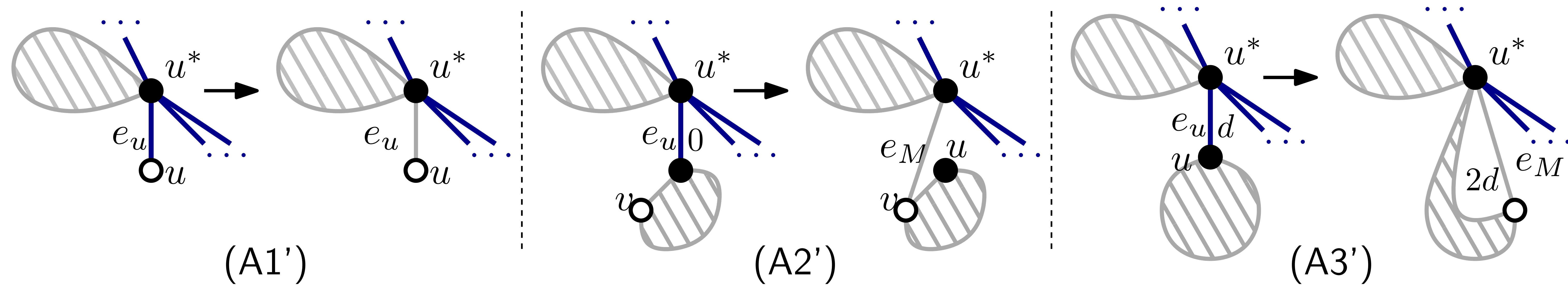
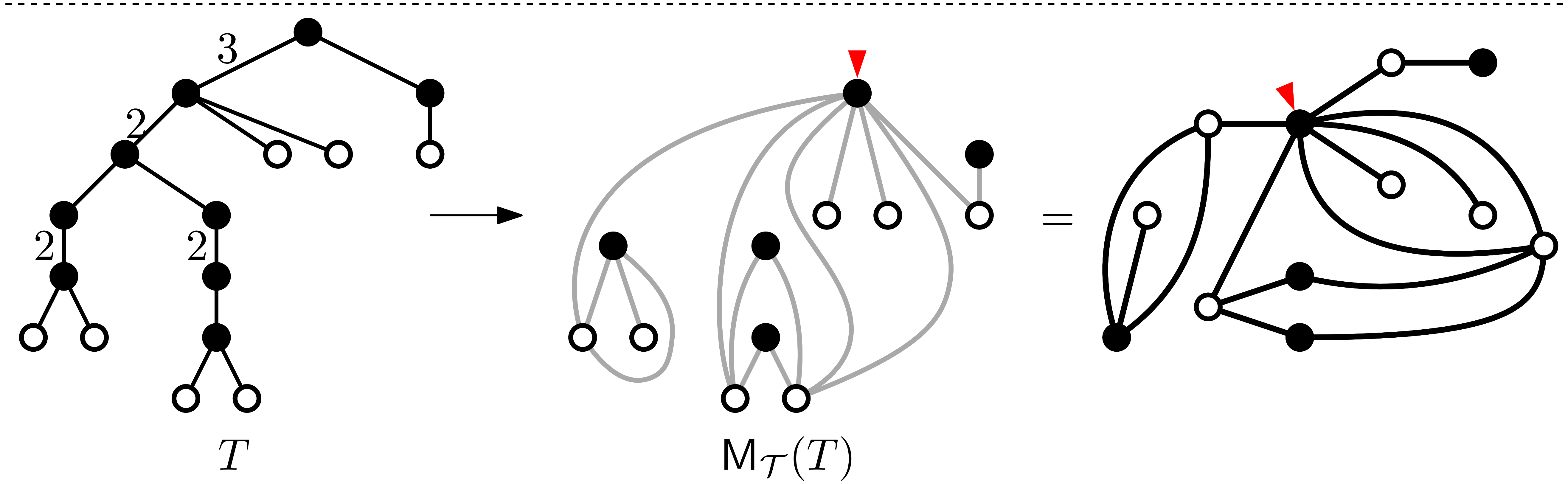
From trees to maps: an exploration

Depth-first exploration of edges, counter-clockwise, starting from the root

Turning edges in (T, ℓ) int edges in M. Walking only on black vertices, except for leaves





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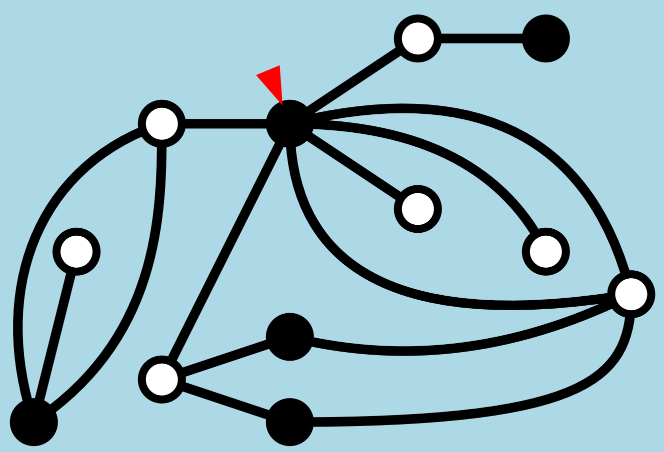

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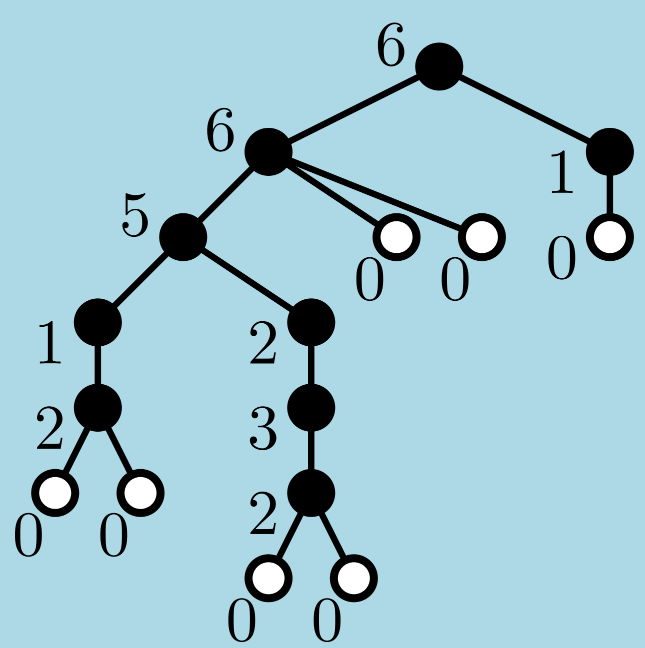

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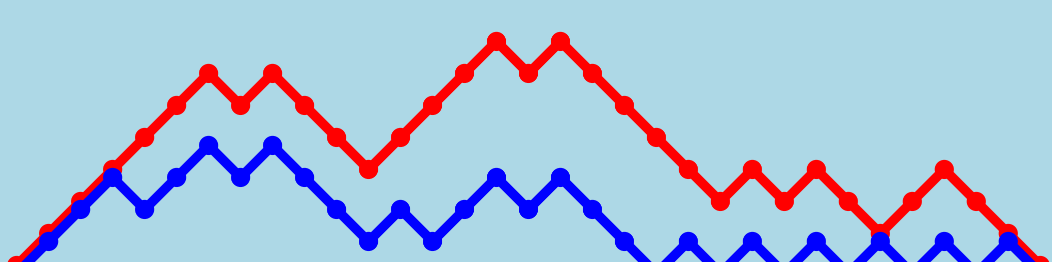


Chapoton's new interval of length $2n+2$

From trees to intervals

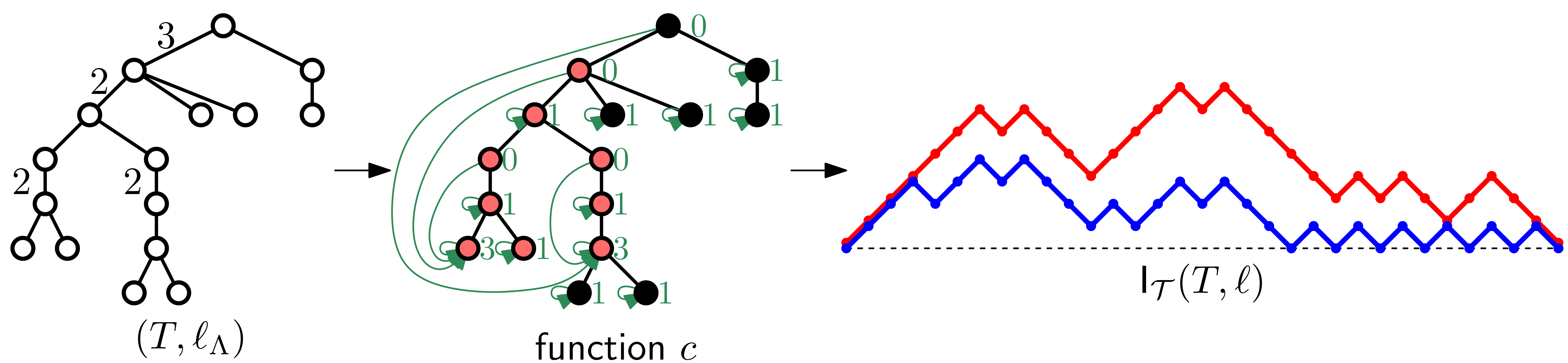
The **certificates** of nodes in (T, ℓ) is defined by a coloring process (reverse preorder):

- Initially all nodes are black;
- v a leaf $\Rightarrow$ the certificate of v is v
- v not a leaf, e its first down edge $\Rightarrow$ visit nodes after v in preorder, color visited nodes red, stop just before $(\ell_\Lambda(e)+1)$-st black node. The certificate of v is the last visited node.

Certificate function c of (T, ℓ): $c(u) = \#$nodes with u as certificate

From (T, ℓ) to $I = [P, Q]$:

- P: concatenation of $ud^{c(v)}$ for v in preorder;
- Q: $uQ'd$ with Q' from contour walk.



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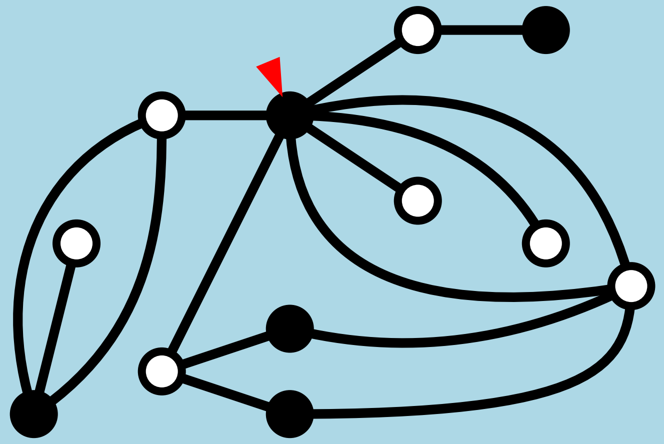

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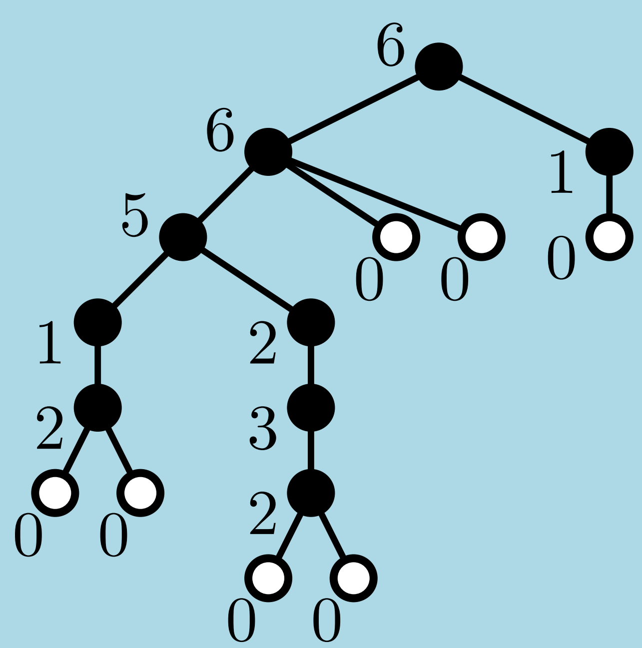

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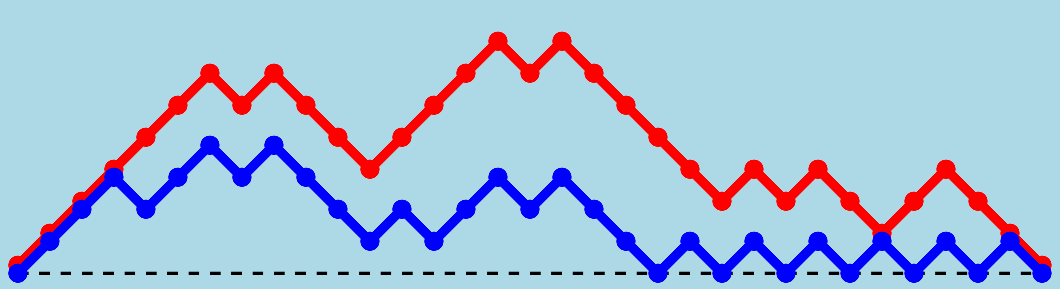

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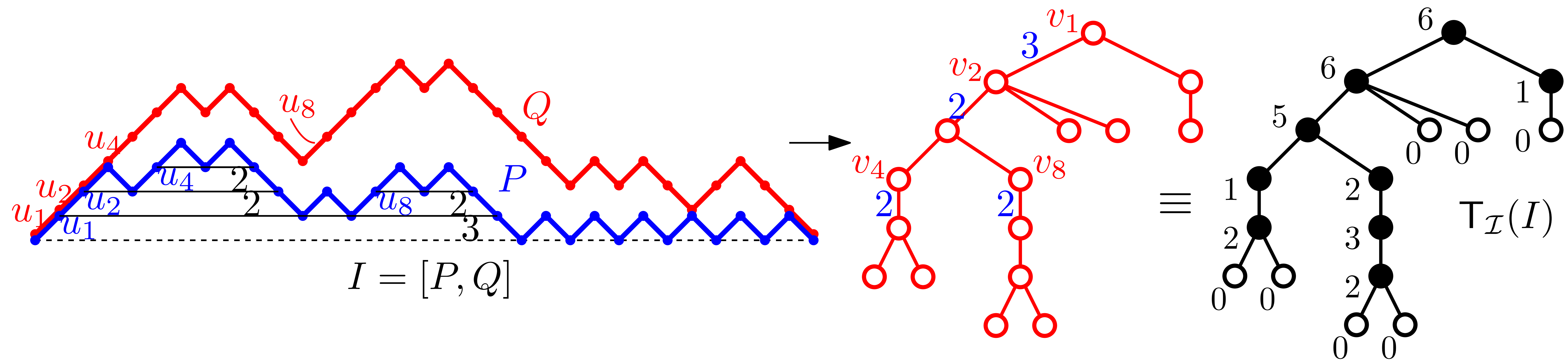
From intervals to trees

From $I = [P, Q]$ to (T, ℓ) (or (T, ℓ_Λ)):

- T: from $Q = uQ'd$;
- ℓ_Λ: from rising contacts of subpath between matching steps.



Statistics correspondence

$$c_{0,0}(I) = \mathrm{lnode}(T, \ell), \quad c_{0,1}(I) = \mathrm{znode}(T, \ell), \quad c_{1,1}(I) = \mathrm{pnode}(T, \ell).$$

For $M \longleftrightarrow (T, \ell) \longleftrightarrow I$:

$$c_{0,0}(I) = \mathrm{white}(M), \quad c_{0,1}(I) = \mathrm{black}(M), \quad c_{1,1}(I) + 1 = \mathrm{face}(M).$$

Symmetries and structures

- "Derecursivified" version of recursive decompositions known to Chapoton and Fusy;
- Bijective explanation of the S_3 symmetry.